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The contracted model of exploded real numbers

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ABSTRACT. In this paper we show that a set of complex numbers u, where $\operatorname{Im} u = \frac{1}{2} \cdot \frac{n}{|n|+1}$, $(n = 0, \pm 1, \pm 2, \dots)$ and

$$(\operatorname{Re} u) \cdot (\operatorname{Im} u) \geq 0$$

is one of the suitable model of exploded real numbers. This model allows the conclusion that the set of exploded real numbers exists.

In [1] we introduced the set of exploded real numbers $\overset{\sqcup}{R}$ with the following postulates and requirements.

Postulate of extension:
The set of real numbers is a proper subset of exploded real numbers. For any real number x there exists one exploded real number which is called exploded x or the exploded of x. Moreover, the set of exploded x is called the set of exploded real numbers.

Postulate of unambiguity:
For any pair of real numbers x and y, their explodeds are equal if and only if x is equal to y.

Postulate of ordering:
For any pair of real numbers x and y, the exploded x is less than exploded y if and only if x is less than y.

Postulate of super-addition:
For any pair of real numbers x and y, the super-sum of their explodeds is exploded of their sum.

Postulate of super-multiplication:
For any pair of real numbers x and y, the super-product of their explodeds is the exploded of their product.

Requirement of equality for exploded real numbers:
If x and y are real numbers then x as an exploded real number equals to y as an exploded real number if they are equal in the traditional sense.

Requirement of ordering for exploded real numbers:
If x and y are real numbers then x as an exploded real number is less than y as an exploded real number if x is less than y in the traditional sense.

Requirement of monotonity of super-addition:
If u and v are arbitrary exploded real numbers and u is less than v then, for any exploded real number w, u superplus w is less than v superplus w.

Requirement of monotonity of super-multiplication:
If u and v are arbitrary exploded real numbers and u is less than v then, for any positive exploded real number w, u super- multiplied by w is less than v super-multiplied by w.

Definition 1. The explosion of real numbers in a contracted sense: for any real number x, its exploded is

$$\overset{\square}{x} = (\operatorname{sgn} x)\left(\operatorname{area\,th}\{|x|\} + \frac{i}{2}\frac{[|x|]}{[|x|]+1}\right), \quad x \in R. \tag{1.1}$$

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Clearly,

$$\operatorname{Im} \overset{\square}{x} = \frac{1}{2}\frac{n}{|n|+1}, \quad \text{where} \quad n \quad \text{is an integer number and} \quad (\operatorname{Re} \overset{\square}{x}) \cdot (\operatorname{Im} \overset{\square}{x}) \geq 0.$$

Theorem 2. The mapping $x \to \overset{\square}{x}$ is mutually unambiguous.

Proof. Obviously, if $x = y \Rightarrow \overset{\square}{x} = \overset{\square}{y}$ ($\operatorname{Re} \overset{\square}{x} = \operatorname{Re} \overset{\square}{y}$ and $\operatorname{Im} \overset{\square}{x} = \operatorname{Im} \overset{\square}{y}$)
Conversely, we assume that $\overset{\square}{x} = \overset{\square}{y}$. Hence,

$$(\operatorname{sgn} x) \operatorname{area th}\{|x|\} = (\operatorname{sgn} y) \operatorname{area th}\{|y|\} \tag{2.1}$$

and

$$(\operatorname{sgn} x)\frac{[|x|]}{[|x|]+1} = (\operatorname{sgn} y)\frac{[|y|]}{[|y|]+1}. \tag{2.2}$$

By (2.2) the cases $|x| \geq 1$ and $|y| < 1$; $|x| < 1$ and $|y| \geq 1$ are not allowed so we have the following two cases

a) $$0 \leq |x|,\ |y| < 1$$

or

b) $$|x|,\ |y| \geq 1,$$

only.

