Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
20 (2004), 239–266
www.emis.de/journals
ISSN 1786-0091

SUMMATION OF FOURIER SERIES

FERENC WEISZ

Dedicated to Professor William R. Wade on his 60-th birthday

Abstract. A general summability method of different orthogonal series is given with the help of an integrable function θ. As special cases the trigonometric Fourier, Walsh-, Walsh-Kaczmarz-, Vilenkin- and Ciesielski-Fourier series and the Fourier transforms are considered. For each orthonormal system a different Hardy space is introduced and the atomic decomposition of these Hardy spaces are presented. A sufficient condition is given for a sublinear operator to be bounded on the Hardy spaces. Under some conditions on θ it is proved that the maximal operator of the θ-means of these Fourier series is bounded from the Hardy space H_p to L_p ($p_0 < p \leq \infty$) and is of weak type (1,1), where $p_0 < 1$ is depending on θ. In the endpoint case $p = p_0$ a weak type inequality is derived. As a consequence we obtain that the θ-means of a function $f \in L_1$ converge a.e. to f. Some special cases of the θ-summation are considered, such as the Cesàro, Fejér, Riesz, de La Vallée-Poussin, Rogosinski, Weierstrass, Picar, Bessel and Riemann summations. Similar results are verified for several-dimensional Fourier series and Hardy spaces and for the multi-dimensional dyadic derivative.

1. Introduction

Lebesgue [38] proved that the Fejér means $\sigma_n f$ of the trigonometric Fourier series of a function $f \in L_1$ converge a.e. to f as $n \to \infty$. It is known that the maximal operator of the Fejér means is of weak type $(1,1)$, i.e.

$$\sup_{\rho>0} \rho\lambda(\sigma_* f > \rho) \leq C\|f\|_1 \qquad (f \in L_1)$$

(see Zygmund [100]) and that σ_* is bounded from the classical H_1 Hardy space to L_1 (see Móricz [42, 43, 44] and Weisz [80, 84]), where $\sigma_* := \sup_{n\in\mathbb{N}} |\sigma_n|$. The author [80, 84, 85] verified that σ_* is also bounded from H_p to L_p whenever $1/2 < p < \infty$. The same results are known for the Walsh system (see Fine [23], Schipp [53], Fujii [26] and Weisz [77]), for the Walsh-Kaczmarz system (see Gát [28] and Simon [59, 58]), for the Vilenkin system (see Simon [57] and Weisz [83]) and for the Ciesielski system (see Weisz [92]).

Butzer and Nessel [7] and recently Bokor, Schipp, Szili and Vértesi [5, 48, 49, 62, 63] considered a general method of summation, the so called θ-summability. They

2000 *Mathematics Subject Classification.* Primary 42B08, 42A24, 42C10, Secondary 42B30, 41A15, 42A38.

Key words and phrases. Fejér means, Cesàro means, θ-summability, trigonometric system, Walsh system, Walsh-Kaczmarz system, Vilenkin system, Ciesielski system, Fourier transforms, Hardy spaces, atomic decomposition, dyadic derivative.

This research was supported by the Hungarian Scientific Research Funds (OTKA) No T043769, T047128, T047132.

proved that if $\hat{\theta}$ can be estimated by a non-increasing integrable function, then the θ-means of a function $f \in L_1(\mathbb{R})$ converge a.e. to f. This convergence result is also proved there for the θ-means of trigonometric Fourier series (see also Stein and Weiss [60]). As special cases they considered the Weierstrass, Picar, Bessel, Fejér, de La Vallée-Poussin and Riesz summations.

In this survey paper we summarize the results appeared in this topic in the last 10–20 years. With the help of an integrable function θ a general summability method (called θ-summability) of different orthogonal series is considered. As special cases the trigonometric Fourier, Walsh-, Walsh-Kaczmarz-, Vilenkin- and Ciesielski-Fourier series and the Fourier transforms are examined. For each orthonormal or biorthonormal system we introduce a different Hardy space. The atomic decomposition of each Hardy space is presented. With the help of the atomic decomposition a sufficient condition is given for a sublinear operator to be bounded from the Hardy space to the L_p space. Under some weak conditions on θ it is proved by the preceding theorem that the maximal operator of the θ-means of these Fourier series is bounded from the Hardy space H_p to L_p $(p_0 < p \leq \infty)$ and is of weak type (1,1), where $p_0 < 1$ is depending on θ. In the endpoint case $p = p_0$ a weak type inequality is derived. For $p < p_0$ the result is not true in general. As a consequence we obtain that the θ-means of a function $f \in L_1$ converge a.e. to f. Some special cases of the θ-summation are considered, such as the Cesàro, Fejér, Riesz, de La Vallée-Poussin, Rogosinski, Weierstrass, Picar, Bessel and Riemann summations. Similar results are verified for several-dimensional Fourier series and Hardy spaces and for the multi-dimensional dyadic derivative.

2. θ-summability of Fourier series

We consider the unit interval $[0,1)$ and the Lebesgue measure λ on it. We also use the notation $|I|$ for the Lebesgue measure of the set I. We briefly write L_p instead of the real $L_p([0,1),\lambda)$ space while the norm (or quasi-norm) of this space is defined by $\|f\|_p := (\int_{[0,1)} |f|^p \, d\lambda)^{1/p}$ $(0 < p \leq \infty)$. The space $L_{p,\infty} = L_{p,\infty}([0,1),\lambda)$ $(0 < p < \infty)$ consists of all measurable functions f for which

$$\|f\|_{p,\infty} := \sup_{\rho>0} \rho\lambda(|f| > \rho)^{1/p} < \infty,$$

while we set $L_{\infty,\infty} = L_\infty$. Note that $L_{p,\infty}$ is a quasi-normed space. It is easy to see that

$$L_p \subset L_{p,\infty} \qquad \text{and} \qquad \|\cdot\|_{p,\infty} \leq \|\cdot\|_p$$

for each $0 < p \leq \infty$.

Let $\mathbb{M}$ denote either $\mathbb{Z}$ or $\mathbb{N}$. Suppose that ϕ_n and ψ_n $(n \in \mathbb{M})$ are real or complex valued **uniformly bounded** functions and

$$\int_0^1 \phi_n \overline{\psi_m} \, d\lambda = \begin{cases} 1, & \text{if } n = m \\ 0, & \text{if } n \neq m. \end{cases}$$

This means that the system

$$\Psi := (\phi_n, \psi_n, n \in \mathbb{M})$$

is *biorthogonal*.

For a function $f \in L_1$ the nth Fourier coefficient with respect to Ψ is defined by

$$\hat{f}(n) := \int_{[0,1)} f \overline{\phi_n} \, d\lambda.$$

Denote by $s_n^\Psi f$ the nth partial sum of the Fourier series of $f \in L_1$, namely,

$$s_n^\Psi f := \sum_{k\in\mathbb{M}, |k|\le n} \hat{f}(k)\psi_k \qquad (n \in \mathbb{N}).$$

Obviously,

$$s_n^\Psi f(x) = \int_0^1 f(t) D_n^\Psi(t,x)\,dt,$$

where the *Dirichlet kernels* are defined by

$$D_n^\Psi(t,x) := \sum_{k\in\mathbb{M}, |k|\le n} \overline{\phi_k}(t)\psi_k(x) \qquad (n \in \mathbb{N}, t, x \in [0,1)).$$

The *Fejér means* $\sigma_n^\Psi f$ $(n \in \mathbb{N})$ of an integrable function f are given by

$$\sigma_n^\Psi f := \frac{1}{n+1}\sum_{k=0}^n s_k^\Psi f = \sum_{k\in\mathbb{M}, |k|\le n} \Big(1 - \frac{|k|}{n+1}\Big)\hat{f}(k)\psi_k.$$

If

$$K_n^\Psi := \frac{1}{n+1}\sum_{k=0}^n D_k^\Psi$$

denotes the n-th *Fejér kernel*, then

$$\sigma_n^\Psi f(x) = \int_0^1 f(t) K_n^\Psi(t,x)\,dt \qquad (f \in L_1, t, x \in [0,1)).$$

The *maximal Fejér operator* is defined by

$$\sigma_*^\Psi f := \sup_{n\in\mathbb{N}} |\sigma_n^\Psi f|.$$

Recall that the *Fourier transform* of an integrable function $f \in L_1(\mathbb{R})$ is defined by

$$\hat{f}(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-\imath tx}\,dx. \tag{1}$$

We are going to introduce the θ-summability, which was considered in Butzer and Nessel [7]. More recently Bokor, Schipp, Szili and Vértesi [48, 5, 49, 62, 63] investigated the uniform convergence of the θ-means and some interpolation problems for continuous functions.

In what follows we consider two types of *θ-summations*. First suppose that the sequence

$$\theta = (\theta(k, n+1), k \in \mathbb{Z}, n \in \mathbb{N})$$

of real numbers is even in the first parameter, more precisely, $\theta(-k, n+1) = \theta(k, n+1)$ for each $k \in \mathbb{Z}, n \in \mathbb{N}$. We suppose that

$$\theta(0, n+1) = 1, \quad \lim_{n\to\infty} \theta(k, n+1) = 1, \quad (\theta(k, n+1))_{k\in\mathbb{Z}} \in \ell_1 \tag{2}$$

for each $n, k \in \mathbb{N}$. For this first type we will investigate the Cesàro summability.

For the other type of θ-summations let $\theta \in L_1(\mathbb{R})$ be an even continuous function satisfying

$$\theta(0) = 1, \qquad \hat{\theta} \in L_1(\mathbb{R}), \qquad \lim_{x\to\infty} \theta(x) = 0, \qquad \Big(\theta(\frac{k}{n+1})\Big)_{k\in\mathbb{Z}} \in \ell_1 \tag{3}$$

for each $n \in \mathbb{N}$. Note that this last condition is satisfied if θ is non-increasing on (c, ∞) for some $c \ge 0$ or if it has compact support. We write

$$\theta(k, n+1) = \theta(\frac{k}{n+1}).$$

We consider several well known summability methods of this type.

Besides (2) or (3) one of the following conditions is always supposed.

- (i) $\theta \in L_1(\mathbb{R})$ and $|t^{i+2}\hat{\theta}^{(i+1)}(t)| \le C$ for all $i = 0, \ldots, N$ where $N \in \mathbb{N}$ and $\hat{\theta}^{(N+1)} \ne 0$. In this case let $p_0 = 1/(N+2)$.
- (ii) $\theta \in L_1(\mathbb{R})$, $|t^{\alpha+1}\hat{\theta}(t)| \le C$ and $|t^{\alpha+1}\hat{\theta}'(t)| \le C$ for some $0 < \alpha \le 1$. Moreover, $|K_n^\theta| \le Cn$ and $|(K_n^\theta)'| \le Cn^2$. Let $p_0 = 1/(\alpha+1)$.
- (iii) θ denotes the (C,α) or Riesz summation for $0 < \alpha \le 1 \le \gamma < \infty$ (see Examples 1 and 3). Let $p_0 = 1/(\alpha+1)$.
- (iv) θ is twice continuously differentiable on $\mathbb{R}$ except of finitely many points, $\theta'' \ne 0$ except of finitely many points and finitely many intervals, the left and right limits $\lim_{x\to y\pm 0} x\theta'(x) \in \mathbb{R}$ does exist at each point $y \in \mathbb{R}$ and $\lim_{x\to\infty} x\theta'(x) = 0$. Let $p_0 = 1/2$.

Butzer and Nessel [7, pp. 248-251] verified that if θ is even, $\lim_{x\to\infty}\theta(x) = 0$ and θ, θ' and $x\theta''(x)$ are integrable functions, then $\hat{\theta} \in L_1(\mathbb{R})$. Using this one can show that $\hat{\theta} \in L_1(\mathbb{R})$ follows from (iv) and from the other conditions of (3) (see Weisz [96, Theorem 4]). Moreover, if

$$\lim_{x\to\infty} x\theta(x) = 0 \tag{4}$$

then (iv) implies (i) with $N = 0$. This can similarly be proved as Lemma 5.3 in Weisz [94].

The θ-*means* of $f \in L_1$ are defined by

$$\begin{aligned} \sigma_n^{\Psi,\theta} f(x) &:= \sum_{k\in\mathbb{M}} \theta(k, n+1)\hat{f}(k)\psi_k(x) \\ &= \int_0^1 f(t) K_n^{\Psi,\theta}(t,x)\, dt, \end{aligned}$$

where the $K_n^{\Psi,\theta}$ kernels satisfy

$$K_n^{\Psi,\theta}(t,x) := \sum_{k\in\mathbb{M}} \theta(k, n+1)\overline{\phi_k}(t)\psi_k(x) \qquad (n \in \mathbb{N}, t, x \in [0,1)),$$

which is well defined by (3). We define the *maximal θ-operator* by

$$\sigma_*^{\Psi,\theta} f := \sup_{n\in\mathbb{N}} |\sigma_n^{\Psi,\theta} f| \qquad (f \in L_1).$$

If $\theta(x) := (1 - |x|) \vee 0$, then we get the Fejér kernels and means.

The constants C are absolute constants and the constants C_p are depending only on p and may denote different constants in different contexts.

Under some conditions we have proved in [96] that if the maximal Fejér-operator σ_*^Ψ is bounded on a quasi-normed space then so is $\sigma_*^{\Psi,\theta}$. Let $\mathbf{X}$ and $\mathbf{Y}$ be two complete quasi-normed spaces of measurable functions, L_∞ be continuously embedded into $\mathbf{X}$ and L_∞ be dense in $\mathbf{X}$. Suppose that if $0 \le f \le g$, $f, g \in \mathbf{Y}$ then $\|f\|_{\mathbf{Y}} \le \|g\|_{\mathbf{Y}}$. If $f_n, f \in \mathbf{Y}$, $f_n \ge 0$ $(n \in \mathbb{N})$ and $f_n \nearrow f$ a.e. as $n \to \infty$, then assume that $\|f - f_n\|_{\mathbf{Y}} \to 0$. Note that the spaces L_p and $L_{p,\infty}$ $(0 < p \le \infty)$ satisfy these properties.

