Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
20 (2004), 233–238
www.emis.de/journals
ISSN 1786-0091

REMARKS ON GENERALIZED RAMANUJAN SUMS AND EVEN FUNCTIONS

LÁSZLÓ TÓTH

Dedicated to Professor W.R. Wade on his 60th birthday

Abstract. We prove a simple formula for the main value of r-even functions and give applications of it. Considering the generalized Ramanujan sums $c_A(n,r)$ involving regular systems A of divisors we show that it is not possible to develop a Fourier theory with respect to $c_A(n,r)$, like in the the usual case of classical Ramanujan sums $c(n,r)$.

1. Introduction

Ramanujan's trigonometric sum $c(n,r)$ is defined as the sum of n-th powers of the r-th primitive roots of unity, i.e.

$$c(n,r) = \sum_{\substack{k \pmod r \\ (k,r)=1}} \exp(2\pi ikn/r),$$

where $r, n \in \mathbb{N} \equiv \{1,2,3,\ldots\}$. In his original paper [R18] S. Ramanujan proved, among others, that for every $n \in \mathbb{N}$,

$$\begin{aligned} \frac{\sigma(n)}{n} &= \frac{\pi^2}{6}\sum_{r=1}^{\infty}\frac{c(n,r)}{r^2} \\ &= \frac{\pi^2}{6}\left(1+\frac{(-1)^n}{2^2}+\frac{2\cos(2\pi n/3)}{3^2}+\frac{2\cos(\pi n/2)}{4^2}+\cdots\right), \end{aligned} \tag{1}$$

where $\sigma(n)$ stands for the sum of the positive divisors of n. Formula (1) shows how the values of $\sigma(n)/n$ fluctuate harmonically about their mean value $\pi^2/6$. Here the main value of a function $f\colon \mathbb{N} \to \mathbb{C}$ is defined by $M(f) = \lim_{x\to\infty} \sum_{n\le x} f(n)$, if this limit exists.

The orthogonality relations

$$M(c(\cdot,r)c(\cdot,s)) = \delta_{r,s}\phi(r), \tag{2}$$

where $\delta_{r,s}$ is the Kronecker-symbol and $\phi(r) = c(r,r)$ is Euler's arithmetical function, suggest to have expansions, convergent pointwise or in other sense, of functions f of the form

$$f(n) = \sum_{r=1}^{\infty} a_r c(n,r), \quad n \in \mathbb{N},$$

2000 *Mathematics Subject Classification.* 11A25, 11L03, 11N37.

Key words and phrases. Ramanujan sum, r-even function, mean value, regular system of divisors.

Supported partially by the Hungarian National Foundation for Scientific Research under grant OTKA T 031877.

where the coefficients a_r are given by

$$a_r = \frac{1}{\phi(r)} M(f c_r).$$

A Fourier analysis of arithmetical functions, with respect to Ramanujan sums, parallel to periodic and almost periodic functions, was developed by several authors, cf. J. Delsarte [D45], W. Schwarz and J. Spilker [SchS74], A. Hildebrand [H84], G. Gát [G91], L. Lucht [L95], see also the books [K75], [SchS94].

Ramanujan's sum $c(n,r)$ is an example of an r-even function (even function (mod r)), i.e. a function $f(n,r)$ such that $f(n,r) = f(gcd(n,r),r)$ for every $n,r \in \mathbb{N}$. This concept was introduced by E. Cohen [C55]. For a fixed r the set $\mathcal{E}_r$ of r-even functions $f\colon \mathbb{N} \to \mathbb{C}$ is a complex Hilbert space of finite dimension $\tau(r)$ = the number of divisors of r, under the inner product

$$\langle f, g\rangle = \frac{1}{r} \sum_{d|r} \phi(d) f(r/d) \overline{g(r/d)},$$

and $(c'(\cdot,q))_{q|r}$, $c'(n,q) = \frac{1}{\sqrt{\phi(q)}} c(n,q)$ is an orthonormal basis for $\mathcal{E}_r$. The main value $M(c(\cdot,r))$ exists for every $r \geq 1$, it is $M(c(\cdot,r)) = \delta_{r,1}$, hence $M(f)$ exists for each $f \in \mathcal{E}_r$. If $\mathcal{E} = \cup_{r\in\mathbb{N}} \mathcal{E}_r$, then $\mathcal{E}$ is a dense subalgebra of the algebra $\mathbb{C}^{\mathbb{N}}$.