In the case a) exception of $x = y = 0$, $|x| < 1$ and $y = 0$; $x = 0$ and $|y| < 1$ is not allowed. (See (2.1).) Otherwise we can see that $\{|x|\}$ and $\{|y|\}$ are positive numbers, so (2.1) gives that $\operatorname{sgn} x = \operatorname{sgn} y$.

In the case b) we have that $\frac{[|x|]}{[|x|]+1}$ and $\frac{[|y|]}{[|y|]+1}$ are positive numbers so, (2.2) gives that $\operatorname{sgn} x = \operatorname{sgn} y$.

Collecting these, for all allowed cases of the pairs x, y we obtain

$$\operatorname{sgn} x = \operatorname{sgn} y. \tag{2.3}$$

Using (2.3) we can see that (2.2) yields

$$[|x|] = [|y|]. \tag{2.4}$$

Using (2.3) again by (2.1) we get

$$\{|x|\} = \{|y|\}. \tag{2.5}$$

By (2.4) and (2.5) we have that $|x| = |y|$ and finally (2.3) gives that $x = y$. ■

Remark. Theorem 2 shows that the Postulate of unambiguity is fulfilled.

Theorem 3. If u is a complex number such that $\operatorname{Im} u = \frac{1}{2}\frac{n}{|n|+1}$, $n = 0, \pm 1, \pm 2, \ldots$ and $(\operatorname{Re} u) \cdot (\operatorname{Im} u) \geq 0$, then

$$\overset{\square}{\overline{\frac{\operatorname{Im} u}{\frac{1}{2} - |\operatorname{Im} u|} + \operatorname{th} \operatorname{Re} u}} = u.$$

Proof. It is easy to see that

$$\frac{\operatorname{Im} u}{\frac{1}{2} - |\operatorname{Im} u|} = n \tag{3.2}$$

is valid. First let be $n = 1, 2, 3, \ldots$. Now we have that $\operatorname{Re} u \geq 0$ and by (1.1)

$$\overset{\square}{\overline{n + \operatorname{th} \operatorname{Re} u}} = \operatorname{area th}(\operatorname{th} \operatorname{Re} u) + \frac{i}{2}\frac{n}{n+1} = \operatorname{Re} u + i \operatorname{Im} u = u.$$

For $n = 0$, u is a real number, so $\operatorname{Re} u = u$. Using (1.1) we have:

$$\overset{\square}{\text{th }u} = (\operatorname{sgn} u)\operatorname{area\,th}\{|\text{ th } u|\} = (\operatorname{sgn} u)\operatorname{area\,th}|\text{ th } u| =$$

$$= (\operatorname{sgn} u)\operatorname{area\,th}(\operatorname{th}|u|) = (\operatorname{sgn} u)|u| = u.$$

Finally, for $n = -1, -2, -3$ we have that $\operatorname{Re} u \leq 0$ and by (1.1)

$$\overset{\square}{n + \operatorname{th}\operatorname{Re} u} = -\Big(\operatorname{area\,th}(-\operatorname{th}\operatorname{Re} u) + \frac{i}{2}\frac{|n|}{|n|+1}\Big) = \operatorname{Re} u + i\operatorname{Im} u = u.$$

$\blacksquare$

Theorem 3 and (1.2) yield

Corollary 4. The complex number u is an exploded real number in a contracted sense, if and only if $\operatorname{Im} u = \frac{1}{2}\frac{n}{|n|+1}$, $n = 0, \pm 1, \pm 2, \ldots$, and $(\operatorname{Re} u)\cdot(\operatorname{Im} u) \geq 0$.

We denote the set of exploded real numbers, in a contracted sense, by $\overset{\square}{R}$.

$$\overset{\square}{R} = \{u \in \mathbf{C} : u = \operatorname{Re} u + i\operatorname{Im} u,\ \operatorname{Im} u = \frac{1}{2}\frac{n}{|n|+1}, n \text{ is integer and } (\operatorname{Re} u)\cdot(\operatorname{Im} u) \geq 0.\}$$

Definition 5. For any set $S \subseteq \mathbf{R}$, the exploded S is: $\overset{\square}{S} = \{u \in \mathbf{C} : \quad u = \overset{\square}{x}$ such that $x \in S\}$. Considering the open interval $(-1, 1)$ by Definitions 1 and 5 we obtain

Corollary 6. $\overset{\square}{(-1,1)} = \mathbf{R}$.