Theorem 1. *Assume that (3) and (iv) are satisfied. Moreover, suppose that*

$$\int_0^1 |K_n^\Psi(t,x)|\, dt \le C \qquad (n \in \mathbb{N}, x \in [0,1)) \tag{5}$$

and

$$|D_n^\Psi(t,x)| \le \frac{C}{|t-x|} \qquad (t, x \in [0,1), t \ne x) \tag{6}$$

for all $n \in \mathbb{N}$. *If* $\sigma_*^\Psi : \mathbf{X} \to \mathbf{Y}$ *is bounded, i.e.*

$$\|\sigma_*^\Psi f\|_{\mathbf{Y}} \le C\|f\|_{\mathbf{X}} \qquad (f \in \mathbf{X} \cap L_\infty), \tag{7}$$

then $\sigma_*^{\Psi,\theta}$ *is also bounded,*

$$\|\sigma_*^{\Psi,\theta} f\|_{\mathbf{Y}} \le C\|f\|_{\mathbf{X}} \qquad (f \in \mathbf{X}). \tag{8}$$

Obviously, (5) yields that σ_* is bounded on L_∞, namely,

$$\|\sigma_*^\Psi f\|_\infty \le C\|f\|_\infty \qquad (f \in L_\infty).$$

If Ψ do not satisfy (6) then we suppose a little bit more on θ.

Theorem 2. *Instead of (6) assume (4). Then Theorem 1 holds also.*

For the question, how to prove (7) for Hardy spaces, see Section 5.

3. Some summability methods

In this section we consider several summability methods introduced in the book of Butzer and Nessel [7] and some other popular ones as special cases of the θ-summation. Of course, there are a lot of other summability methods which could be considered as special cases. It is easy to see that (2), (3) and (4) are satisfied all in the next examples. The elementary computations are left to the reader.

Example 1. (C, α) **or Cesàro summation.** Let

$$\theta_1(k, n+1) = \begin{cases} \frac{A_{n-|k|}^\alpha}{A_n^\alpha} & \text{if } |k| \le n \\ 0 & \text{if } |k| \ge n+1 \end{cases}$$

where

$$A_n^\alpha := \binom{n+\alpha}{n} = \frac{(\alpha+1)(\alpha+2)\dots(\alpha+n)}{n!}$$

$(0 < \alpha < \infty)$. The Cesàro operators are given by

$$\begin{aligned} \sigma_n^{\Psi,\theta_1} f(x) &:= \frac{1}{A_n^\alpha} \sum_{k \in \mathbb{M}, |k| \le n} A_{n-|k|}^\alpha \hat{f}(k)\psi_k(x) \\ &= \frac{1}{A_n^\alpha} \sum_{k=0}^n A_{n-k}^{\alpha-1} s_k^\Psi f. \end{aligned}$$

If $\alpha = 1$ then we get

Example 2. Fejér summation. Let

$$\theta_2(x) = \begin{cases} 1 - |x| & \text{if } |x| \le 1 \\ 0 & \text{if } |x| > 1. \end{cases}$$

σ_n^{Ψ,θ_2} is the nth Fejér operator:

$$\begin{aligned} \sigma_n^{\Psi,\theta_2} f(x) &:= \sum_{k \in \mathbb{M}, |k| \le n} \Big(1 - \frac{|k|}{n+1}\Big) \hat{f}(k)\psi_k(x) \\ &= \frac{1}{n+1} \sum_{k=0}^n s_k^\Psi f(x). \end{aligned}$$

It is known that

$$\hat{\theta}_2(x) = \frac{1}{\sqrt{2\pi}} \Big(\frac{\sin x/2}{x/2}\Big)^2$$

and

$$|\hat{\theta}_2'(x)| \le \frac{C}{x^2}.$$

Hence (i) with $N=0$ and (ii) with $\alpha=1$ are valid.

The Fejér summation can also be generalized in the next way.

Example 3. Riesz summation. Let

$$\theta_3(x) := \begin{cases} (1-|x|^\gamma)^\alpha & \text{if } |x|\le 1\\ 0 & \text{if } |x|>1 \end{cases}$$

for some $0\le \alpha,\gamma<\infty$. The Riesz operators are given by

$$\sigma_n^{\Psi,\theta_3} f(x) := \sum_{k\in\mathbb{M},|k|\le n} \Big(1-\Big|\frac{k}{n+1}\Big|^\gamma\Big)^\alpha \hat f(k)\psi_k(x).$$

The Riesz means are called *typical means* if $\gamma=1$, *Bochner-Riesz means* if $\gamma=2$ and *Fejér means* if $\alpha=\gamma=1$. If $1\le\alpha<\infty$ and $0<\gamma<\infty$ then θ_3 satisfies (iv) and if $0<\alpha\le 1\le\gamma<\infty$ then (ii) is true (see Weisz [87]).

Example 4. de La Vallée-Poussin summation. Let

$$\theta_4(x) = \begin{cases} 1 & \text{if } |x|\le 1/2\\ -2|x|+2 & \text{if } 1/2<|x|\le 1\\ 0 & \text{if } |x|>1 \end{cases}$$

and

$$\sigma_n^{\Psi,\theta_4} f(x) := \sum_{k\in\mathbb{M},|k|\le n} \Big(\Big(-2\frac{|k|}{n+1}+2\Big)\wedge 1\Big)\hat f(k)\psi_k(x).$$

One can show that

$$\sigma_{2n+1}^{\Psi,\theta_4} f = 2\sigma_{2n+1}^{\Psi,\theta_2} f - \sigma_n^{\Psi,\theta_2} f$$

and since $\theta_4(x)=2\theta_2(x)-\theta_2(2x)$, we have

$$|\hat\theta_4(x)| \le \frac{C}{x^2}, \qquad |\hat\theta_4'(x)| \le \frac{C}{x^2}.$$

Hence we get the conditions (i) with $N=0$, (ii) with $\alpha=1$ and (iv). Note that we could generalize this summation if we take in the definition of θ_4 another number than $1/2$.

Example 5. Jackson-de La Vallée-Poussin summation. Let

$$\theta_5(x) = \begin{cases} 1-3x^2/2+3|x|^3/4 & \text{if } |x|\le 1\\ (2-|x|)^3/4 & \text{if } 1<|x|\le 2\\ 0 & \text{if } |x|>2 \end{cases}$$

and

$$\begin{aligned}\sigma_n^{\Psi,\theta_5} f(x) \quad := \quad & \sum_{k\in\mathbb{M},|k|\le 2n+1} \Bigg(\Big(1-\frac{3}{2}\Big(\frac{|k|}{n+1}\Big)^2+\frac{3}{4}\Big(\frac{|k|}{n+1}\Big)^3\Big)\\ & \wedge \frac{1}{4}\Big(2-\frac{|k|}{n+1}\Big)^3\Bigg)\hat f(k)\psi_k(x).\end{aligned}$$

One can find in Butzer and Nessel [7] that

$$\hat\theta_5(x) = \frac{3}{\sqrt{8\pi}}\Big(\frac{\sin x/2}{x/2}\Big)^4.$$

Therefore we can show by elementary computations that

$$|\hat\theta_5^{(i)}(x)| \le \frac{C}{x^4}, \qquad (i=0,1,2,3),$$

and so (i) with $N=2$, (ii) and (iv) are true.

Example 6. The summation method of cardinal B-splines. For $m \geq 2$ let

$$M_m(x) := \frac{1}{(m-1)!} \sum_{k=0}^{l} (-1)^k \binom{m}{k} (x-k)^{m-1}$$

$(x \in [l, l+1), l = 0, 1, \ldots, m-1)$ and

$$\theta_6(x) = \frac{M_m(m/2 + mx/2)}{M_m(m/2)}.$$

Note that θ_6 is even and $\theta_6(x) = 0$ for $|x| \geq 1$ (see also Schipp and Bokor [48]). Then

$$\sigma_n^{\Psi,\theta_6} f(x) := \sum_{k \in \mathbb{M}, |k| \leq n} \frac{M_m(\frac{m}{2} + \frac{m}{2}\frac{k}{n+1})}{M_m(\frac{m}{2})} \hat{f}(k) \psi_k(x).$$

It is shown in Schipp and Bokor [48] that

$$\hat{\theta}_6(x) = \frac{1}{\pi m M_m(m/2)} \Big(\frac{\sin x/m}{x/m} \Big)^m.$$

It is easy to see that

$$|\hat{\theta}_6^{(i)}(x)| \leq \frac{C}{x^m}, \qquad (i = 0, 1, \ldots, m-1).$$

Thus (i) with $N = m-2$, (ii) and (iv) are satisfied.

Example 7. This example generalizes Examples 4, 5, 6. Let

$$0 = \alpha_0 < \alpha_1 < \ldots < \alpha_m$$

and $\beta_0, \ldots, \beta_m$ ($m \in \mathbb{N}$) be real numbers, $\beta_0 = 1$, $\beta_m = 0$. Suppose that θ_7 is even, $\theta_7(\alpha_j) = \beta_j$ $(j = 0, 1, \ldots, m)$, $\theta_7(x) = 0$ for $x \geq \alpha_m$, θ_7 is a polynomial on the interval $[\alpha_{j-1}, \alpha_j]$ $(j = 1, \ldots, m)$. In this case (iv) is true.

Example 8. Rogosinski summation. Let

$$\theta_8(x) = \begin{cases} \cos \pi x/2 & \text{if } |x| \leq 1 \\ 0 & \text{if } |x| > 1 \end{cases}$$

and

$$\sigma_n^{\Psi,\theta_8} f(x) := \sum_{k \in \mathbb{M}, |k| \leq n} \cos \Big(\frac{\pi k}{2(n+1)} \Big) \hat{f}(k) \psi_k(x).$$

Since

$$\hat{\theta}_8(x) = \frac{\sin(x - \pi/2)}{2(x^2 - (\pi/2)^2)}$$

(see e.g. Schipp and Bokor [48]), we can verify that

$$|\hat{\theta}_8(x)| \leq \frac{C}{x^2}, \qquad |\hat{\theta}_8'(x)| \leq \frac{C}{x^2}$$

and so (i), (ii) and (iv) are satisfied.

Example 9. Weierstrass summation. Let

$$\theta_1(x) = e^{-|x|^\gamma}$$

for some $0 < \gamma < \infty$. The θ-means are given by

$$\sigma_n^{\Psi,\theta_9} f(x) := \sum_{k \in \mathbb{M}} e^{-(\frac{|k|}{n+1})^\gamma} \hat{f}(k) \psi_k(x) (n \in \mathbb{N}).$$

Of course, we can take another index set than $\mathbb{N}$. For example we can change $(\frac{1}{n+1})^\gamma$ by t:

$$V_t^{\Psi,\theta_9} f(x) := \sum_{k \in \mathbb{M}} e^{-t|k|^\gamma} \hat{f}(k) \psi_k(x),$$

or e^{-t} by r:

$$W_r^{\Psi,\theta_9} f(x) := \sum_{k\in\mathbb{M}} r^{|k|^\gamma} \hat{f}(k)\psi_k(x).$$

θ_9 satisfies (i) for all $N \in \mathbb{N}$ and (iv). One can compute that

$$|\hat{\theta}_9(x)| \le \frac{C}{x^2} \qquad (x \in (0,\infty)) \tag{9}$$

if $\gamma \ge 1$. Thus θ_9 satisfies also (ii) with $\alpha = 1$ if $1 \le \gamma < \infty$. Note that if $\gamma = 1$ then we obtain the Abel means (see e.g. [7]).

Example 10. Generalized Picar and Bessel summations. Let

$$\theta_{10}(x) = \frac{1}{(1+|x|^\gamma)^\alpha}$$

for some $0 < \alpha, \gamma < \infty$ such that $\alpha\gamma > 1$. The θ-means are given by

$$\sigma_n^{\Psi,\theta_{10}} f(x) := \sum_{k\in\mathbb{M}} \frac{1}{\Big(1 + (\frac{|k|}{n+1})^\gamma\Big)^\alpha} \hat{f}(k)\psi_k(x).$$

Since (9) is true in this case, too, one can show (see Weisz [94, p. 201]) that θ_{10} satisfies (iv) and (i) for $N = -[-\alpha\gamma] - 2$ if $1 < \alpha\gamma < \infty$ and (ii) for $\alpha = 1$ if $2 < \alpha\gamma < \infty$. Originally the summation is called Picar if $\alpha = 1$ and Bessel if $\gamma = 2$.

Example 11. Let

$$\theta_{11}(x) := \begin{cases} 1 & \text{if } |x| \le 1 \\ |x|^{-\alpha} & \text{if } |x| > 1 \end{cases}$$

for some $1 < \alpha < \infty$. We have

$$\sigma_n^{\Psi,\theta_{11}} f(x) := \sum_{k\in\mathbb{M},|k|\le n} \hat{f}(k)\psi_k(x) + \sum_{k\in\mathbb{M},|k|>n} \Big|\frac{k}{n+1}\Big|^{-\alpha} \hat{f}(k)\psi_k(x).$$

We can prove as in Example 10 that θ_{11} satisfies (iv) and (i) for $N = -[-\alpha] - 2$ if $1 < \alpha < \infty$ and (ii) if $2 < \alpha < \infty$.