For these and various other properties of $c(n,r)$ and of r-even functions see [K75], [McC86], [SchS94].

The Ramanujan sum $c(n,r)$ has been generalized in several directions. One of the generalizations, due to P.J. McCarthy [McC68], notation $c_A(n,r)$, is involving regular systems A of divisors, see Section 2, and it has all nice algebraic properties of the usual kind.

The following question can be formulated. Is it possible to develop a Fourier theory concerning the generalized sums $c_A(n,r)$, analogous to the usual one?

The aims of this paper are the following:

- to prove a simple formula for the main value of r-even functions (Proposition 1), this result seems to have not been appeared in the literature, and to give applications of it,
- to compute the main value of $c_A(\cdot,r)$ for an arbitrary regular system A (Proposition 2),
- to show that the answer is negative for the question formulated above (Propositions 3 and 4).

2. Regular convolutions

Let $A(n)$ be a subset of the set of positive divisors of n for each $n \in \mathbb{N}$. The A-convolution of the functions $f, g\colon \mathbb{N} \to \mathbb{C}$ is given by

$$(f *_A g)(n) = \sum_{d\in A(n)} f(d) g(n/d), \quad n \in \mathbb{N}.$$

The system $A = (A(n))_{n\in\mathbb{N}}$ of divisors is called regular, cf. [N63], if

(a) $(\mathbb{C}^{\mathbb{N}}, +, *_A)$ is a commutative ring with unity,
(b) the A-convolution of multiplicative functions is multiplicative (recall that function f is multiplicative if $f(mn) = f(m)f(n)$ whenever $gcd(m,n) = 1$),
(c) the constant 1 function has an inverse μ_A (generalized Möbius function) with respect to $*_A$ and $\mu_A(p^a) \in \{-1, 0\}$ for every prime power $p^a (a \geq 1)$.

It can be shown, cf. [N63], [McC86], that $*_A$ is regular iff

(i) $A(mn) = \{de : d \in A(m), e \in A(n)\}$ for every $m,n \in \mathbb{N}, (m,n) = 1$,

(ii) for every prime power $p^a (a \geq 1)$ there exists a divisor $t = t_A(p^a)$ of a, called the type of p^a with respect to A, such that $A(p^{it}) = \{1, p^t, p^{2t}, \ldots, p^{it}\}$ for every $i \in \{0, 1, \ldots, a/t\}$.

Examples of regular systems of divisors are $A = D$, where $D(n)$ is the set of all positive divisors of n, and $A = U$, where $U(n)$ is the set of divisors d of n such that $(d, n/d) = 1$ (unitary divisors). For every prime power p^a one has $t_D(p^a) = 1$ and $t_U(p^a) = a$. Here $*_D$ and $*_U$ are the Dirichlet convolution and the unitary convolution, respectively. For properties of regular convolutions and related arithmetical functions we refer to [N63], [McC86], [S78], [T97].

The following generalization of $c(n,r)$ is due to P.J. McCarthy [McC68], see also [McC86]. For a regular system A of divisors and $r, n \in \mathbb{N}$ let

$$c_A(n,r) = \sum_{\substack{k \pmod r \\ (k,r)_A = 1}} \exp(2\pi i k n/r),$$

where $(k,r)_A = \max\{d \in \mathbb{N} : d|k, d \in A(r)\}$, and let $c_A(r,r) \equiv \phi_A(r)$ be the generalized Euler function. For $A = U$ the functions $c_U(n,r) \equiv c^*(n,r)$ and $\phi_U(r) \equiv \phi^*(r)$ were introduced by E. Cohen [C60].