So, we can see that the Postulate of extension is fulfilled.

Definition 7. The compression of exploded real numbers: for any exploded real number u, its compressed is

$$\underset{\square}{u} = \frac{\operatorname{Im} u}{\frac{1}{2} - |\operatorname{Im} u|} + th\ \operatorname{Re} u, \quad u \in \overset{\square}{R}. \tag{7.1}$$

By (3.1) and (7.1) we have the identity

$$\overset{\square}{(\underset{\square}{u})} = u, \quad u \in \overset{\square}{R}. \tag{7.2}$$

Definition 8. For set $S \subseteq \overset{\square}{\mathbf{R}}$, the compressed of S is: $\underset{\square}{S} = \{x \in \mathbf{R} : x = \underset{\square}{u}$, such that $u \in S\}$.

Theorem 9. For any real number x the identity

$$\underset{\square}{(\overset{\square}{x})} = x, \quad x \in \mathbf{R} \tag{9.1}$$

holds.

Definitions 5 and 8 with (7.2) and (9.1) yield

Corollary 10.

$$\underset{\square}{(\overset{\square}{S})} = S, \quad S \subseteq \mathbf{R} \tag{10.1}$$

and

$$\overset{\square}{(\underset{\square}{S})} = S, \quad S \subseteq \overset{\sqcup}{R}. \tag{10.2}$$

Definition 11. For any $x, y \in R$ we say that $\overset{\square}{x} \overset{\overset{\square}{R}}{<} \overset{\square}{y}$ if $\operatorname{Im} \overset{\square}{x} < \operatorname{Im} \overset{\square}{y}$ or if $\operatorname{Im} \overset{\square}{x} = \operatorname{Im} \overset{\square}{y}$ then $\operatorname{Re} \overset{\square}{x} < \operatorname{Re} \overset{\square}{y}$.

Definition 12. For any $x, y \in \mathbf{R}$ we say that $\overset{\square}{x} \overset{\overset{\square}{R}}{>} \overset{\square}{y}$ if $\overset{\square}{y} \overset{\overset{\square}{R}}{<} \overset{\square}{x}$.

Theorem 13. For any x the inequality $\overset{\square}{x} \overset{\overset{\square}{R}}{<} \overset{\square}{y}$ holds if and only if $x < y$.

Proof.

Necessity. Let us assume that $\overset{\square}{x} \overset{\overset{\square}{R}}{<} \overset{\square}{y}$. By Definition 11 we consider two cases:

Case 1. $\operatorname{Im} \overset{\square}{x} < \operatorname{Im} \overset{\square}{y}$, that is, by (1.1) we have

$$(\operatorname{sgn} x)\frac{[|x|]}{[|x|]+1} < (\operatorname{sgn} y)\frac{[|y|]}{[|y|]+1} \tag{13.1}$$

Now, if $x \geq y$ then considering the monotonity of the function $f(x) = (\operatorname{sgn} x)\frac{[|x|]}{[|x|]+1}$ we have that $f(x) \geq f(y)$ which contradicts (13.1). So, $x < y$.

Case 2. $\operatorname{Im} \overset{\square}{x} = \operatorname{Im} \overset{\square}{y}$ and $\operatorname{Re} \overset{\square}{x} < \operatorname{Re} \overset{\square}{y}$. Now we have (2.2) and

$$(\operatorname{sgn} x) \operatorname{area\,th}\{|x|\} < (\operatorname{sgn} y) \operatorname{area\,th}\{|y|\} \tag{13.2}$$

moreover, x and y are not integer numbers. If $x = 0$ then $y > 0$, if $y = 0$ then $x < 0$. Otherwise, $\operatorname{area\,th}\{|x|\}$, $\operatorname{area\,th}\{|y|\} > 0$. Inequality $\operatorname{sgn} x > \operatorname{sgn} y$ is not allowed.