Example 12. Riemann summation. Let

$$\theta_{12}(x) = \Big(\frac{\sin x/2}{x/2}\Big)^2 = \sqrt{2\pi}\,\hat{\theta}_2(x).$$

Then

$$\hat{\theta}_{12}(x) = \sqrt{2\pi}\,\theta_2(x) = \sqrt{2\pi}\max(0, 1-|x|)$$

and so

$$|\hat{\theta}_{12}'(x)| = \sqrt{2\pi}\; 1_{(-1,1)}(x) \le C/x^2.$$

The Riemann means are given by

$$\sigma_n^{\Psi,\theta_{12}} f(x) := \sum_{k\in\mathbb{M}} \Big(\frac{\sin k/(2(n+1))}{k/(2(n+1))}\Big)^2 \hat{f}(k)\psi_k(x).$$

If we change $1/(n+1)$ to μ then we get the usual form of the Riemann summation,

$$V_\mu^{\Psi,\theta_{12}} f(x) := \sum_{k\in\mathbb{M}} \Big(\frac{\sin k\mu/2}{k\mu/2}\Big)^2 \hat{f}(k)\psi_k(x) \qquad (\mu \in (0,\infty)).$$

Thus (i) with $N = 0$, (ii) and (iv) are true. Note that the Riemann summation was considered in Bari [1], Zygmund [100], Gevorkyan [32, 33, 34] and also in Weisz [82, 86].

4. Orthonormal systems

In this section we consider five orthonormal or biorthogonal systems and the Fourier transforms.

4.1. Trigonometric system. The trigonometric system is defined by

$$\mathcal{T} := (\exp(2\pi\imath n\cdot), n \in \mathbb{Z}),$$

where $\imath := \sqrt{-1}$. In this case

$$D_n^{\mathcal{T}}(t,x) = \sum_{|k|\le n} e^{-2\pi\imath kt} e^{2\pi\imath kx} = \sum_{|k|\le n} e^{2\pi\imath k(x-t)} \qquad (n \in \mathbb{N}, t, x \in [0,1)).$$

For this last expression we use the notation $D_n^{\mathcal{T}}(x-t)$. So $D_n^{\mathcal{T}}(x-t) := D_n^{\mathcal{T}}(t,x)$. Similarly, $K_n^{\mathcal{T},\theta}(x-t) := K_n^{\mathcal{T},\theta}(t,x)$. The inequalities (5) and (6) are proved e.g. in Zygmund [100] or Torchinsky [65].

4.2. Walsh system. Let

$$r(x) := \begin{cases} 1 & \text{if } x \in [0, \frac{1}{2}) \\ -1 & \text{if } x \in [\frac{1}{2}, 1) \end{cases}$$

extended to $\mathbb{R}$ by periodicity of period 1. The *Rademacher system* $(r_n, n \in \mathbb{N})$ is defined by

$$r_n(x) := r(2^n x) \qquad (x \in [0,1), n \in \mathbb{N}).$$

The *Walsh functions* are given by

$$w_n(x) := \prod_{k=0}^{\infty} r_k(x)^{n_k} \qquad (x \in [0,1), n \in \mathbb{N})$$

where $n = \sum_{k=0}^{\infty} n_k 2^k$, $(0 \le n_k < 2)$. Let

$$\mathcal{W} := (w_n, n \in \mathbb{N}).$$

Since $w_n(t)w_n(x) = w_n(x \dot{+} t) = w_n(x \dot{-} t)$, we use also the notation $D_n^{\mathcal{W}}(x-t)$ and $K_n^{\mathcal{W},\theta}(x-t)$. For the definition of the dyadic addition $\dot{+}$ see Schipp, Wade, Simon and Pál [50]. Conditions (5) and (6) are proved in Schipp, Wade, Simon and Pál [50] and Fine [22, 23].

4.3. Walsh-Kaczmarz system. The *Kaczmarz rearrangement* of the Walsh system is also considered. For $n \in \mathbb{N}$ there is a unique s such that $n = 2^s + \sum_{k=0}^{s-1} n_k 2^k$, $(0 \le n_k < 2)$. Define

$$\kappa_n(x) := r_s(x) \prod_{k=0}^{s-1} r_{s-k-1}(x)^{n_k} \qquad (x \in [0,1), n \in \mathbb{N})$$

and $\kappa_0 := 1$. Let

$$\mathcal{K} := (\kappa_n, n \in \mathbb{N}).$$

It is easy to see that $\kappa_{2^n} = w_{2^n} = r_n$ $(n \in \mathbb{N})$ and

$$\{\kappa_k : k = 2^n, \ldots, 2^{n+1}-1\} = \{w_k : k = 2^n, \ldots, 2^{n+1}-1\}.$$

We use again the notation $D_n^{\mathcal{K}}(x-t)$ and $K_n^{\mathcal{K},\theta}(x-t)$. Inequality (5) is proved in Gát [28] and Simon [59]. Note that (6) is not true for the Walsh-Kaczmarz system (see Shneider [54]).

4.4. **Vilenkin system.** The Walsh system is generalized as follows. We need a sequence $(p_n, n \in \mathbb{N})$ of natural numbers whose terms are at least 2. We suppose always that this sequence is **bounded**. Introduce the notations $P_0 = 1$ and

$$P_{n+1} := \prod_{k=0}^{n} p_k \qquad (n \in \mathbb{N}).$$

Every point $x \in [0,1)$ can be written in the following way:

$$x = \sum_{k=0}^{\infty} \frac{x_k}{P_{k+1}}, \qquad 0 \le x_k < p_k,\ x_k \in \mathbb{N}.$$

If there are two different forms, choose the one for which $\lim_{k\to\infty} x_k = 0$. The functions

$$r_n(x) := \exp \frac{2\pi \imath x_n}{p_n} \qquad (n \in \mathbb{N})$$

are called *generalized Rademacher functions*.

The *Vilenkin system* is given by

$$v_n(x) := \prod_{k=0}^{\infty} r_k(x)^{n_k}$$

where $n = \sum_{k=0}^{\infty} n_k P_k$, $0 \le n_k < p_k$. Recall that the functions corresponding to the sequence $(2, 2, \ldots)$ are the Rademacher and Walsh functions (see Vilenkin [66] or Schipp, Wade, Simon and Pál [50]). Let

$$\mathcal{V} := (v_n, n \in \mathbb{N}).$$

Again, $D_n^{\mathcal{V}}(x-t) := D_n^{\mathcal{V}}(t,x)$ and $K_n^{\mathcal{V},\theta}(x-t) := K_n^{\mathcal{V},\theta}(t,x)$. The inequalities (5) and (6) are due to Simon [57].

4.5. **Ciesielski system.** The Walsh system can be generalized also in the following way. First we introduce the spline systems as in Ciesielski [16, 15]. Let us denote by D the differentiation operator and define the integration operators

$$Gf(t) := \int_0^t f\, d\lambda, \qquad Hf(t) := \int_t^1 f\, d\lambda.$$

Define the χ_n, $n = 1, 2, \ldots$, *Haar system* by $\chi_1 := 1$ and

$$\chi_{2^n+k}(x) := \begin{cases} 2^{n/2}, & \text{if } x \in ((2k-2)2^{-n-1}, (2k-1)2^{-n-1}) \\ -2^{n/2}, & \text{if } x \in ((2k-1)2^{-n-1}, (2k)2^{-n-1}) \\ 0, & \text{otherwise} \end{cases}$$

for $n, k \in \mathbb{N}$, $0 < k \le 2^n$, $x \in [0,1)$.

Let $m \ge -1$ be a fixed integer. Applying the Schmidt orthonormalization to the linearly independent functions

$$1, t, \ldots, t^{m+1}, G^{m+1}\chi_n(t), \qquad n \ge 2,$$

we get the *spline system* $(f_n^{(m)}, n \ge -m)$ of order m. For $0 \le k \le m+1$ and $n \ge k - m$ define the splines

$$f_n^{(m,k)} := D^k f_n^{(m)}, \qquad g_n^{(m,k)} := H^k f_n^{(m)}$$

of order (m,k). Let us normalize these functions and introduce a more unified notation,

$$h_n^{(m,k)} := \begin{cases} f_n^{(m,k)} \|f_n^{(m,k)}\|_2^{-1} & \text{for } 0 \le k \le m+1 \\ g_n^{(m,-k)} \|f_n^{(m,-k)}\|_2 & \text{for } 0 \le -k \le m+1. \end{cases}$$

The system $(h_i^{(m,k)}, h_i^{(m,-k)}, i \ge |k| - m\}$ is biorthogonal. We get the Haar system if $m = -1$, $k = 0$ and the *Franklin system* if $m = 0$, $k = 0$.

Starting with the spline system $(h_n^{(m,k)}, n \geq |k| - m)$ we define the *Ciesielski system* $(c_n^{(m,k)}, n \geq |k| - m - 1)$ in the same way as the Walsh system arises from the Haar system, namely,

$$c_n^{(m,k)} := h_{n+1}^{(m,k)} \qquad (n = |k| - m - 1, \dots, 0)$$

and

$$c_{2^\nu+i}^{(m,k)} := \sum_{j=1}^{2^\nu} A_{i+1,j}^{(\nu)} h_{2^\nu+j}^{(m,k)} \qquad (0 \leq i \leq 2^\nu - 1).$$

As mentioned before,

$$c_n^{(-1,0)} = w_n \qquad (n \in \mathbb{N})$$

is the usual Walsh system. It is known (see Schipp, Wade, Simon, Pál [50] or Ciesielski, Simon, Sjölin [13]) that

$$A_{i+1,j}^{(\nu)} = A_{j,i+1}^{(\nu)} = 2^{-\nu/2} w_i\left(\frac{2j-1}{2^{\nu+1}}\right).$$

The system

$$\mathcal{C} := \mathcal{C}^{(m,k)} := (c_n^{(m,k)}, c_n^{(m,-k)}, n \geq |k| - m - 1)$$

is uniformly bounded and biorthogonal whenever $|k| \leq m + 1$.

For the Ciesielski systems we have to modify slightly the definitions of partial sums, θ-means and kernel functions as follows.

Let

$$\begin{aligned} s_n^{\mathcal{C}} f &:= \sum_{j=|k|-m-1}^{n} \hat{f}(j) c_j^{(m,-k)} \qquad (n \in \mathbb{N}), \\ D_n^{\mathcal{C}}(t,x) &:= \sum_{j=|k|-m-1}^{n} c_j^{(m,k)}(t) c_j^{(m,-k)}(x) \qquad (n \in \mathbb{N}, t, x \in [0,1)), \end{aligned}$$

$$\begin{aligned} \sigma_n^{\mathcal{C}} f &:= \frac{1}{n+1} \sum_{k=0}^{n} s_n^{\mathcal{C}} f = \sum_{j=|k|-m-1}^{-1} \hat{f}(j) c_j^{(m,-k)} + \sum_{j=0}^{n} \Big(1 - \frac{|j|}{n+1}\Big) \hat{f}(j) c_j^{(m,-k)}, \\ K_n^{\mathcal{C}} &:= \frac{1}{n+1} \sum_{k=0}^{n} D_n^{\mathcal{C}} \qquad (n \in \mathbb{N}), \end{aligned}$$

$$\begin{aligned} \sigma_n^{\mathcal{C},\theta} f(x) &:= \sum_{j=|k|-m-1}^{-1} \hat{f}(j) c_j^{(m,-k)} + \sum_{j=0}^{n} \theta\Big(\frac{j}{n+1}\Big) \hat{f}(j) c_j^{(m,-k)}, \\ K_n^{\mathcal{C},\theta}(t,x) &:= \sum_{j=|k|-m-1}^{-1} c_j^{(m,k)}(t) c_j^{(m,-k)} + \sum_{j=0}^{n} \theta\Big(\frac{j}{n+1}\Big) c_j^{(m,k)}(t) c_j^{(m,-k)}. \end{aligned}$$

Inequalities (5) and (6) are due to the author [92, 96].

4.6. **Fourier transforms.** The definition (1) of the Fourier transform can be extended to $f \in L_p(\mathbb{R})$ $(1 \leq p \leq 2)$ (see e.g. Butzer and Nessel [7]). It is known that if $f \in L_p(\mathbb{R})$ $(1 \leq p \leq 2)$ and $\hat{f} \in L_1(\mathbb{R})$ then

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(u) e^{\imath x u} \, du \qquad (x \in \mathbb{R}).$$

This motivates the definition of the *Dirichlet integral* $s_t^{\mathcal{F}} f$ $(t > 0)$:

$$s_t^{\mathcal{F}} f(x) := \frac{1}{\sqrt{2\pi}} \int_{-t}^{t} \hat{f}(u) e^{\imath x u} \, du$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(u) D_t^{\mathcal{F}}(x-u)\,du = (f * D_t^{\mathcal{F}})(x),$$

where $*$ denotes the convolution and

$$D_t^{\mathcal{F}}(x) := \frac{1}{\sqrt{2\pi}} \int_{-t}^{t} e^{\imath xu}\,du$$

is the *Dirichlet kernel.* It is easy to see that

$$|D_t^{\mathcal{F}}(x)| \le \frac{C}{x} \qquad (t>0, x \ne 0).$$

The *Fejér means* $\sigma_T^{\mathcal{F}} f$ are defined by

$$\begin{aligned} \sigma_T^{\mathcal{F}} f(x) &:= \frac{1}{T} \int_0^T s_t^{\mathcal{F}} f(x)\,dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{-T}^{T} \Big(1 - \frac{|t|}{T}\Big) \hat{f}(t) e^{\imath xt}\,dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(u) K_T^{\mathcal{F}}(x-u)\,du = (f * K_T^{\mathcal{F}})(x) \qquad (T>0) \end{aligned}$$

where

$$K_T^{\mathcal{F}}(u) := \frac{1}{T} \int_0^T D_t^{\mathcal{F}}(u)\,dt = \frac{2\sqrt{2}}{\sqrt{\pi}} \frac{\sin^2 \frac{Tu}{2}}{Tu^2}$$

is the *Fejér kernel.* Remark that

$$\int_{\mathbb{R}} K_T^{\mathcal{F}}(u)\,du = \sqrt{2\pi} \qquad (T>0)$$

(see Zygmund [100, Vol. II. pp. 250-251]).

