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SOME SUBCLASSES OF α-UNIFORMLY CONVEX FUNCTIONS

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Abstract. In this paper we define some subclass of α - uniformly convex functions with respect to a convex domain included in right half plane D.

1. Introduction

Let $\mathcal{H}(U)$ be the set of functions which are regular in the unit disc U,

$$A = \{f \in \mathcal{H}(U) : f(0) = f'(0) - 1 = 0\},$$

$\mathcal{H}_u(U) = \{f \in \mathcal{H}(U) : f \text{ is univalent in } U\}$ and $S = \{f \in A : f \text{ is univalent in } U\}$.

Let consider the integral operator $L_a \colon A \to A$ defined as:

$$f(z) = L_a F(z) = \frac{1+a}{z^a} \int_0^z F(t) \cdot t^{a-1} dt, \quad a \in \mathbb{C}, \quad \text{Re } a \geq 0. \tag{1}$$

In the case $a = 1, 2, 3, \ldots$ this operator was introduced by S.D. Bernardi and it was studied by many authors in different general cases.

Let D^n be the Sălăgean differential operator (see [10]) defined as:

$$D^n \colon A \to A, \quad n \in \mathbb{N} \text{ and } D^0 f(z) = f(z)$$
$$D^1 f(z) = Df(z) = zf'(z), \quad D^n f(z) = D(D^{n-1} f(z)).$$

2. Preliminary results

Definition 2.1 ([4])**.** Let $\alpha \in [0,1]$ and $f \in A$. We say that f is α - uniformly convex function if:

$$\text{Re}\left\{(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} \geq \left|(1-\alpha)\left(\frac{zf'(z)}{f(z)} - 1\right) + \alpha\frac{zf''(z)}{f'(z)}\right|, \ z \in U.$$

We denote this class with UM_α.

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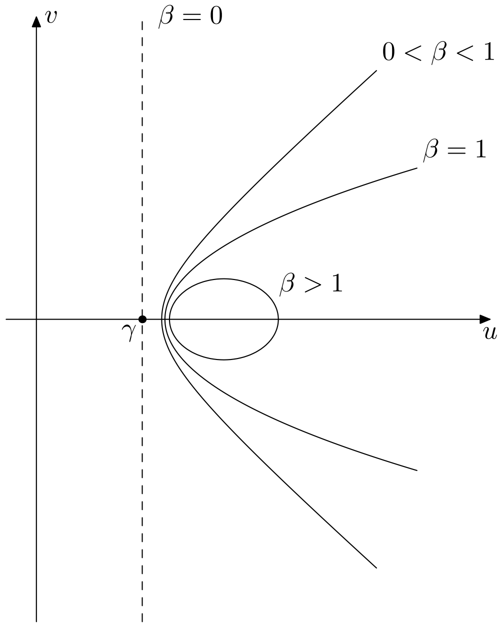

FIGURE 1

Remark 2.1. Geometric interpretation: $f \in UM_\alpha$ if and only if

$$J(\alpha, f; z) = (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)$$

take all values in the parabolic region $\Omega = \{w : |w-1| \leq \text{Re } w\} = \{w = u+iv : v^2 \leq 2u-1\}$. We have $UM_0 = SP$, where the class SP was introduced by F. Ronning in [9] and $UM_\alpha \subset M_\alpha$, where M_α is the well know class of α - convex functions introduced by P.T. Mocanu in [8].

Definition 2.2 ([1])**.** Let $\alpha \in [0,1]$ and $n \in \mathbb{N}$. We say that $f \in A$ is in the class $UD_{n,\alpha}(\beta, \gamma)$, $\beta \geq 0$, $\gamma \in [-1, 1)$, $\beta + \gamma \geq 0$ if

$$\text{Re}\left[(1-\alpha)\frac{D^{n+1}f(z)}{D^n f(z)} + \alpha\frac{D^{n+2}f(z)}{D^{n+1}f(z)}\right] \geq \beta\left|(1-\alpha)\frac{D^{n+1}f(z)}{D^n f(z)} + \alpha\frac{D^{n+2}f(z)}{D^{n+1}f(z)} - 1\right| + \gamma.$$

Remark 2.2. Geometric interpretation: $f \in UD_{n,\alpha}(\beta, \gamma)$ if and only if

$$J_n(\alpha, f; z) = (1-\alpha)\frac{D^{n+1}f(z)}{D^n f(z)} + \alpha\frac{D^{n+2}f(z)}{D^{n+1}f(z)}$$

take all values in the convex domain included in right half plane $D_{\beta,\gamma}$, where $D_{\beta,\gamma}$ is an elliptic region for $\beta > 1$, a parabolic region for $\beta = 1$, a hyperbolic region for $0 < \beta < 1$, the half plane $u > \gamma$ for $\beta = 0$. (Figure 1.)

We have $UD_{0,\alpha}(1, 0) = UM_\alpha$.

The next theorem is result of the so called "admissible functions method" introduced by P.T. Mocanu and S.S. Miller (see [5], [6], [7]).