If $\operatorname{sgn} x < \operatorname{sgn} y$ then $x < y$, obviously.

If $\operatorname{sgn} x = \operatorname{sgn} y = 1$, then (2.2) yields that $[x] = [y]$ and (13.2) gives that $\{x\} < \{y\}$, so $0 < x < y$.

If $\operatorname{sgn} x = \operatorname{sgn} y = -1$, then (2.2) yields that $[|x|] = [|y|]$ and the identity $[|x|] = -([x]+1)$ shows that $[x] = [y]$. Inequality (13.2) gives that $\{|x|\} > \{|y|\}$. Hence, by identity $\{|x|\} = -(\{x\} - 1)$ we have that $\{x\} < \{y\}$. So, $x < y < 0$ is obtained.

Collecting the cases we have

$$x < y. \tag{13.3}$$

Sufficiency. Let us assume that $x < y$. Considering the monotonity of the function $f(x) = (\operatorname{sgn} x)\frac{[|x|]}{[|x|]+1}$, we have

$$(\operatorname{sgn} x)\frac{[|x|]}{[|x|]+1} < (\operatorname{sgn} y)\frac{[|y|]}{[|y|]+1} \tag{13.4}$$

or

$$(\operatorname{sgn} x)\frac{[|x|]}{[|x|]+1} = (\operatorname{sgn} y)\frac{[|y|]}{[|y|]+1}. \tag{13.5}$$

In case of (13.4), Definition 11 and (1.1) show that $\overset{\square}{x} \overset{\overset{\square}{R}}{<} \overset{\square}{y}$.

In case of (13.5) the cases $|x| \geq 1$ and $|y| < 1$; $|x| < 1$ and $|y| \geq 1$ are not allowed. So, we have the following two cases

a) $$0 \leq |x|, |y| < 1$$

or

b) $$|x|, |y| \geq 1,$$

only.

In the case a) if $x = 0$ then $y > 0$, if $y = 0$ then $x < 0$. Otherwise, $0 < |x|,\ |y| < 1$. Clearly, $[|x|] = [|y|] = 0$, so $\{|x|\} = |x|$, $\{|y|\} = |y|$. The inequality $x < y$ implies that $\operatorname{sgn} x \le \operatorname{sgn} y$.

If $\operatorname{sgn} x < \operatorname{sgn} y$ then $-1 < x < 0 < y < 1$. So,

$$(\operatorname{sgn} x)\,\mathrm{area\,th}\{|x|\} < 0 < (\operatorname{sgn} x)\,\mathrm{area\,th}\{|y|\}$$

and Definition 11 by (1.1) gives that $\overset{\square}{x} < \overset{\square}{y}$.

If $\operatorname{sgn} x = \operatorname{sgn} y = 1$ then $0 < x < y < 1$. So,

$$0 < (\operatorname{sgn} x)\,\mathrm{area\,th}\{|x|\} < (\operatorname{sgn} y)\,\mathrm{area\,th}\{|y|\}$$

and Definition 11 by (1.1) gives that $\overset{\square}{x} < \overset{\square}{y}$.

If $\operatorname{sgn} x = \operatorname{sgn} y = -1$, then $-1 < x < y < 0$. Hence, $0 < |y| < |x| < 1$ and $0 < \{|y|\} < \{|x|\} < 1$. So

$$0 > (\operatorname{sgn} y)\,\mathrm{area\,th}\{|y|\} > (\operatorname{sgn} x)\,\mathrm{area\,th}\{|x|\} > -1$$

and Definition 11 by (1.1) gives that $\overset{\square}{x} < \overset{\square}{y}$.

In the case b) (13.5) yields

$$[|x|] = [|y|].$$

Integer x and y are not allowed.