The *θ-means* of $f \in L_p(\mathbb{R})$ $(1 \le p \le 2)$ are defined by

$$\begin{aligned} \sigma_T^{\mathcal{F},\theta} f(x) &:= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \Big(\theta(\frac{t}{T})\Big) \hat{f}(t) e^{\imath xt}\,dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(u) K_T^{\mathcal{F},\theta}(x-u)\,du \qquad (x \in \mathbb{R}, T>0), \end{aligned}$$

where

$$K_T^{\mathcal{F},\theta}(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \theta\Big(\frac{t}{T}\Big) e^{\imath xt}\,dt.$$

The definition of the θ-means can be extended to tempered distributions as follows:

$$\sigma_T^{\mathcal{F},\theta} f := f * K_T^{\mathcal{F},\theta} \qquad (T>0).$$

One can show that $\sigma_T^{\mathcal{F},\theta} f$ is well defined for all tempered distributions $f \in H_p^{\mathcal{F}}$ $(0 < p \le \infty)$ and for all functions $f \in L_p$ $(1 \le p \le \infty)$ (cf. Stein [61]). Note that the Hardy spaces $H_p^{\mathcal{F}}$ are defined in the next section.

The *maximal Fejér* and *θ-operators* are defined by

$$\sigma_*^{\mathcal{F},\theta} f := \sup_{T>0} |\sigma_T^{\mathcal{F},\theta} f|.$$

If $\theta(x) := (1-|x|) \vee 0$, then we get the maximal Fejér operator. In this case we leave the θ in the notation. Now Theorem 1 reads as follows (see Weisz [96]).

Theorem 3. *If (3) and (iv) hold and if*

$$\|\sigma_*^{\mathcal{F}} f\|_{\mathbf{Y}} \le C \|f\|_{\mathbf{X}} \qquad (f \in \mathbf{X} \cap L_\infty),$$

then

$$\|\sigma_*^{\mathcal{F},\theta} f\|_{\mathbf{Y}} \le C \|f\|_{\mathbf{X}} \qquad (f \in \mathbf{X}),$$

where $\mathbf{X}$ *and* $\mathbf{Y}$ *is defined in Theorem 1.*

For the trigonometric system and for Fourier transforms we will suppose one of the conditions (i)–(iv), for the Walsh and Vilenkin systems we will suppose (iii) or (iv) and for the Walsh-Kaczmarz and Ciesielski systems (iv).

5. Hardy spaces

For different function systems different Hardy spaces are considered. In order to have a common notation for the dyadic, Vilenkin and classical Hardy spaces we define the *Poisson kernels* $P_t^{\mathcal{G}}$ ($\mathcal{G} \in \{\mathcal{T}, \mathcal{W}, \mathcal{K}, \mathcal{V}, \mathcal{C}, \mathcal{F}\}$). Set

$$\begin{aligned}
P_t^{\mathcal{T}}(x) &:= \sum_{j=-\infty}^{\infty} e^{-t|j|} e^{2\pi \imath j x} \qquad (x \in \mathbb{R}, t > 0),\\
P_t^{\mathcal{F}}(x) &:= \frac{ct}{(t+|x|^2)} \qquad (x \in \mathbb{R}, t > 0),\\
P_t^{\mathcal{W}}(x) &:= P_t^{\mathcal{K}}(x) := 2^n 1_{[0,2^{-n})}(x) \quad \text{if } n \le t < n+1 \qquad (x \in \mathbb{R}),\\
P_t^{\mathcal{V}}(x) &:= P_n 1_{[0,P_n^{-1})}(x) \quad \text{if } n \le t < n+1 \qquad (x \in \mathbb{R}),\\
P_t^{\mathcal{C}}(x) &:= \begin{cases} P_t^{\mathcal{F}}(x) & \text{if } k \le m \\ P_t^{\mathcal{W}}(x) & \text{if } k = m+1 \end{cases} \qquad (x \in \mathbb{R}).
\end{aligned}$$

We remark that the numbers m and k are appeared in the definition of the Ciesielski systems.

For a tempered distribution f the *non-tangential maximal function* is defined by

$$f_*^{\mathcal{G}}(x) := \sup_{t>0} |(f * P_t^{\mathcal{G}})(x)| \qquad (x \in \mathbb{R})$$

where $\mathcal{G} \in \{\mathcal{T}, \mathcal{W}, \mathcal{K}, \mathcal{V}, \mathcal{C}, \mathcal{F}\}$.

For $0 < p < \infty$ the *Hardy space* $H_p^{\mathcal{G}}(\mathbb{R})$ consists of all tempered distributions f for which

$$\|f\|_{H_p^{\mathcal{G}}(\mathbb{R})} := \|f_*^{\mathcal{G}}\|_p < \infty.$$

Now let $H_p^{\mathcal{F}} := H_p^{\mathcal{F}}(\mathbb{R})$ and

$$H_p^{\mathcal{G}} := H_p^{\mathcal{G}}([0,1)) := \{f \in H_p^{\mathcal{G}}(\mathbb{R}) : \operatorname{supp} f \subset [0,1)\},$$

where $\mathcal{G} \in \{\mathcal{T}, \mathcal{W}, \mathcal{K}, \mathcal{V}, \mathcal{C}\}$. Define $H_\infty^{\mathcal{G}} := L_\infty$.

Note that $H_p^{\mathcal{W}}$ is the dyadic Hardy space. It is known (see Stein [61], Weisz [94]) that

$$H_p^{\mathcal{G}} \sim L_p \qquad (1 < p \le \infty).$$

The intervals $[k2^{-n}, (k+1)2^{-n})$, $(0 \le k < 2^n)$ (resp. $[kP_n^{-1}, (k+1)P_n^{-1})$, $(0 \le k < P_n)$) are called dyadic (resp. Vilenkin) intervals.

Now some boundedness theorems for Hardy spaces are given. To this end we introduce the definition of the atoms. The atomic decomposition is a useful characterization of the Hardy spaces by the help of which some boundedness results, duality theorems, maximal inequalities and interpolation results can be proved. The atoms are relatively simple and easy to handle functions. If we have an atomic decomposition, then we have to prove several theorems for atoms, only. A first version of the atomic decomposition was introduced by Coifman and Weiss [17] in the classical case and by Herz [35] in the martingale case.

A function $a \in L_\infty$ is called a *p-atom* for the $H_p^{\mathcal{T}}$ space if

(a) $\operatorname{supp} a \subset I$, $I \subset [0,1)$ is a generalized interval,
(b) $\|a\|_\infty \le |I|^{-1/p}$,
(c) $\int_I a(x) x^j \, dx = 0$, where $j \le [1/p - 1]$.

Under a generalized interval we mean an interval $[(a,b)]$ or a set $[(0,a)]\cup[(b,1)]$ $(0\le a<b\le 1)$. For the space $H_p^{\mathcal{C}}$ we suppose only that $I\subset[0,1)$ is an interval. For $H_p^{\mathcal{F}}$ we consider intervals $I\subset\mathbb{R}$. For $H_p^{\mathcal{W}}$ and $H_p^{\mathcal{K}}$ (resp. for $H_p^{\mathcal{V}}$) we assume that $I\subset[0,1)$ is a dyadic (resp. Vilenkin) interval and instead of (c) we suppose

(c') $\int_I a(x)\,dx=0$.

The basic result of atomic decomposition is the following one.

Theorem 4. *A tempered distribution f is in $H_p^{\mathcal{G}}$ $(0<p\le 1,\ \mathcal{G}\in\{\mathcal{T},\mathcal{W},\mathcal{K},\mathcal{V},\mathcal{C},\mathcal{F}\})$ if and only if there exist a sequence $(a^k,k\in\mathbb{N})$ of p-atoms for $H_p^{\mathcal{G}}$ and a sequence $(\mu_k,k\in\mathbb{N})$ of real numbers such that*

$$\begin{aligned}&\sum_{k=0}^{\infty}\mu_k a^k=f\quad\text{in the sense of distributions,}\\&\sum_{k=0}^{\infty}|\mu_k|^p<\infty.\end{aligned}\tag{10}$$

Moreover,

$$\|f\|_{H_p^{\mathcal{G}}}\sim\inf\Big(\sum_{k=0}^{\infty}|\mu_k|^p\Big)^{1/p}\tag{11}$$

where the infimum is taken over all decompositions of f of the form (10).

For the Walsh, Walsh-Kaczmarz and Vilenkin systems the first sum in (10) is taken in the sense of martingales. The proof of this theorem can be found e.g. in Latter [37], Lu [39], Coifman and Weiss [17], Coifman [18], Wilson [98, 99] and Stein [61] in the classical case and in Weisz [75, 74] for martingale Hardy spaces.

If I is an interval then let $I^r=2^rI$ be an interval with the same center as I, for which $I\subset I^r$ and $|I^r|=2^r|I|$ $(r\in\mathbb{N})$.

The following result gives a sufficient condition for V to be bounded from $H_p^{\mathcal{G}}$ to L_p. For $p_0=1$ it can be found in Schipp, Wade, Simon and Pál [50] and in Móricz, Schipp and Wade [41], for $p_0<1$ see Weisz [80].

Theorem 5. *Suppose that*

$$\int_{[0,1)\setminus I^r}|Va|^{p_0}\,d\lambda\le C_{p_0}$$

for all p_0-atoms a and for some fixed $r\in\mathbb{N}$ and $0<p_0\le 1$. If the sublinear operator V is bounded from L_{p_1} to L_{p_1} $(1<p_1\le\infty)$ then

$$\|Vf\|_p\le C_p\|f\|_{H_p^{\mathcal{G}}}\qquad(f\in H_p^{\mathcal{G}})\tag{12}$$

for all $p_0\le p\le p_1$. Moreover, if $p_0<1$ then the operator V is of weak type $(1,1)$, i.e. if $f\in L_1$ then

$$\lambda(|Vf|>\rho)\le\frac{C}{\rho}\|f\|_1\qquad(\rho>0).\tag{13}$$

Note that (13) can be obtained from (12) by interpolation. For the basic definitions and theorems on interpolation theory see Bergh and Löfström [3] and Bennett and Sharpley [2] or Weisz [74, 94]. This theorem can be regarded also as an alternative tool to the Calderon-Zygmund decomposition lemma for proving weak type $(1,1)$ inequalities. In many cases this theorem can be applied better and more simply than the Calderon-Zygmund decomposition lemma.

We formulate also a weak version of this theorem.

Theorem 6. *Suppose that*

$$\sup_{\rho>0} \rho^p \ \lambda\Big(\{|Va| > \rho\} \cap \{[0,1) \setminus I^r\}\Big) \le C_p$$

for all p-atoms a and for some fixed $r \in \mathbb{N}$ and $0 < p < 1$. If the sublinear operator V is bounded from L_{p_1} to L_{p_1} $(1 < p_1 \le \infty)$, then

$$\|Vf\|_{p,\infty} \le C_p \|f\|_{H_p^{\mathcal{G}}} \quad (f \in H_p^{\mathcal{G}}).$$

Using these two theorems and Theorems 1, 2 and 3 we can prove the next result (see Weisz [90, 97, 94, 96]).

Theorem 7. *Besides (3) we suppose one of the conditions (i)–(iv) for the trigonometric system and for Fourier transforms, (iii) or (iv) for the Walsh and Vilenkin systems, (iv) for the Ciesielski system, (iv) and (4) for the Walsh-Kaczmarz system. If $p_0 < p \le \infty$ and $\mathcal{G} \in \{\mathcal{T}, \mathcal{W}, \mathcal{K}, \mathcal{V}, \mathcal{C}\}$ then*

$$\|\sigma_*^{\mathcal{G},\theta} f\|_p \le C_p \|f\|_{H_p^{\mathcal{G}}} \qquad (f \in H_p^{\mathcal{G}}), \tag{14}$$

where $p_0 < 1$ is defined in the conditions (i)–(iv). Moreover,

$$\|\sigma_*^{\mathcal{G},\theta} f\|_{p_0,\infty} \le C_{p_0} \|f\|_{H_{p_0}^{\mathcal{G}}} \qquad (f \in H_{p_0}^{\mathcal{G}}). \tag{15}$$

In particular, if $f \in L_1$ then

$$\sup_{\rho>0} \rho \, \lambda(\sigma_*^{\mathcal{G},\theta} f > \rho) \le C \|f\|_1. \tag{16}$$

For the Fejér summability inequalities (14) and (16) were proved by Móricz [42, 43, 44, ($p = 1$)] and Weisz [80, 84, 85] for the trigonometric system, Schipp [53] and Weisz [77] for the Walsh system, Gát [28] and Simon [59, 58] for the Walsh-Kaczmarz system, Simon [57] and Weisz [83] for the Vilenkin system and by Weisz [92] for the Ciesielski system.

Note that (14) is not true for $p \le p_0$ in general, there are counterexamples in Colzani, Taibleson and Weiss [19] and Simon [58] for the trigonometric and Walsh systems. For $p = p_0$ (15) is weaker than (14) and, in general, for $p < p_0$ (15) does not hold either.