Theorem 2.1. *Let h convex in U and* $\mathrm{Re}[\beta h(z)+\delta]>0$, $z\in U$. *If $p\in\mathcal{H}(U)$ with $p(0)=h(0)$ and p satisfied the Briot-Bouquet differential subordination* $p(z)+\dfrac{zp'(z)}{\beta p(z)+\delta}\prec h(z)$, *then* $p(z)\prec h(z)$.

Definition 2.3 ([3])**.** The function $f\in A$ is n-starlike with respect to convex domain included in right half plane D if the differential expression $\dfrac{D^{n+1}f(z)}{D^n f(z)}$ takes values in the domain D.

If we consider $q(z)$ an univalent function with $q(0)=1$, Re $q(z)>0$, $q'(0)>0$ which maps the unit disc U into the convex domain D we have:

$$\frac{D^{n+1}f(z)}{D^n f(z)}\prec q(z).$$

We note by $S_n^*(q)$ the set of all these functions.

3. Main results

Let $q(z)$ be an univalent function with $q(0)=1$, $q'(0)>0$, which maps the unit disc U into a convex domain included in right half plane D.

Definition 3.1. Let $f\in A$ and $\alpha\in[0,1]$. We say that f is α-uniform convex function with respect to D, if

$$J(\alpha,f;z)=(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\prec q(z).$$

We denote this class with $UM_\alpha(q)$.

Remark 3.1. Geometric interpretation: $f\in UM_\alpha(q)$ if and only if $J(\alpha,f;z)$ take all values in the convex domain included in right half plan D.

Remark 3.2. We have $UM_\alpha(q)\subset M_\alpha$, where M_α is the well know class of α-convex function. If we take $D=\Omega$ (see Remark 2.1) we obtain the class UM_α.

Remark 3.3. From the above definition it easily results that $q_1(z)\prec q_2(z)$ implies $UM_\alpha(q_1)\subset UM_\alpha(q_2)$.

Theorem 3.1. *For all $\alpha,\alpha'\in[0,1]$ with $\alpha<\alpha'$ we have $UM_{\alpha'}(q)\subset UM_\alpha(q)$.*

Proof. From $f\in UM_{\alpha'}(q)$ we have

$$J(\alpha',f;z)=(1-\alpha')\frac{zf'(z)}{f(z)}+\alpha'\left(1+\frac{zf''(z)}{f'(z)}\right)\prec q(z), \tag{2}$$

where $q(z)$ is univalent in U with $q(0)=1$, $q'(0)>0$, and maps the unit disc U into the convex domain included in right half plane D.

With notation $\dfrac{zf'(z)}{f(z)}=p(z)$, where $p(z)=1+p_1z+\dots$ we have:

$$J(\alpha',f;z)=p(z)+\alpha'\cdot\frac{zp'(z)}{p(z)}.$$

From (2) we have $p(z)+\alpha'\cdot\dfrac{zp'(z)}{p(z)}\prec q(z)$ with $p(0)=q(0)$ and Re $q(z)>0$, $z\in U$.

In this conditions from Theorem 2.1, with $\delta = 0$, we obtain $p(z) \prec q(z)$, or $p(z)$ take all values in D.

If we consider the function $g\colon [0, \alpha'] \to \mathbb{C}$, $g(u) = p(z) + u \cdot \dfrac{zp'(z)}{p(z)}$, with $g(0) = p(z) \in D$ and $g(\alpha') = J(\alpha', f; z) \in D$. Since the geometric image of $g(\alpha)$ is on the segment obtained by the union of the geometric image of $g(0)$ and $g(\alpha')$, we have $g(\alpha) \in D$ or $p(z) + \alpha \dfrac{zp'(z)}{p(z)} \in D$.

Thus $J(\alpha, f; z)$ take all values in D, or $J(\alpha, f; z) \prec q(z)$. This means $f \in UM_\alpha(q)$. □

Theorem 3.2. *If $F(z) \in UM_\alpha(q)$ then $f(z) = L_a(F)(z) \in S_0^*(q)$, where L_a is the integral operator defined by (1) and $\alpha \in [0, 1]$.*

Proof. From (1) we have

$$(1 + a)F(z) = af(z) + zf'(z).$$

With notation $\dfrac{zf'(z)}{f(z)} = p(z)$, where $p(z) = 1 + p_1 z + \dots$ we have

$$\frac{zF'(z)}{F(z)} = p(z) + \frac{zp'(z)}{p(z) + a}.$$

If we denote $\dfrac{zF'(z)}{F(z)} = h(z)$, with $h(0) = 1$, we have from $F(z) \in UM_\alpha(q)$ (see Definition 3.1):

$$h(z) + \alpha \cdot \frac{zh'(z)}{h(z)} \prec q(z),$$

where $q(z)$ is univalent un U with $q(0) = 1$, $q'(z) > 0$ and maps the unit disc U into the convex domain included in right half plane D.

From Theorem 2.1 we obtain $h(z) \prec q(z)$ or $p(z) + \dfrac{zp'(z)}{p(z) + a} \prec q(z)$.