If $\operatorname{sgn} x = \operatorname{sgn} y = 1$, then the identity $\{|x|\} = x - [|x|]$ by $x < y$ implies that $\{|x|\} < \{|y|\}$. Hence, $(\operatorname{sgn} x)\,\mathrm{area\,th}\{|x|\} < (\operatorname{sgn} y)(\mathrm{area\,th}\{|y|\})$. So, Definition 11 by (1.1) gives that $\overset{\square}{x} < \overset{\square}{y}$.

The case $\operatorname{sgn} x = 1$ and $\operatorname{sgn} y = -1$ is not allowed.

If $\operatorname{sgn} x = -1$ and $\operatorname{sgn} y = 1$ then $(\operatorname{sgn} x)\,\mathrm{area\,th}\{|x|\} < (\operatorname{sgn} y)(\mathrm{area\,th}\{|y|\})$. So Definition 11 by (1.1) gives that $\overset{\square}{x} < \overset{\square}{y}$.

If $\operatorname{sgn} x = \operatorname{sgn} y = -1$, then identity $\{|x|\} = -x - [|x|]$. So, inequality $x < y$ implies $-x > -y > 1$. Hence, $\{|x|\} > \{|y|\}$. So, $(\operatorname{sgn} x)\,\mathrm{area\,th}\{|x|\} < (\operatorname{sgn} y)\,\mathrm{area\,th}\{|y|\}$ and Definition 11 by (1.1) gives that $\overset{\square}{x} < \overset{\square}{y}$.

Remark. Theorem 13 shows that the Postulate of ordering is fulfilled.

Theorem 14. If $x, y \in \mathbf{R}$ then $x \overset{\overset{\square}{R}}{<} y \Longleftrightarrow x < y$.

Proof. Identity (7.2) and Theorem 13 show that $x \overset{\overset{\square}{R}}{<} y \Longleftrightarrow \underset{\square}{x} < \underset{\square}{y}$. By (7.1) we have that $\underset{\square}{x} = th\ x$ and $\underset{\square}{y} = th\ y$. Using the strict monotonity of the function th we have that $\underset{\square}{x} < \underset{\square}{y} \Longleftrightarrow x < y$.

Remark. Theorem 14 shows that the Requirement of ordering is fulfilled.

Remark 15. By Theorem 14 we may use $u < v$ instead of $u \overset{\overset{\square}{R}}{<} v$ for any $u, v \in \overset{\square}{R}$. Theorem 13 with identity (7.2) gives

Theorem 16. (*Monotonity of compression*) For any $u, v \in \overset{\square}{R}$ the inequality $\underset{\square}{u} < \underset{\square}{v}$ holds if and only if $u < v$. Moreover, Theorem 13 yields the following corollaries:

Corollary 17. The relation "$<$" is irreflixive, anti- symmetrical and transitive.

Corollary 18. (*Trichotomity)*) For any $x, y \in \mathbf{R}$ from among relations $\overset{\square}{x} < \overset{\square}{y}$, $\overset{\square}{x} = \overset{\square}{y}$ and $\overset{\square}{x} > \overset{\square}{y}$ one and only one is true.

Definition 19. (*Super-addition*) For any $x, y \in \mathbf{R}$, the super-sum of $\overset{\square}{x}$ and $\overset{\square}{y}$ is

$$\overset{\square}{x} \,-\!\!\oplus\!\!-\, \overset{\square}{y} = (\operatorname{sgn}(x+y))\Big(\mathrm{area\,th}\{|x+y|\} + \frac{i}{2}\,\frac{[|x+y|]}{[|x+y|]+1}\Big). \tag{19.1}$$

By Definition 1 the identity

$$(19.2) \qquad \overset{\square}{x} \oplus \overset{\square}{y} = \overset{\square}{x+y}, \quad x, y \in \mathbf{R} \quad (\mathit{See\ Postulate\ of\ super-addition})$$

is obvious.