Inequality (16) and the usual density argument of Marcinkiewicz and Zygmund [40] implies

Corollary 1. *Under the conditions of Theorem 7 if $f \in L_1$ then*

$$\sigma_n^{\mathcal{G},\theta} f \to f \qquad \text{a.e. as } \ n \to \infty,$$

where $\mathcal{G} \in \{\mathcal{T}, \mathcal{W}, \mathcal{K}, \mathcal{V}, \mathcal{C}\}$. Moreover,

$$\sigma_T^{\mathcal{F},\theta} f \to f \qquad \text{a.e. as } \ T \to \infty.$$

6. θ-summability of multi-dimensional Fourier series

In this section the preceding results are generalized for d-dimensional Fourier series. For a set $\mathbb{X} \ne \emptyset$ let $\mathbb{X}^d$ be its Cartesian product $\mathbb{X} \times \ldots \times \mathbb{X}$ taken with itself d-times. The d-dimensional biorthogonal system

$$\Psi^d = \Psi \otimes \cdots \otimes \Psi$$

is defined by the Kronecker product of the one-dimensional biorthogonal system $\Psi := (\phi_n, \psi_n, n \in \mathbb{M})$ taken with itself d-times. Then

$$\Psi^d := (\phi_n, \psi_n, n \in \mathbb{M}^d),$$

where $\phi_n := \phi_{n_1} \otimes \cdots \otimes \phi_{n_d}$ and $\psi_n := \psi_{n_1} \otimes \cdots \otimes \psi_{n_d}$ $(n = (n_1, \ldots, n_d))$. This means that we take the Kronecker product of the same function systems. We define the d-dimensional trigonometric ($\mathcal{T}^d$), Walsh ($\mathcal{W}^d$), Vilenkin ($\mathcal{V}^d$) and Ciesielski ($\mathcal{C}^d$)

systems in this way. In the definition of the d-dimensional Vilenkin (resp. Ciesielski) systems we allow different one-dimensional Vilenkin (resp. Ciesielski) systems. The more-dimensinal Walsh-Kaczmarz system is not considered in this section.

For $f \in L_1[0,1)^d$ the Fourier coefficients with respect to Ψ^d are defined by

$$\hat{f}(n) := \int_{[0,1)^d} f\overline{\phi}_n \, d\lambda \qquad (n \in \mathbb{M}^d).$$

Let

$$\begin{aligned} s_n^{\Psi^d} f(x) &:= \sum_{k \in \mathbb{M}^d, |k| \le n} \hat{f}(k) \psi_k(x) \\ &= \int_{[0,1)^d} f(t) (D_{n_1}^{\Psi}(t_1, x_1) \cdots D_{n_d}^{\Psi}(t_d, x_d)) \, dt \qquad (x \in [0,1)^d, n \in \mathbb{N}^d), \end{aligned}$$

where $k \le n$ ($k, n \in \mathbb{N}^d$) means that $k_i \le n_i$ for all $i = 1, \ldots, d$.

The *Fejér means* $\sigma_n^{\Psi^d} f$ ($n \in \mathbb{N}^d$) of $f \in L_1[0,1)^d$ are given by

$$\begin{aligned} \sigma_n^{\Psi^d} f(x) &:= \frac{1}{\prod_{i=1}^d (n_i + 1)} \sum_{j=1}^d \sum_{k_j=0}^{n_j} s_k^{\Psi^d} f(x) \\ &= \sum_{k \in \mathbb{M}^d, |k| \le n} \Big(\prod_{i=1}^d \Big(1 - \frac{|k_i|}{n_i + 1} \Big) \Big) \hat{f}(k) \psi_k(x) \\ &= \int_{[0,1)^d} f(t) (K_{n_1}^{\Psi}(t_1, x_1) \cdots K_{n_d}^{\Psi}(t_d, x_d)) \, dt \end{aligned}$$

($x \in [0,1)^d, n \in \mathbb{N}^d$).

In case each θ_i ($i = 1, \ldots, d$) satisfies (3) and one of the conditions (i)–(iv), the *θ-means* of $f \in L_1[0,1)^d$ are defined by

$$\begin{aligned} \sigma_n^{\Psi^d, \theta} f(x) &:= \sum_{k \in \mathbb{M}^d} \Big(\prod_{i=1}^d \theta_i \Big(\frac{k_i}{n_i + 1} \Big) \Big) \hat{f}(k) \psi_k(x) \\ &= \int_{[0,1)^d} f(t) (K_{n_1}^{\Psi, \theta_1}(t_1, x_1) \cdots K_{n_d}^{\Psi, \theta_d}(t_d, x_d)) \, dt, \end{aligned}$$

($x \in [0,1)^d, n \in \mathbb{N}^d$). For the Ciesielski system we have to take again some necessary modifications in the above definitions. We define the restricted and non-restricted *maximal Fejér* and *θ-operators* by

$$\sigma_{\square}^{\Psi^d} f := \sup_{\substack{2^{-\tau} \le n_j / n_k \le 2^{\tau} \\ j,k = 1, \ldots, d}} |\sigma_n^{\Psi^d} f|, \qquad \sigma_*^{\Psi^d} f := \sup_{n \in \mathbb{N}^d} |\sigma_n^{\Psi^d} f|$$

and

$$\sigma_{\square}^{\Psi^d, \theta} f := \sup_{\substack{2^{-\tau} \le n_j / n_k \le 2^{\tau} \\ j,k = 1, \ldots, d}} |\sigma_n^{\Psi^d, \theta} f|, \qquad \sigma_*^{\Psi^d, \theta} f := \sup_{n \in \mathbb{N}^d} |\sigma_n^{\Psi^d, \theta} f|,$$

respectively, where $\tau \ge 0$ is given.

In the more-dimensional case the *Fourier transform* of a function $f \in L_1(\mathbb{R}^d)$ is introduced by

$$\hat{f}(u) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(x) e^{-\imath u \cdot x} \, dx \qquad (u \in \mathbb{R}^d),$$

where $u \cdot x = \sum_{k=1}^d u_k x_k$ ($u, x \in \mathbb{R}^d$).

The *Fejér* and *θ-means* of $f \in L_p(\mathbb{R}^d)$ $(1 \le p \le 2)$ are defined by

$$\begin{aligned}\sigma_T^{\mathcal{F}^d,\theta} f(x) &:= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \Big(\prod_{i=1}^d \Big(\theta_i(\frac{t_i}{T_i}) \Big) \Big) \hat{f}(t) e^{\imath x \cdot t}\, dt \\ &= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(u)(K_{T_1}^{\mathcal{F},\theta_1}(x_1 - u_1) \dots K_{T_d}^{\mathcal{F},\theta_d}(x_d - u_d))\, du,\end{aligned}$$

$(x \in \mathbb{R}^d, T \in \mathbb{R}_+^d)$. The definition of the θ-means can be extended to tempered distributions as follows:

$$\sigma_T^{\mathcal{F}^d,\theta} f := f * (K_{T_1}^{\mathcal{F},\theta_1} \otimes \dots \otimes K_{T_d}^{\mathcal{F},\theta_d}) \qquad (T \in \mathbb{R}_+^d).$$

Again, $\sigma_T^{\mathcal{F}^d,\theta} f$ is well defined for all tempered distributions $f \in H_p^{\mathcal{F}^d}$ $(0 < p \le \infty)$ and for all functions $f \in L_p(\mathbb{R}^d)$ $(1 \le p \le \infty)$ (cf. Stein [61]).

For a given $\tau \ge 0$ the restricted and non-restricted *maximal Fejér* and *θ-operators* are given by

$$\sigma_\square^{\mathcal{F}^d,\theta} f := \sup_{\substack{2^{-\tau} \le T_j/T_k \le 2^\tau \\ j,k=1,\dots,d}} |\sigma_T^{\mathcal{F}^d,\theta} f|, \qquad \sigma_*^{\mathcal{F}^d,\theta} f := \sup_{T \in \mathbb{R}_+^d} |\sigma_T^{\mathcal{F}^d,\theta} f|.$$

If each $\theta_i(x) := (1 - |x|) \vee 0$ $(i = 1, \dots, d)$, then we get the Fejér means.

Theorems 1, 2 and 3 hold in the more-dimensional case, too.

Theorem 8. *Assume that (3) and (iv) hold for each θ_i $(i = 1, \dots, d)$. If (5) and (6) are satisfied then*

$$\|\sigma_*^{\mathcal{G}^d} f\|_{\mathbf{Y}} \le C \|f\|_{\mathbf{X}} \qquad (f \in \mathbf{X} \cap L_\infty),$$

implies

$$\|\sigma_*^{\mathcal{G}^d,\theta} f\|_{\mathbf{Y}} \le C \|f\|_{\mathbf{X}} \qquad (f \in \mathbf{X}),$$

where $\mathbf{X}$ and $\mathbf{Y}$ are defined in Theorem 1 and $\mathcal{G} \in \{\mathcal{T}, \mathcal{W}, \mathcal{V}, \mathcal{C}, \mathcal{F}\}$. If we assume (4) instead of (6), then the same holds.

In the more-dimensional case we define three kinds of Hardy spaces. Let

$$f_*^{\mathcal{G}^d,\square}(x) := \sup_{t>0} |(f * (P_t^{\mathcal{G}} \otimes \dots \otimes P_t^{\mathcal{G}}))(x)|$$

$$f_*^{\mathcal{G}^d}(x) := \sup_{t \in \mathbb{R}_+^d} |(f * (P_{t_1}^{\mathcal{G}} \otimes \dots \otimes P_{t_d}^{\mathcal{G}}))(x)|$$

$$f_*^{\mathcal{G}_i^d}(x) := \sup_{t_k > 0, k=1,\dots,d; k \ne i} |(f * (P_{t_1}^{\mathcal{G}} \otimes \dots \otimes P_{t_{i-1}}^{\mathcal{G}} \otimes P_{t_{i+1}}^{\mathcal{G}} \otimes \dots \otimes P_{t_d}^{\mathcal{G}}))(x)|$$

$(x \in \mathbb{R}^d, i = 1, \dots, d)$. For $0 < p < \infty$ the *Hardy spaces* $H_p^{\mathcal{G}^d,\square}(\mathbb{R} \times \dots \times \mathbb{R})$, $H_p^{\mathcal{G}^d}(\mathbb{R} \times \dots \times \mathbb{R})$ and $H_p^{\mathcal{G}_i^d}(\mathbb{R} \times \dots \times \mathbb{R})$ consist of all tempered distributions f for which

$$\|f\|_{H_p^{\mathcal{G}^d,\square}(\mathbb{R} \times \dots \times \mathbb{R})} := \|f_*^{\mathcal{G}^d,\square}\|_p < \infty,$$

$$\|f\|_{H_p^{\mathcal{G}^d}(\mathbb{R} \times \dots \times \mathbb{R})} := \|f_*^{\mathcal{G}^d}\|_p < \infty$$

and

$$\|f\|_{H_p^{\mathcal{G}_i^d}(\mathbb{R} \times \dots \times \mathbb{R})} := \|f_*^{\mathcal{G}_i^d}\|_p < \infty,$$

respectively, where $\mathcal{G} \in \{\mathcal{T}, \mathcal{W}, \mathcal{V}, \mathcal{C}, \mathcal{F}\}$ and $i = 1, \dots, d$.

Now let

$$H_p^{\mathcal{F}^d,\square} := H_p^{\mathcal{F}^d,\square}(\mathbb{R} \times \dots \times \mathbb{R}), \quad H_p^{\mathcal{F}^d} := H_p^{\mathcal{F}^d}(\mathbb{R} \times \dots \times \mathbb{R}), \quad H_p^{\mathcal{F}_i^d} := H_p^{\mathcal{F}_i^d}(\mathbb{R} \times \dots \times \mathbb{R})$$

and

$$H_p^{\mathcal{G}^d} := H_p^{\mathcal{G}^d}([0,1) \times \dots \times [0,1)) := \{f \in H_p^{\mathcal{G}^d}(\mathbb{R} \times \dots \times \mathbb{R}) : \text{supp } f \subset [0,1)^d\},$$

where $\mathcal{G} \in \{\mathcal{T}, \mathcal{W}, \mathcal{V}, \mathcal{C}\}$. We define $H_p^{\mathcal{G}^d,\Box}$ and $H_p^{\mathcal{G}_i^d}$ analogously ($i = 1, \ldots, d$). For $\mathcal{G} \in \{\mathcal{T}, \mathcal{W}, \mathcal{V}, \mathcal{C}, \mathcal{F}\}$,

$$H_p^{\mathcal{G}^d,\Box} \sim H_p^{\mathcal{G}^d} \sim H_p^{\mathcal{G}_i^d} \sim L_p \qquad (1 < p \le \infty)$$

(see Stein [61], Weisz [94]). Moreover $H_1^{\mathcal{G}_i^d} \supset L(\log L)^{d-1}$, namely,

$$\|f\|_{H_1^{\mathcal{G}_i^d}} \le C + C\||f|(\log^+ |f|)^{d-1}\|_1,$$

where $\mathcal{G} \in \{\mathcal{T}, \mathcal{W}, \mathcal{V}, \mathcal{C}\}$ and $i = 1, \ldots, d$.

6.1. **The Hardy spaces $H_p^{\mathcal{G}^d,\Box}$.** To obtain some boundedness result for the operator $\sigma_\Box^{\mathcal{G}^d,\theta}$ we consider the Hardy space $H_p^{\mathcal{G}^d,\Box}$. Now the situation is similar to the one-dimensional case. A dyadic (resp. Vilenkin) rectangle is the Kronecker product of dyadic (resp. Vilenkin) intervals.

A function $a \in L_\infty$ is a *cube p-atom* for the $H_p^{\mathcal{T}^d,\Box}$ space if

(a) supp $a \subset I$, $I \subset [0,1)^d$ is a generalized cube,
(b) $\|a\|_\infty \le |I|^{-1/p}$,
(c) $\int_I a(x)x^j \, dx = 0$, for all multi-indices $j = (j_1, \ldots, j_d)$ with $|j| \le [d(1/p - 1)]$.

We suppose for $H_p^{\mathcal{C}^d,\Box}$ that $I \subset [0,1)^d$ is a cube, for $H_p^{\mathcal{F}^d,\Box}$ that $I \subset \mathbb{R}^d$ is a cube, for $H_p^{\mathcal{W}^d,\Box}$ (resp. for $H_p^{\mathcal{V}^d,\Box}$) that $I \subset [0,1)^d$ is a dyadic (resp. Vilenkin) cube. Furthermore, in case $\mathcal{G} \in \{\mathcal{W}, \mathcal{V}\}$, for $H_p^{\mathcal{G}^d,\Box}$ we assume instead of (c)

(c') $\int_I a(x) \, dx = 0$.