$c_A(n,r)$ preserves the basic properties of $c(n,r)$. For example, for every regular A and $r, n \in \mathbb{N}$ one has

$$c_A(n,r) = \sum_{d|n, d \in A(r)} d\mu_A(r/d), \tag{3}$$

hence $c_A(n,r)$ is integer-valued and it is multiplicative in r. Note that

$$c_A(n,r) = \sum_{d|r, \gamma_A(r)|d} c(n,d), \tag{4}$$

where γ_A is multiplicative and $\gamma_A(p^a) = p^{a-t+1}$ for every prime power p^a $(a \geq 1)$, here $t = t_A(p^a)$ (generalized core function), see [McC68], Th. 2.

The function f is called A-even (mod r) if $f(n,r) = f((n,r)_A, r)$ for each $n, r \in \mathbb{N}$, cf. [McC68]. Let $\mathcal{E}_{A,r}$ denote the set of functions $f(n,r)$ which are A-even (mod r). Then $\mathcal{E}_{A,r} \subset \mathcal{E}_r$ for every A and $r \in \mathbb{N}$. For example, $c_A(n,r)$ is A-even (mod r). Let $\mathcal{E}_A = \cup_{r \in \mathbb{N}} \mathcal{E}_{A,r}$.

3. Results

Proposition 1. *Let $r \in \mathbb{N}$ and $f \in \mathcal{E}_r$. Then*

$$M(f) = \frac{1}{r}(f *_D \phi)(r) \equiv \frac{1}{r} \sum_{e|r} f(e)\phi(r/e).$$

Moreover, for every $x \geq 1$ and every $\varepsilon > 0$,

$$\frac{1}{K_f} \left| \sum_{n \leq x} f(n) - M(f)x \right| \leq C_\varepsilon r^{1+\varepsilon},$$

where $|f(n)| \leq K_f, n \in \mathbb{N}$ and C_ε is a constant depending only on ε.

Proof. Write f in the form

$$f(n) = \sum_{q|r} h(q) c(n,q), \quad n \in \mathbb{N},$$

where the Fourier coefficients $h(q)$, are given for every $q|r$ by

$$h(q) = \frac{1}{r\phi(q)} \sum_{e|r} \phi(e) f(r/e) c(r/e, q) = \frac{1}{r} \sum_{e|r} f(r/e) c(r/q, e),$$

and use that $\sum_{n\le x} c_q(n) = \delta_{q,1}x + R_q(x)$, where $|R_q(x)| \le q^{1+\varepsilon}$, cf. [McC86], Ch. 2, [K75], Ch. 7. Then

$$\sum_{n\le x} f(n) = \sum_{n\le x}\sum_{q|r} h(q)c(n,q) = \sum_{q|r} h(q)\sum_{n\le x} c(n,q)$$
$$= \sum_{q|r} h(q)(\delta_{q,1}x + R_q(x)) = h(1)x + \sum_{q|r} h(q)R_q(x),$$

where $h(1) = \frac{1}{r}(f *_D \phi)(r)$, $|h(q)| \le \frac{1}{r}\sum_{e|r} |f(r/e)||c(r/q,e)| \le K_f\frac{1}{r}\sum_{e|r} e = K_f\sigma(r)/r$ and

$$|\sum_{q|r} h(q)R_q(x)| \le K_f(\sigma(r)/r)\sum_{q|r} q^{1+\varepsilon}$$

and the result follows by the usual estimates $\sigma(r) \le r\tau(r)$ and $\tau(r) \ll r^{\varepsilon}$. □

As an application, consider the function $\phi(s,d,n)$ defined by $\phi(s,d,n) = \#\{k \in \mathbb{N}\cap[1,n] : (s+(k-1)d,n) = 1\}$, where $s,d \in \mathbb{N}, (s,d) = 1$. Note that $\phi(1,1,n) = \phi(n)$ is the Euler function. T. Maxsein [M90] pointed out that $\phi(s,\cdot,n)$ is an n-even function and determined its main value:

$$M(\phi(s,\cdot,n)) = n\prod_{p|n}(1 - 1/p + 1/p^2),$$

where the product is over the prime divisors p of n. This follows from Proposition 1 by easy computations.