Definition 20. (*Super-multiplication*) For any $x, y \in \mathbf{R}$, the super-multiplication of $\overset{\square}{x}$ and $\overset{\square}{y}$ is

$$(20.1) \qquad \overset{\square}{x} \odot \overset{\square}{y} = (\operatorname{sgn}(x \cdot y))(\operatorname{area\,th}\{|x \cdot y|\} + \frac{i}{2}\frac{[|x \cdot y|]}{[|x \cdot y|] + 1}.$$

By Definition 1 the identity

$$(20.2) \qquad \overset{\square}{x} \odot \overset{\square}{y} = \overset{\square}{x \cdot y}, \quad x, y \in \mathbf{R} \quad (\mathit{See\ Postulate\ of\ super-multiplication})$$

is obvious.

Remark 21. Using identities (19.2) and (20.2) we find that the field $(\mathbf{R}, +, \cdot)$ is isomorphic with the algebraic structure $(\overset{\square}{R}, \oplus, \odot)$; so the latter is also a field with the operations super-addition and super-multiplication. By (19.1) we can see that the additive unit element of $\overset{\square}{R}$ is $\overset{\square}{0} = 0$. The additive inverse element of $\overset{\square}{x}$ is $\overset{\square}{-x}$ for which, by (1.1), the identity

$$(21.1) \qquad \overset{\square}{-x} = -\overset{\square}{x}, \quad x \in \mathbf{R}$$

holds. By (20.1) we can see that the multiplicative unit element of $\overset{\square}{R}$ is $\overset{\square}{1} = \frac{i}{4}$. The multiplicative inverse element of $\overset{\square}{x} \neq 0$ is $\overset{\square}{(\frac{1}{x})}$.

Remark 22. By (7.1) we have that for any $u \in \overset{\square}{R}$ the identity

$$(22.1) \qquad \underset{\square}{-u} = -\underset{\square}{u}, \quad u \in \overset{\square}{R}$$

holds. Moreover, denoting $\overset{\square}{x} = u$ and $\overset{\square}{y} = v$, the identities (19.2) and (20.2) by (9.1) yield the identities

$$(22.2) \qquad u \oplus v = \overset{\square}{\underset{\square}{u} + \underset{\square}{v}}\,(u, v \in \overset{\square}{R})$$

and

$$(22.3) \qquad u \odot v = \overset{\square}{\underset{\square}{u} \cdot \underset{\square}{v}}\,(u, v \in \overset{\square}{R}),$$

respectively.

Definition 23. The exploded real number u is called positive if $u > 0$ and negative if $u < 0$. (These are extensions of the familiar positivity and negativity of real numbers.)

Theorem 24. (*Monotonity of super-addition*) Let u, v and w be arbitrary exploded real numbers. If $u < v$ then

$$u \odot w < v \oplus w.$$

Proof. Using (22.2), Theorem 16, Theorem 13 and (22.2) again, we have that

$$u \oplus w = \overset{\square}{\underset{\square}{u} + \underset{\square}{w}} < \overset{\square}{\underset{\square}{v} + \underset{\square}{w}} = v \oplus w.$$

Theorem 25. (*Monotonity of super-multiplication*) Let u, v be arbitrary and w positive exploded real numbers. If $u < v$ then $u \odot w < v \odot w$.

Proof. First, we mention that by Theorem 16 and Definition 23 with Definition (7.1) $\underset{\square}{w} > \underset{\square}{0} = 0$ is obtained. Moreover, using (22.3), Theorem 16, Theorem 13 and (22.3) again, we have that

$$u \odot w = \overline{\underset{\square}{u} \cdot \underset{\square}{w}} < \overline{\underset{\square}{v} \cdot \underset{\square}{w}} = v \odot w.$$

Remark 26. Considering Remark 21, Theorem 24 and Theorem 25 we can see that $(\overset{\sqcup}{R}, \oplus, \odot)$ is an ordered field.

Reference

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