The basic result of atomic decomposition is the following one.

Theorem 9. *A tempered distribution f is in $H_p^{\mathcal{G}^d,\Box}$ ($0 < p \le 1, \mathcal{G} \in \{\mathcal{T}, \mathcal{W}, \mathcal{V}, \mathcal{C}, \mathcal{F}\}$) if and only if there exist a sequence $(a^k, k \in \mathbb{N})$ of cube p-atoms for $H_p^{\mathcal{G}^d,\Box}$ and a sequence $(\mu_k, k \in \mathbb{N})$ of real numbers such that*

$$\sum_{k=0}^{\infty} \mu_k a^k = f \quad \textit{in the sense of distributions,} \qquad \sum_{k=0}^{\infty} |\mu_k|^p < \infty. \tag{17}$$

Moreover,

$$\|f\|_{H_p^{\mathcal{G}^d,\Box}} \sim \inf \Big(\sum_{k=0}^{\infty} |\mu_k|^p \Big)^{1/p} \tag{18}$$

where the infimum is taken over all decompositions of f of the form (17).

Again, for the Walsh and Vilenkin systems the first sum in (17) is taken in the sense of martingales.

For a rectangle $R = I_1 \times \ldots \times I_d \subset \mathbb{R}^d$ let $R^r := 2^r R := I_1^r \times \ldots \times I_d^r$ ($r \in \mathbb{N}$). The following result generalizes Theorem 5.

Theorem 10. *Suppose that*

$$\int_{[0,1)^d \setminus I^r} |Va|^{p_0} \, d\lambda \le C_{p_0}$$

for all cube p_0-atoms a and for some fixed $r \in \mathbb{N}$ and $0 < p_0 \le 1$. If the sublinear operator V is bounded from L_{p_1} to L_{p_1} ($1 < p_1 \le \infty$) then

$$\|Vf\|_p \le C_p \|f\|_{H_p^{\mathcal{G}^d,\Box}} \qquad (f \in H_p^{\mathcal{G}^d,\Box}) \tag{19}$$

for all $p_0 \le p \le p_1$. *Moreover, if* $p_0 < 1$ *then the operator* V *is of weak type* $(1,1)$, *i.e. if* $f \in L_1$ *then*

$$\lambda(|Vf| > \rho) \le \frac{C}{\rho}\|f\|_1 \qquad (\rho > 0). \tag{20}$$

Again, (20) follows from (19) by interpolation. The following theorem is due to the author (see Weisz [78, 88, 76, 95, 93, 94]). Let $p_0 := \max\{p_{j,0}, j = 1, \ldots, d\}$, where $p_{j,0}$ is the number defined in (i)–(iv) for θ_j.

Theorem 11. *Assume that (3) holds for each* θ_j $(j = 1, \ldots, d)$. *Furthermore suppose one of the conditions (ii)–(iii) for the trigonometric system and for Fourier transforms, (iii) for the Walsh and Vilenkin systems and for the Ciesielski system we consider the Fejér summability. If* $\max\{p_0, d/(d+1)\} < p \le \infty$ *and* $\mathcal{G} \in \{\mathcal{T}, \mathcal{W}, \mathcal{V}, \mathcal{C}, \mathcal{F}\}$ *then*

$$\|\sigma_\square^{\mathcal{G}^d,\theta} f\|_p \le C_p \|f\|_{H_p^{\mathcal{G}^d,\square}} \qquad (f \in H_p^{\mathcal{G}^d,\square}).$$

In particular, if $f \in L_1$ *then*

$$\sup_{\rho>0} \rho\, \lambda(\sigma_\square^{\mathcal{G}^d,\theta} f > \rho) \le C\|f\|_1.$$

The last weak type inequality implies

Corollary 2. *Under the conditions of Theorem 11 if* $f \in L_1$ *then*

$$\sigma_n^{\mathcal{G}^d,\theta} f \to f \qquad a.e.$$

as $n \to \infty$ *and* $2^{-\tau} \le n_j/n_k \le 2^\tau$ $(j, k = 1, \ldots, d)$, *where* $\mathcal{G} \in \{\mathcal{T}, \mathcal{W}, \mathcal{V}, \mathcal{C}\}$. *Moreover,*

$$\sigma_T^{\mathcal{F}^d,\theta} f \to f \qquad a.e.$$

as $T \to \infty$ *whenever* $2^{-\tau} \le T_j/T_k \le 2^\tau$ $(j, k = 1, \ldots, d)$.

This corollary was proved first by Marcinkiewicz and Zygmund [40] for the trigonometric Fourier series and for the Fejér means.

6.2. **The Hardy spaces $H_p^{\mathcal{G}^d}$.** In the investigation of the operator $\sigma_*^{\mathcal{G}^d,\theta}$ we use the Hardy spaces $H_p^{\mathcal{G}^d}$. The atomic decomposition for $H_p^{\mathcal{G}^d}$ is much more complicated. One reason of this is that the support of an atom is not a rectangle but an open set. Moreover, here we have to choose the atoms from L_2 instead of L_∞. This atomic decomposition was proved by Chang and Fefferman [10, 11, 12, 20, 21], Bernard [4] and Weisz [73, 81, 89, 94]. For an open set $F \subset [0,1)^d$ denote by $\mathcal{M}(F)$ the maximal dyadic subrectangles of F. First we define the atoms for the Hardy space defined for the Fourier transforms. Taking the obvious changes we get the atoms for the trigonometric system and for the Ciesielski system.

A function $a \in L_2$ is a *p-atom* for the $H_p^{\mathcal{F}^d}$ space if

(a) supp $a \subset F$ for some open set $F \subset \mathbb{R}^d$ with finite measure,
(b) $\|a\|_2 \le |F|^{1/2-1/p}$,
(c) a can be further decomposed into the sum of "elementary particles" $a_R \in L_\infty$, $a = \sum_R a_R$, where $R \subset F$ are dyadic rectangles, such that
 (α) supp $a_R \subset 5R$,
 (β) for all $i = 1, \ldots, d$ and R we have

$$\int_{\mathbb{R}} a_R(x) x_i^k \, dx_i = 0 \qquad (k \le N(p) := [2/p - 3/2]),$$

(γ) $a_R \in C^{N(p)+1}$ such that $\|a_R\|_\infty \le d_R$ and

$$\left\|\partial_1^{k_1} \dots \partial_d^{k_d} a_R\right\|_\infty \le \frac{d_R}{|I_1|^{k_1} \cdots |I_d|^{k_d}}$$

for all $0 \le k_i \le N(p)+1$ $(i = 1, \dots, d)$ with

$$\sum_R d_R^2 |R| \le C_p |F|^{1-2/p},$$

where $R = I_1 \times \ldots \times I_d$.

Moreover, a can also be decomposed into the sum of "elementary particles" $\alpha_R \in L_2$, $a = \sum_{R \in \mathcal{M}(F^{(1)})} \alpha_R$, satisfying

(d) supp $\alpha_R \subset 5R$,

(e) for all $i = 1, \dots, d$ and $R \in \mathcal{M}(F^{(1)})$,

$$\int_{\mathbb{R}} \alpha_R(x) x_i^k \, dx_i = 0 \qquad (k \le N(p)),$$

(f) for every disjoint partition $\mathcal{P}_l$ $(l = 1, 2, \dots)$ of $\mathcal{M}(F^{(1)})$,

$$\Big(\sum_l \| \sum_{R \in \mathcal{P}_l} \alpha_R \|_2^2\Big)^{1/2} \le |F|^{1/2-1/p},$$

where $F^{(1)} := \{M_s(1_F) > 1/100\}$ and M_s is the strong maximal function

$$M_s f(x) := \sup_{x \in R} \frac{1}{|R|} \int_R |f| \, d\lambda \qquad (x \in \mathbb{R}^d),$$

the supremum is taken over all rectangles $R \subset \mathbb{R}^d$ with sides parallel to the axes.

This definition is a little bit simpler for the dyadic and Vilenkin Hardy spaces. A function $a \in L_2$ is a *p-atom* for the $H_p^{\mathcal{W}^d}$ space if

(a) supp $a \subset F$ for some open set $F \subset [0,1)^d$,

(b) $\|a\|_2 \le |F|^{1/2-1/p}$,

(c) a can be further decomposed into the sum of "elementary particles" $a_R \in L_2$, $a = \sum_{R \in \mathcal{M}(F)} a_R$ in L_2, satisfying

(d) supp $a_R \subset R \subset F$,

(e) for all $i = 1, \dots, d$ and $R \in \mathcal{M}(F)$ we have

$$\int_{[0,1)} a_R(x) \, d\lambda(x_i) = 0,$$

(f) for every disjoint partition $\mathcal{P}_l$ $(l = 1, 2, \dots)$ of $\mathcal{M}(F)$,

$$\Big(\sum_l \| \sum_{R \in \mathcal{P}_l} a_R \|_2^2\Big)^{1/2} \le |F|^{1/2-1/p}.$$

Note that for the Vilenkin system we take instead of the (maximal) dyadic rectangle (maximal) Vilenkin rectangle.

Theorem 12. *A tempered distribution f is in $H_p^{\mathcal{G}^d}$ $(0 < p \le 1, \mathcal{G} \in \{\mathcal{T}, \mathcal{W}, \mathcal{V}, \mathcal{C}, \mathcal{F}\})$ if and only if there exist a sequence $(a^k, k \in \mathbb{N})$ of p-atoms for $H_p^{\mathcal{G}^d}$ and a sequence $(\mu_k, k \in \mathbb{N})$ of real numbers such that*

$$\begin{aligned} &\sum_{k=0}^\infty \mu_k a^k = f \quad \textit{in the sense of distributions,} \\ &\sum_{k=0}^\infty |\mu_k|^p < \infty. \end{aligned} \tag{21}$$

Moreover,

$$\|f\|_{H_p^{\mathcal{G}^d}} \sim \inf \Big(\sum_{k=0}^{\infty} |\mu_k|^p \Big)^{1/p}$$

where the infimum is taken over all decompositions of f of the form (21).

The corresponding results to Theorems 4 and 12 for the $H_p^{\mathcal{G}^d}$ space are much more complicated. First we consider the two-dimensional case. Since the definition of the p-atom is very complex, to obtain a usable condition about the boundedness of the operators, we have to introduce simpler atoms.

If $d = 2$, a function $a \in L_2[0,1)^2$ is called a *simple p-atom* for the $H_p^{\mathcal{T}^d}$ and $H_p^{\mathcal{C}^d}$ spaces, if

(a) supp $a \subset R$, $R \subset [0,1)^2$ is a rectangle,
(b) $\|a\|_2 \le |R|^{1/2-1/p}$,
(c) $\int_{[0,1)} a(x) x_i^k \, d\lambda(x_i) = 0$, for $i = 1, 2$ and $k \le [2/p - 3/2]$.

For Fourier transforms we change the unit interval by $\mathbb{R}$. For the Walsh and Vilenkin system instead of (c) we assume

(c') $\int_{[0,1)} a(x) \, d\lambda(x_i) = 0$ for $i = 1, 2$

and we use dyadic and Vilenkin rectangles.

Note that $H_p^{\mathcal{G}^d}$ cannot be decomposed into rectangle p-atoms, a counterexample can be found in Weisz [74]. However, the following result says that for an operator V to be bounded from $H_p^{\mathcal{G}^d}$ to L_p $(0 < p \le 1)$ it is enough to check V on simple p-atoms and the boundedness of V on L_2.

Theorem 13. *Suppose that $d = 2$, $0 < p_0 \le 1$ and there exists $\eta > 0$ such that*

$$\int_{[0,1)^2 \setminus R^r} |Va|^{p_0} \, d\lambda \le C_{p_0} 2^{-\eta r}, \tag{22}$$

for all simple p_0-atoms a and for all $r \ge 1$. If the sublinear operator V is bounded from L_2 to L_2, then

$$\|Vf\|_p \le C_p \|f\|_{H_p^{\mathcal{G}^d}} \qquad (f \in H_p^{\mathcal{G}^d}) \tag{23}$$

for all $p_0 \le p \le 2$. In particular, if $p_0 < 1$ then the operator V is of weak type $(H_1^{\mathcal{G}_i^d}, L_1)$, i.e. if $f \in H_1^{\mathcal{G}_i^d}$ for some $i = 1, \ldots, d$ then

$$\sup_{\rho > 0} \rho \lambda(|Vf| > \rho) \le C \|f\|_{H_1^{\mathcal{G}_i^d}}. \tag{24}$$

Inequality (24) follows from (23) by interpolation. In some sense the space $H_1^{\mathcal{G}_i^d}$ plays the role of the one-dimensional L_1 space.

Theorem 13 for two-dimensional classical Hardy spaces is due to Fefferman [20] and for martingale Hardy spaces to Weisz [81]. Unfortunately, the proof of this theorem works for two dimensions, only. In the proof we decreased the dimension by 1 and we used the fact that every one-dimensional open set can be decomposed into the disjoint union of maximal dyadic intervals, which is obviously not true for higher dimensions. Journé [36] even verified that the preceding result do not hold for dimensions greater than 2. So there are fundamental differences between the theory in the two-parameter and three- or more-parameter cases. Fefferman asked in [21] whether one can find sufficient conditions for the sublinear operator to be bounded from $H_p^{\mathcal{G}^d}$ to L_p in higher dimensions. In what follows we answer this question.

Now let us extend the definition of the two-parameter simple atoms.