Proposition 1 applies also for $f(n) = c_A(n,r)$, which is an r-even function. However, a better error term can be obtained and the computations are simpler by a direct proof using representation (3). We have

Proposition 2. *For every regular system A and $r \in \mathbb{N}$,*

$$M(c_A(\cdot,r)) = \delta_{r,1}$$

and

$$|\sum_{n\le x} c_A(n,r) - \delta_{r,1}x| \le \psi_A(r),$$

where ψ_A is multiplicative and $\psi_A(p^a) = p^a + p^{a-t}$ for every prime power p^a $(a \ge 1)$, where $t = t_A(p^a)$ (generalized Dedekind function).

Proof. Using (3),

$$\sum_{n\le x} c_A(n,r) = \sum_{\substack{n\le x \\ d|n, d\in A(r)}} d\mu_A(r/d) = \sum_{d\in A(r)} d\mu_A(r/d)\sum_{\substack{n\le x\\ d|n}} 1 = \sum_{d\in A(r)} d\mu_A(r/d)[x/d]$$
$$= x\sum_{d\in A(r)}\mu_A(r/d) - \sum_{d\in A(r)} d\mu_A(r/d)(x/d - [x/d])$$
$$= x\delta_{r,1} + R_A(r),$$

where

$$|R_A(r)| \le \sum_{d\in A(r)} d|\mu_A(r/d)| = \psi_A(r).$$

□

Note that $\psi_A(r) \le \sigma(r) < Cr\ln\ln r$ for every $r \in \mathbb{N}$, with a suitable constant C.

The following result shows that for every system $A \ne D$ the orthogonality relations (2) are violated.

Proposition 3. *For every regular system A,*

$$M(c_A(\cdot,r)c_A(\cdot,s)) = \begin{cases} \phi_A(r), & \text{if } r = s, \\ 0, & \text{if } rs > 1, (r,s) = 1, \end{cases}$$

but for $A \neq D$ there exist r, s such that $r \neq s$ and $M(c_A(\cdot,r)c_A(\cdot,s)) \neq 0$.

Proof. Let A be arbitrary. Applying (4) and (2) we obtain

$$M(c_A(\cdot,r)c_A(\cdot,s)) = \sum_{\substack{d|r,\gamma_A(r)|d \\ e|s,\gamma_A(s)|e}} M(c(\cdot,r)c(\cdot,s)) = \sum_{\substack{d|r,\gamma_A(r)|d \\ d|s,\gamma_A(s)|d}} \phi(d). \tag{5}$$

Using that $\gamma_A(k) > 1$ for $k > 1$ we get the first part of the desired result.

Now let $A \neq D$. Then there exists a prime power p^a such that $t \equiv t_A(p^a) > 1$. Therefore $A(p^t) = \{1, p^t\}$. Let $r = p$, $s = p^t$, then $r \neq s$ and the last sum in (5) has one single term, namely $\phi(p) = p - 1 \neq 0$. $\square$

Proposition 4. *$\mathcal{E}_A$ is a vector space if and only if $A = D$.*

Proof. We show that $\mathcal{E}_A$ is not a vector space for $A \neq D$.