Using the hypothesis and the construction of the function $q(z)$ we obtain from Theorem 2.1 $\dfrac{zf'(z)}{f(z)} = p(z) \prec q(z)$ or $f(z) \in S_0^*(q) \subset S^*$. □

Definition 3.2. Let $f \in A$, $\alpha \in [0, 1]$ and $n \in \mathbb{N}$. We say that f is $\alpha - n$-uniformly convex function with respect to D if

$$J_n(\alpha, f; z) = (1 - \alpha)\frac{D^{n+1}f(z)}{D^n f(z)} + \alpha\frac{D^{n+2}f(z)}{D^{n+1}f(z)} \prec q(z).$$

We denote this class with $UD_{n,\alpha}(q)$.

Remark 3.4. Geometric interpretation: $f \in UD_{n,\alpha}(q)$ if and only if $J_n(\alpha, f; z)$ take all values in the convex domain included in right half plane D.

Remark 3.5. We have $UD_{0,\alpha}(q) = UM_\alpha(q)$ and if in the above definition we consider $D = D_{\beta,\gamma}$ (see Remark 2.2) we obtain the class $UD_{n,\alpha}(\beta, \gamma)$.

Remark 3.6. It is easy to see that $q_1(z) \prec q_2(z)$ implies $UD_{n,\alpha}(q_1) \subset UD_{n,\alpha}(q_2)$.

Theorem 3.3. *For all $\alpha, \alpha' \in [0, 1]$ with $\alpha < \alpha'$ we have $UD_{n,\alpha'}(q) \subset UD_{n,\alpha}(q)$.*

Proof. From $f \in UD_{n,\alpha'}(q)$ we have:

$$J_n(\alpha', f; z) = (1-\alpha')\frac{D^{n+1}f(z)}{D^n f(z)} + \alpha' \frac{D^{n+2}f(z)}{D^{n+1}f(z)} \prec q(z), \tag{3}$$

where $q(z)$ is univalent in U with $q(0) = 1$, $q'(0) > 0$, and maps the unit disc U into the convex domain included in right half plane D.

With notation $\dfrac{D^{n+1}f(z)}{D^n f(z)} = p(z)$, where $p(z) = 1 + p_1 z + \dots$ we have

$$J_n(\alpha', f; z) = p(z) + \alpha' \cdot \frac{zp'(z)}{p(z)}.$$

From (3) we have $p(z) + \alpha' \cdot \dfrac{zp'(z)}{p(z)} \prec q(z)$ with $p(0) = q(0)$ and Re $q(z) > 0$, $z \in U$. In this condition from Theorem 2.1 we obtain $p(z) \prec q(z)$, or $p(z)$ take all values in D.

If we consider the function

$$g\colon [0, \alpha'] \to \mathbb{C}, \quad g(u) = p(z) + u \cdot \frac{zp'(z)}{p(z)},$$

with $g(0) = p(z) \in D$ and $g(\alpha') = J_n(\alpha', f; z) \in D$, it easy to see that

$$g(\alpha) = p(z) + \alpha \frac{zp'(z)}{p(z)} \in D.$$

Thus we have $J_n(\alpha, f; z) \prec q(z)$ or $f \in UD_{n,\alpha}(q)$. □

Theorem 3.4. *If $F(z) \in UD_{n,\alpha}(q)$ then $f(z) = L_a(F)(z) \in S_n^*(q)$, where L_a is the integral operator defined by (1).*

Proof. From (1) we have $(1+a)F(z) = af(z) + zf'(z)$. By means of the application of the linear operator D^{n+1} we obtain:

$$(1+a)D^{n+1}F(z) = aD^{n+1}f(z) + D^{n+1}(zf'(z))$$

or

$$(1+a)D^{n+1}F(z) = aD^{n+1}f(z) + D^{n+2}f(z).$$

With notation $\dfrac{D^{n+1}f(z)}{D^n f(z)} = p(z)$, where $p(z) = 1 + p_1 z + \dots$, we have:

$$\frac{D^{n+1}F(z)}{D^n F(z)} = p(z) + \frac{1}{p(z) + a} \cdot zp'(z).$$

If we denote $\dfrac{D^{n+1}F(z)}{D^n F(z)} = h(z)$, with $h(0) = 1$, we have from $F \in UD_{n,\alpha}(q)$:

$$h(z) + \alpha \frac{zh'(z)}{h(z)} \prec q(z),$$

where $q(z)$ is univalent in U with $q(0) = 1$, $q'(0) > 0$, and maps the unit disc U into the convex domain included in right half plane D.

From Theorem 2.1 we obtain $h(z) \prec q(z)$ or $p(z) + \dfrac{zp'(z)}{p(z) + a} \prec q(z)$.

Using the hypothesis we obtain from Theorem 2.1 $p(z) \prec q(z)$ or $f(z) \in S_n^*(q)$. □

Remark 3.7. If we consider $D = D_{\beta,\gamma}$ in Theorem 3.3 and Theorem 3.4 we obtain the main results from [1] and if we take $D = D_{\beta,\gamma}$ and $\alpha = 0$ in Theorem 3.4 we obtain the Theorem 3.1 from [2].

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