Let $d \geq 3$. A function $a \in L_2[0,1)^d$ is called a *simple p-atom* for the $H_p^{\mathcal{T}^d}$ and $H_p^{\mathcal{C}^d}$ spaces, if there exist intervals $I_i \subset [0,1)$, $i = 1, \ldots, j$ for some $1 \leq j \leq d-1$ such that

(a) $\operatorname{supp} a \subset I_1 \times \ldots I_j \times A$ for some measurable set $A \subset [0,1)^{d-j}$,
(b) $\|a\|_2 \leq (|I_1| \cdots |I_j||A|)^{1/2-1/p}$,
(c) $\int_{I_i} a(x) x_i^k \, dx_i = \int_A a \, d\lambda = 0$ for $i = 1, \ldots, j$ and $k \leq [2/p - 3/2]$.

Of course if $a \in L_2$ satisfies these conditions for another subset of $\{1, \ldots, d\}$ than $\{1, \ldots, j\}$, then it is also called simple p-atom.

For the other Hardy spaces we take the obvious changes, for example for the dyadic Hardy space we suppose instead of (c) that

(c') $\int_{I_i} a \, d\lambda = \int_A a \, d\lambda = 0$ for all $i = 1, \ldots, j$.

As in the two-parameter case, $H_p^{\mathcal{G}^d}$ cannot be decomposed into simple p-atoms. It is easy to see that the condition (22) can also be formulated as follows:

$$\int_{(I_1^r)^c \times I_2} |Va|^{p_0} \, d\lambda + \int_{(I_1^r)^c \times I_2^c} |Va|^{p_0} \, d\lambda \leq C_{p_0} 2^{-\eta r}$$

and the corresponding inequality holds for the dilation of I_2, where H^c denotes the complement of the set H and $R = I_1 \times I_2$. For higher dimensions we generalize this form. The next theorem is due to the author [89, 94].

Theorem 14. *Let $d \geq 3$. Suppose that the operators V_n are linear for every $n \in \mathbb{N}^d$ and*

$$V^* := \sup_{n \in \beta^d} |V_n|$$

is bounded on L_2. Suppose that there exist $\eta_1, \ldots, \eta_d > 0$, such that for all simple p_0-atoms a and for all $r_1 \ldots, r_d \geq 1$

$$\int_{(I_1^{r_1})^c \times \ldots \times (I_j^{r_j})^c} \int_A |Va|^{p_0} \, d\lambda \leq C_{p_0} 2^{-\eta_1 r_1} \cdots 2^{-\eta_j r_j}.$$

If $j = d-1$ and $A = I_d \subset [0,1)$ is an interval, then we assume also that

$$\int_{(I_1^{r_1})^c \times \ldots \times (I_{d-1}^{r_{d-1}})^c} \int_{(I_d)^c} |Va|^{p_0} \, d\lambda \leq C_{p_0} 2^{-\eta_1 r_1} \cdots 2^{-\eta_{d-1} r_{d-1}}.$$

Then

$$\|V^* f\|_p \leq C_p \|f\|_{H_p^{\mathcal{G}^d}} \qquad (f \in H_p^{\mathcal{G}^d})$$

for all $p_0 \leq p \leq 2$. In particular, if $p_0 < 1$ and $f \in H_1^{\mathcal{G}_i^d}$ for some $i = 1, \ldots, d$ then

$$\lambda(|Vf| > \rho) \leq \frac{C}{\rho} \|f\|_{H_1^{\mathcal{G}_i^d}} \qquad (\rho > 0). \tag{25}$$

Applying Theorems 8, 13 and 14 we can prove the next result (see Weisz [90, 97, 94, 96]).

Theorem 15. *Assume that (3) holds for each θ_j $(j = 1, \ldots, d)$. Furthermore suppose one of the conditions (i)–(iv) for the trigonometric system and for Fourier transforms, (iii) or (iv) for the Walsh and Vilenkin systems and (iv) for the Ciesielski system. If $p_0 < p \leq \infty$ and $\mathcal{G} \in \{\mathcal{T}, \mathcal{W}, \mathcal{K}, \mathcal{V}, \mathcal{C}\}$ then*

$$\|\sigma_*^{\mathcal{G}^d, \theta} f\|_p \leq C_p \|f\|_{H_p} \qquad (f \in H_p^{\mathcal{G}^d}).$$

In particular, if $f \in H_1^{\mathcal{G}_i^d}$ and $i = 1, \ldots, d$ then

$$\sup_{\rho > 0} \rho \, \lambda(\sigma_*^{\mathcal{G}^d, \theta} f > \rho) \leq C \|f\|_{H_1^{\mathcal{G}_i^d}}.$$

Corollary 3. *Under the conditions of Theorem 15 if* $f \in H_1^{\mathcal{G}_i^d} (\supset L(\log L)^{d-1})$, $\mathcal{G} \in \{\mathcal{T}, \mathcal{W}, \mathcal{V}, \mathcal{C}\}$, $i = 1, \ldots, d$, *then*

$$\sigma_n^{\mathcal{G}^d,\theta} f \to f \qquad a.e., \quad as \quad n \to \infty.$$

Moreover, if $f \in H_1^{\mathcal{F}_i^d}$ *then*

$$\sigma_T^{\mathcal{F}^d,\theta} f \to f \qquad a.e., \quad as \quad T \to \infty.$$

Gát [30, 31] proved for the Fejér means and for Walsh-Fourier series that this corollary do not hold for $f \in L_1$.

7. The d-dimensional dyadic derivative

The one-dimensional differentiation theorem

$$f(x) = \lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f(t)\,dt \qquad \text{a.e.} \qquad (f \in L_1[0,1))$$

is well known. In the multi-dimensional case

$$f(x) = \lim_{h\to 0} \frac{1}{\prod_{j=1}^d h_j} \int_{x_1}^{x_1+h_1} \ldots \int_{x_d}^{x_d+h_d} f(t)\,dt \qquad \text{a.e.}$$

if $f \in L(\log L)^{d-1}[0,1)^d$. If $\tau^{-1} \le |h_i/h_j| \le \tau$, then it holds for all $f \in L_1[0,1)^d$ (see Zygmund [100]).

In this section the dyadic analogue of this result will be formulated. Butzer and Wagner [8] introduced the concept of the *dyadic derivative* as follows. For each function f defined on $[0,1)$ set

$$(\mathbf{d}_n f)(x) := \sum_{j=0}^{n-1} 2^{j-1}(f(x) - f(x \dot{+} 2^{-j-1})),$$

$(x \in [0,1))$. This definition was extended to the multi-dimensional case by Butzer and Engels [6],

$$\begin{aligned}(\mathbf{d}_n f)(x) \quad &:= \quad \sum_{i=1}^{d} \sum_{j_i=0}^{n_i-1} 2^{j_1+\ldots+j_d-d} \\ &\qquad \times \sum_{\epsilon_i=0}^{1} (-1)^{\epsilon_1+\ldots+\epsilon_d} f(x_1 \dot{+} \epsilon_1 2^{-j_1-1}, \ldots, x_d \dot{+} \epsilon_d 2^{-j_d-1}),\end{aligned}$$

$n \in \mathbb{N}^d, x \in [0,1)^d$. Then f is said to be *dyadically differentiable* at $x \in [0,1)^d$ if $(\mathbf{d}_n f)(x)$ converges as $n \to \infty$. It was verified by Butzer and Wagner [9] that every Walsh function is dyadically differentiable and

$$\lim_{n\to\infty} (\mathbf{d}_n w_k)(x) = k w_k(x) \qquad (x \in [0,1), k \in \mathbb{N}).$$

The d-dimensional version follows easily from this,

$$\lim_{n\to\infty} (\mathbf{d}_n w_k)(x) = \Big(\prod_{i=1}^d k_i\Big) w_k(x) \qquad (x \in [0,1)^d, n, k \in \mathbb{N}^d).$$

Let W be the function whose Walsh-Fourier coefficients satisfy

$$\hat{W}(k) := \begin{cases} 1 & \text{if } k = 0 \\ 1/k & \text{if } k \in \mathbb{N}, k \ne 0. \end{cases}$$

The d-dimensional *dyadic integral* of $f \in L_1[0,1)^d$ is introduced by

$$\mathbf{I}f(x) \quad := \quad f * (W \times \ldots \times W)(x)$$

$$:= \int_0^1 \cdots \int_0^1 f(t) W(x_1 \dot{+} t_1) \cdots W(x_d \dot{+} t_d)\, dt.$$

Notice that $W \in L_2[0,1) \subset L_1[0,1)$, so $\mathbf{I}$ is well defined on $L_1[0,1)^d$.

For a given $\tau \geq 0$ we will consider the *restricted and non-restricted maximal operators*

$$\mathbf{I}_\square f := \sup_{|n_i - n_j| \leq \tau, i,j=1,\ldots,d} |\mathbf{d}_n(\mathbf{I}f)|, \qquad \mathbf{I}_* f := \sup_{n \in \mathbb{N}^d} |\mathbf{d}_n(\mathbf{I}f)|.$$

Theorem 16. *Suppose that $f \in H_p^{\mathcal{W}^d,\square} \cap L_1$ and*

$$\int_0^1 f(x)\, dx_i = 0 \quad (i = 1, \ldots, d). \tag{26}$$

Then

$$\|\mathbf{I}_\square f\|_p \leq C_p \|f\|_{H_p^{\mathcal{W}^d,\square}}$$

for all $d/(d+1) < p < \infty$. Especially, if $f \in L_1$, then

$$\lambda(\mathbf{I}_\square f > \rho) \leq \frac{C}{\rho} \|f\|_1 \qquad (\rho > 0). \tag{27}$$

Corollary 4. *If $\tau \geq 0$ is arbitrary and if $f \in L_1[0,1)^d$ satisfies the condition (26) then*

$$\mathbf{d}_n(\mathbf{I}f) \to f \quad \text{a.e.} \quad \text{as} \quad n \to \infty \quad \text{and} \quad |n_i - n_j| \leq \tau, \ i,j = 1, \ldots, d.$$

Theorem 16 and Corollary 4 are due to the author [79, 94]. In the one-dimensional case (27) and Corollary 4 was proved by Schipp [52] and in the two-dimensional case by Gát [29].

We note that without the condition (26) we can prove Theorem 16 only for $p = 1$.

Theorem 17. *If $p \geq 1$, then*

$$\|\mathbf{I}_\square f\|_p \leq C_p \|f\|_{H_p^{\mathcal{W}^d,\square}} \qquad (f \in H_p^{\mathcal{W}^d,\square}).$$

For the operator $\mathbf{I}_*$ the following theorem was verified in Weisz [91, 94].

Theorem 18. *If (26) is satisfied and $1/2 < p < \infty$ then*

$$\|\mathbf{I}_* f\|_p \leq C_p \|f\|_{H_p} \qquad (f \in H_p).$$

In particular, if $f \in H_1^{\mathcal{W}_i^d}$ for some $i = 1, \ldots, d$ then

$$\sup_{\rho > 0} \rho\, \lambda(\mathbf{I}_* f > \rho) \leq C \|f\|_{H_1^{\mathcal{W}_i^d}}.$$

Corollary 5. *If $f \in H_p^{\mathcal{W}_i^d} (\supset L(\log L)^{d-1})$ satisfies (26), then*

$$\mathbf{d}_n(\mathbf{I}f) \to f \qquad \textit{a.e., as} \quad n \to \infty.$$

Note that this result for $f \in L \log L$ is due to Schipp and Wade [51] in the two-dimensional case.

Similarly to the dyadic derivative we can define the Vilenkin derivative (see Onneweer [45]) and we can prove similar results (see Pál and Simon [46, 47], Gát and Nagy [27] and Simon and Weisz [56, 55].

References

[1] N. K. Bari. *Trigonometric series*. Fizmatgiz, Moskva, 1961. (in Russian).

[2] C. Bennett and R. Sharpley. *Interpolation of operators*, volume 129 of *Pure and Applied Mathematics*. Academic Press, New York, 1988.

[3] J. Bergh and J. Löfström. *Interpolation spaces, an introduction*. Springer, Berlin, 1976.

[4] A. Bernard. Espaces H_1 de martingales à deux indices. Dualité avec les martingales de type BMO. *Bull. Sc. math.*, 103:297–303, 1979.

[5] J. Bokor and F. Schipp. Approximate identification in Laguerre and Kautz bases. *Automatica*, 34:463–468, 1998.

[6] P. L. Butzer and W. Engels. Dyadic calculus and sampling theorems for functions with multidimensional domain. *Information and Control.*, 52:333–351, 1982.

[7] P. L. Butzer and R. J. Nessel. *Fourier Analysis and Approximation*. Birkhäuser Verlag, Basel, 1971.

[8] P. L. Butzer and H. J. Wagner. Walsh series and the concept of a derivative. *Applic. Anal.*, 3:29–46, 1973.

[9] P. L. Butzer and H. J. Wagner. On dyadic analysis based on the pointwise dyadic derivative. *Analysis Math.*, 1:171–196, 1975.

[10] S.-Y. A. Chang and R. Fefferman. A continuous version of duality of H^1 with BMO on the bidisc. *Ann. of Math.*, 112:179–201, 1980.

[11] S.-Y. A. Chang and R. Fefferman. The Calderon-Zygmund decomposition on product domains. *Amer. J. Math.*, 104:455–468, 1982.

[12] S.-Y. A. Chang and R. Fefferman. Some recent developments in Fourier analysis and H^p-theory on product domains. *Bull. Amer. Math. Soc.*, 12:1–43, 1985.

[13] Z. Ciesielski, P. Simon, and P. Sjölin. Equivalence of Haar and Franklin bases in L_p spaces. *Studia Math.*, 60:195–210, 1977.

[14] Z. Ciesielski. A bounded orthonormal system of polygonals. *Studia Math.*, 31:339–346, 1968.

[15] Z. Ciesielski. Constructive function theory and spline systems. *Studia Math.*, 53:277–302, 1975.