Suppose that $A \neq D$. Then there exists a prime power p^a such that $t \equiv t_A(p^a) > 1$. Hence $A(p^t) = \{1, p^t\}$. Let

$$f(n) = (n,p)_A = \begin{cases} p, \text{ if } p|n, \\ 1, \text{ otherwise}, \end{cases}$$

$$g(n) = (n,p^t)_A = \begin{cases} p^t, \text{ if } p^t|n, \\ 1, \text{ otherwise}. \end{cases}$$

Then $f, g \in \mathcal{E}_A$ and suppose that $h = f + g \in \mathcal{E}_A$, i.e. there exists $r \in \mathbb{N}$ such that $f + g \in \mathcal{E}_{A,r}$. Here

$$h(n) = f(n) + g(n) = \begin{cases} p + p^t, \text{ if } p^t|n, \\ 1 + p, \text{ if } p|n, p^t \not| n, \\ 2, \text{ if } p \not| n. \end{cases}$$

From $1 + p = h(p) = h((p,r)_A)$ we have $(p,r)_A = p$, $p \in A(r)$ and from $p + p^t = h(p^t) = h((p^t,r)_A)$ we obtain $(p^t,r)_A = p^t$, $p^t \in A(r)$.

Let $r = p^k s$, where $p \not| s$. Then $p \in A(p^k)$ and $p^t \in A(p^k)$, therefore $p \in A(p^t)$, in contradiction with $A(p^t) = \{1, p^t\}$. $\square$

References

[C55] E. Cohen. A class of arithmetical functions. *Proc. Natl. Acad. Sci. U.S.A.*, 41:939–944, 1955.

[C60] E. Cohen. Arithmetical functions associated with the unitary divisors of an integer. *Math. Z.*, 74:66–80, 1960.

[D45] J. Delsarte. Essai sur l'application de la théorie des fonctions presque-périodiques á l'arithmétique. *Ann. Sci. École Norm. Sup.*, 62:185–204, 1945.

[G91] G. Gát. On almost even arithmetical functions via orthonormal systems on vilenkin groups. *Acta Arith.*, 60:105–123, 1991.

[H84] A. Hildebrand. Über die punktweise Konvergenz von Ramanujan-Entwiklungen zahlentheoretischer Funktionen. *Acta Arith.*, 44:109–140, 1984.

[K75] J. Knopfmacher. *Abstract Analytic Number Theory.* North Holland Publ. Co., 1975.

[L95] L. Lucht. Weighted relationship theorems and Ramanujan expansions. *Acta Arith.*, 70:25–42, 1995.

[M90] T. Maxsein. A note on a generalization of Euler's φ-function. *Indian J. Pure Appl. Math.*, 21:691–694, 1990.

[McC68] P. J. McCarthy. Regular arithmetical convolutions. *Portugal. Math.*, 27:1–13, 1968.

[McC86] P. J. McCarthy. *Introduction to Arithmetical Functions.* Springer- Verlag, 1986.

[N63] W. Narkiewicz. On a class of arithmetical convolutions. *Colloq. Math.*, 10:81–94, 1963.

[R18] S. Ramanujan. On certain trigonometric sums and their applications in the theory of numbers. *Transactions Cambridge Phil. Soc.*, 22:259–276, 1918.

[S78] V. Sita Ramaiah. Arithmetical sums in regular convolutions. *J. Reine Angew. Math.*, 303/304:265–283, 1978.

[SchS74] W. Schwarz and J. Spilker. Mean values and Ramanujan expansions of almost even arithmetical functions. In *Topics in Number Theory*, volume 13 of *Colloq. Math. Soc. J. Bolyai*, pages 315–357, 1974.

[SchS94] W. Schwarz and J. Spilker. *Arithmetical Functions*, volume 184 of *London Math. Soc. Lecture Notes Series*. Cambridge Univ. Press, 1994.

[T97] L. Tóth. Asymptotic formulae concerning arithmetical functions defined by cross-convolutions, I. Divisor-sum functions and Euler-type functions. *Publ. Math. Debrecen*, 50:159–176, 1997.

Institute of Mathematics and Computer Science,
University of Pécs,
Ifjúság u. 6, 7624 Pécs,
Hungary
E-mail address: ltoth@ttk.pte.hu