[16] Z. Ciesielski. Equivalence, unconditionality and convergence a.e. of the spline bases in L_p spaces. In *Approximation Theory, Banach Center Publications*, volume 4, pages 55–68. PWN-Polish Scientific Publishers, Warsaw, 1979.

[17] R. R. Coifman and G. Weiss. Extensions of Hardy spaces and their use in analysis. *Bull. Amer. Math. Soc.*, 83:569–645, 1977.

[18] R. R. Coifman. A real variable characterization of H^p. *Studia Math.*, 51:269–274, 1974.

[19] L. Colzani, M. H. Taibleson, and G. Weiss. Maximal estimates for Cesàro and Riesz means on spheres. *Indiana Univ. Math. J.*, 33:873–889, 1984.

[20] R. Fefferman. Calderon-Zygmund theory for product domains: H^p spaces. *Proc. Nat. Acad. Sci. USA*, 83:840–843, 1986.

[21] R. Fefferman. Some recent developments in Fourier analysis and H^p theory on product domains II. In *Function Spaces and Applications, Proceedings, 1986*, volume 1302 of *Lecture Notes in Math.*, pages 44–51. Springer, Berlin, 1988.

[22] N. J. Fine. On the Walsh functions. *Trans. Amer. Math. Soc.*, 65:372–414, 1949.

[23] N. J. Fine. Cesàro summability of Walsh-Fourier series. *Proc. Nat. Acad. Sci. USA*, 41:558–591, 1955.

[24] S. Fridli. On the rate of convergence of Cesàro means of Walsh-Fourier series. *J. Appr. Theory*, 76:31–53, 1994.

[25] S. Fridli. Coefficient condition for L_1-convergence of Walsh-Fourier series. *J. Math. Anal. Appl.*, 210:731–741, 1997.

[26] N. Fujii. A maximal inequality for H^1-functions on a generalized Walsh-Paley group. *Proc. Amer. Math. Soc.*, 77:111–116, 1979.

[27] G. Gát and K. Nagy. The fundamental theorem of two-parameter pointwise derivate on Vilenkin groups. *Analysis Math.*, 25:33–55, 1999.

[28] G. Gát. On $(C, 1)$ summability of integrable functions with respect to the Walsh-Kaczmarz system. *Studia Math.*, 130:135–148, 1998.

[29] G. Gát. On the two-dimensional pointwise dyadic calculus. *J. Appr. Theory.*, 92:191–215, 1998.

[30] G. Gát. On the divergence of the $(C, 1)$ means of double Walsh-Fourier series. *Proc. Amer. Math. Soc.*, 128:1711–1720, 2000.

[31] G. Gát. Divergence of the $(C, 1)$ means of d-dimensional Walsh-Fourier series. *Analysis Math.*, 27:157–171, 2001.

[32] G. G. Gevorkyan. On the uniqueness of trigonometric series. *Mat. Sb.*, 180:1462–1474, 1989. (in Russian).

[33] G. G. Gevorkyan. On the trigonometric series that are summable by Riemann's method. *Mat. Zametki*, 52:17–34, 1992. (in Russian).

[34] G. G. Gevorkyan. On the uniqueness of multiple trigonometric series. *Mat. Sb.*, 184:93–130, 1993. (in Russian).

[35] C. Herz. Bounded mean oscillation and regulated martingales. *Trans. Amer. Math. Soc.*, 193:199–215, 1974.

[36] J.-L. Journé. Two problems of Calderon-Zygmund theory on product spaces. *Ann. Inst. Fourier, Grenoble*, 38:111–132, 1988.

[37] R. H. Latter. A characterization of $H^p(\mathbf{R}^n)$ in terms of atoms. *Studia Math.*, 62:92–101, 1978.

[38] H. Lebesgue. Recherches sur la convergence des séries de Fourier. *Math. Annalen*, 61:251–280, 1905.

[39] S. Lu. *Four lectures on real H^p spaces.* World Scientific, Singapore, 1995.

[40] J. Marcinkiewicz and A. Zygmund. On the summability of double Fourier series. *Fund. Math.*, 32:122–132, 1939.

[41] F. Móricz, F. Schipp, and W. R. Wade. Cesàro summability of double Walsh-Fourier series. *Trans. Amer. Math. Soc.*, 329:131–140, 1992.

[42] F. Móricz. The maximal Fejér operator for Fourier transforms of functions in Hardy spaces. *Acta Sci. Math. (Szeged)*, 62:537–555, 1996.

[43] F. Móricz. The maximal Fejér operator is bounded from $H^1(\mathbf{T})$ to $L^1(\mathbf{T})$. *Analysis*, 16:125–135, 1996.

[44] F. Móricz. The maximal Fejér operator on the spaces H^1 and L^1. In *Approximation Theory and Function Series, Budapest*, volume 5 of *Bolyai Soc. Math. Studies*, pages 275–292, 1996.

[45] C. W. Onneweer. Differentiability for Rademacher series on groups. *Acta Sci. Math. (Szeged)*, 39:121–128, 1977.

[46] J. Pál and P. Simon. On a generalization of the concept of derivative. *Acta Math. Hungar.*, 29:155–164, 1977.

[47] J. Pál and P. Simon. On the generalized Butzer-Wagner type a.e. differentiability of integral function. *Annales Univ. Sci. Budapest, Sectio Math.*, 20:157–165, 1977.

[48] F. Schipp and J. Bokor. L^∞ system approximation algorithms generated by φ summations. *Automatica*, 33:2019–2024, 1997.

[49] F. Schipp and L. Szili. Approximation in $\mathcal{H}_\infty$-norm. In *Approximation Theory and Function Series*, volume 5, pages 307–320. Bolyai Soc. Math. Studies, Budapest, 1996.

[50] F. Schipp, W. R. Wade, P. Simon, and J. Pál. *Walsh series: An introduction to dyadic harmonic analysis.* Adam Hilger, Bristol, New York, 1990.

[51] F. Schipp and W. R. Wade. A fundamental theorem of dyadic calculus for the unit square. *Applic. Anal.*, 34:203–218, 1989.

[52] F. Schipp. Über einen Ableitungsbegriff von P.L. Butzer and H.J. Wagner. *Mat. Balkanica.*, 4:541–546, 1974.

[53] F. Schipp. Über gewissen Maximaloperatoren. *Ann. Univ. Sci. Budapest Sect. Math.*, 18:189–195, 1975.

[54] A. A. Shneider. On series with respect to the Walsh functions with monotone coefficients. *Izv. Akad. Nauk SSSR Ser. Mat.*, 12:179–192, 1948. (in Russian).

[55] P. Simon and F. Weisz. On the two-parameter Vilenkin derivative. *Math. Pannonica*, 12:105–128, 2000.

[56] P. Simon and F. Weisz. Hardy spaces and the generalization of the dyadic derivative. In L. Leindler, F. Schipp, and J. Szabados, editors, *Functions, Series, Operators, Alexits Memorial Conference, Budapest (Hungary), 1999*, pages 367–388, 2002.

[57] P. Simon. Investigations with respect to the Vilenkin system. *Ann. Univ. Sci. Budapest Sect. Math.*, 27:87–101, 1985.

[58] P. Simon. Cesàro summability with respect to two-parameter Walsh systems. *Monatsh. Math.*, 131:321–334, 2000.

[59] P. Simon. On the Cesáro summability with respect to the Walsh-Kaczmarz system. *J. Appr. Theory*, 106:249–261, 2000.

[60] E. M. Stein and G. Weiss. *Introduction to Fourier analysis on Euclidean spaces.* Princeton Univ. Press, Princeton, N.J., 1971.

[61] E. M. Stein. *Harmonic analysis: Real-variable methods, orthogonality and oscillatory integrals.* Princeton Univ. Press, Princeton, N.J., 1993.

[62] L. Szili and P. Vértesi. On uniform convergence of sequences of certain linear operators. *Acta Math. Hungar.*, 91:159–186, 2001.

[63] L. Szili. On the summability of trigonometric interpolation processes. *Acta Math. Hungar.*, 91:131–158, 2001.

[64] L. Szili. Uniform convergent discrete processes on the roots of four kinds of Chebyshev polynomials. *Annales Univ. Sci. Budapest Eötvös, Sect. Math.*, 45:35–62, 2002.

[65] A. Torchinsky. *Real-variable methods in harmonic analysis.* Academic Press, New York, 1986.

[66] N. J. Vilenkin. On a class of complete orthonormal systems. *Izv. Akad. Nauk. SSSR, Ser. Math.*, 11:363–400, 1947.

[67] W. R. Wade. Decay of Walsh series and dyadic differentiation. *Trans. Amer. Math. Soc.*, 277:413–420, 1983.

[68] W. R. Wade. A growth estimate for Cesàro partial sums of multiple Walsh-Fourier series. In *Coll. Math. Soc. J. Bolyai 49, Alfred Haar Memorial Conference, Budapest (Hungary), 1985*, pages 975–991. North-Holland, Amsterdam, 1986.

[69] W. R. Wade. Harmonic analysis on Vilenkin groups. In *Fourier Analysis and Applications*, NAI Publications, pages 339–370, 1996.

[70] W. R. Wade. Dyadic harmonic analysis. *Contemporary Math.*, 208:313–350, 1997.

[71] W. R. Wade. Growth of Cesàro means of double Vilenkin-Fourier series of unbounded type. In *Fourier Analysis of Divergence*, pages 41–50. Birkhauser, Basel, 1999.

[72] W. R. Wade. Summability estimates of double Vilenkin-Fourier series. *Math. Pannonica*, 10:67–75, 1999.

[73] F. Weisz. On duality problems of two-parameter martingale Hardy spaces. *Bull. Sc. math.*, 114:395–410, 1990.

[74] F. Weisz. *Martingale Hardy spaces and their applications in Fourier analysis*, volume 1568 of *Lecture Notes in Math.* Springer, Berlin, 1994.

[75] F. Weisz. Martingale operators and Hardy spaces generated by them. *Studia Math.*, 114:39–70, 1995.

[76] F. Weisz. Cesàro summability of multi-dimensional trigonometric-Fourier series. *J. Math. Anal. Appl.*, 204:419–431, 1996.

[77] F. Weisz. Cesàro summability of one- and two-dimensional Walsh-Fourier series. *Analysis Math.*, 22:229–242, 1996.

[78] F. Weisz. Cesàro summability of two-dimensional Walsh-Fourier series. *Trans. Amer. Math. Soc.*, 348:2169–2181, 1996.

[79] F. Weisz. (H_p, L_p)-type inequalities for the two-dimensional dyadic derivative. *Studia Math.*, 120:271–288, 1996.

[80] F. Weisz. Cesàro summability of one- and two-dimensional trigonometric-Fourier series. *Colloq. Math.*, 74:123–133, 1997.

[81] F. Weisz. Cesàro summability of two-parameter Walsh-Fourier series. *J. Appr. Theory*, 88:168–192, 1997.

[82] F. Weisz. Riemann summability of two-parameter trigonometric-Fourier series. *East J. Appr.*, 3:403–418, 1997.

[83] F. Weisz. Bounded operators on weak Hardy spaces and applications. *Acta Math. Hungar.*, 80:249–264, 1998.

[84] F. Weisz. The maximal Cesàro operator on Hardy spaces. *Analysis*, 18:157–166, 1998.

[85] F. Weisz. The maximal Fejér operator of Fourier transforms. *Acta Sci. Math. (Szeged)*, 64:447–457, 1998.

[86] F. Weisz. Riemann summability of multi-dimensional trigonometric-Fourier series. *Appr. Theory Appl.*, 14:64–74, 1998.

[87] F. Weisz. Riesz means of Fourier transforms and Fourier series on Hardy spaces. *Studia Math.*, 131:253–270, 1998.

[88] F. Weisz. Maximal estimates for the (C, α) means of d-dimensional Walsh-Fourier series. *Proc. Amer. Math. Soc.*, 128:2337–2345, 1999.

[89] F. Weisz. (C, α) means of several-parameter Walsh- and trigonometric-Fourier series. *East J. Approx.*, 6:129–156, 2000.

[90] F. Weisz. θ-summation and Hardy spaces. *J. Appr. Theory*, 107:121–142, 2000.

[91] F. Weisz. The two-parameter dyadic derivative and the dyadic Hardy spaces. *Analysis Math.*, 26:143–160, 2000.

[92] F. Weisz. On the Fejér means of the bounded Ciesielski systems. *Studia Math.*, 146:227–243, 2001.

[93] F. Weisz. Fejér summability of multi-parameter bounded Ciesielski systems. *Analysis Math.*, 28:135–155, 2002.

[94] F. Weisz. *Summability of multi-dimensional Fourier series and Hardy spaces.* Mathematics and Its Applications. Kluwer Academic Publishers, Dordrecht, Boston, London, 2002.

[95] F. Weisz. Summability results of Walsh- and Vilenkin-Fourier series. In L. Leindler, F. Schipp, and J. Szabados, editors, *Functions, Series, Operators, Alexits Memorial Conference, Budapest (Hungary), 1999*, pages 443–464, 2002.

[96] F. Weisz. θ-summability of Fourier series. *Acta Math. Hungar.*, 103:139–176, 2004.
[97] F. Weisz. Weak type inequalities for the Walsh and bounded Ciesielski systems. *Analysis Math.*, 30:147–160, 2004.
[98] J. M. Wilson. A simple proof of the atomic decomposition for $H^p(\mathbf{R})$, $0 < p \leq 1$. *Studia Math.*, 74:25–33, 1982.
[99] J. M. Wilson. On the atomic decomposition for Hardy spaces. *Pac. J. Math.*, 116:201–207, 1985.
[100] A. Zygmund. *Trigonometric series*. Cambridge Press, London, 1959.

Department of Numerical Analysis,
Eötvös L. University,
1117 Budapest, Pázmány P. sétány 1/C.,
Hungary
E-mail address: weisz@ludens.elte.hu