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THE BECKMAN-QUARLES THEOREM FOR MAPPINGS FROM $\mathbb{C}^2$ TO $\mathbb{C}^2$

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Abstract. Let $\varphi\colon \mathbb{C}^2\times\mathbb{C}^2 \to \mathbb{C}$, $\varphi((x_1,x_2),(y_1,y_2)) = (x_1-y_1)^2+(x_2-y_2)^2$. We say that $f\colon \mathbb{C}^2 \to \mathbb{C}^2$ preserves distance $d \geq 0$, if for each $X, Y \in \mathbb{C}^2$ $\varphi(X,Y) = d^2$ implies $\varphi(f(X), f(Y)) = d^2$. We prove that each unit-distance preserving mapping $f\colon \mathbb{C}^2 \to \mathbb{C}^2$ has a form $I\circ(\gamma,\gamma)$, where $\gamma\colon \mathbb{C} \to \mathbb{C}$ is a field homomorphism and $I\colon \mathbb{C}^2 \to \mathbb{C}^2$ is an affine mapping with orthogonal linear part. We prove an analogous result for mappings from $\boldsymbol{K}^2$ to $\boldsymbol{K}^2$, where $\boldsymbol{K}$ is a field such that $\mathrm{char}(\boldsymbol{K}) \notin \{2,3,5\}$ and -1 is a square.

The classical Beckman-Quarles theorem states that each unit-distance preserving mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$ ($n \geq 2$) is an isometry, see [1]–[5]. Let $\varphi\colon \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}$, $\varphi((x_1,x_2),(y_1,y_2)) = (x_1-y_1)^2 + (x_2-y_2)^2$. We say that $f\colon \mathbb{C}^2 \to \mathbb{C}^2$ preserves distance $d \geq 0$, if for each $X, Y \in \mathbb{C}^2$ $\varphi(X,Y) = d^2$ implies $\varphi(f(X), f(Y)) = d^2$. If $f\colon \mathbb{C}^2 \to \mathbb{C}^2$ and for each $X, Y \in \mathbb{C}^2$ $\varphi(X,Y) = \varphi(f(X), f(Y))$, then f is an affine mapping with orthogonal linear part; it follows from a general theorem proved in [3, 58 ff], see also [4, p. 30]. The author proved in [9]: each unit-distance preserving mapping $f\colon \mathbb{C}^2 \to \mathbb{C}^2$ satisfies

$$\varphi(X,Y) = \varphi(f(X), f(Y)) \tag{1}$$

for all $X, Y \in \mathbb{C}^2$ with rational $\varphi(X,Y)$.

Theorem 1. *If $f\colon \mathbb{C}^2 \to \mathbb{C}^2$ preserves unit distance, $f((0,0)) = (0,0)$, $f((1,0)) = (1,0)$ and $f((0,1)) = (0,1)$, then there exists a field homomorphism $\rho\colon \mathbb{R} \to \mathbb{C}$ satisfying $\forall x_1, x_2 \in \mathbb{C}$:*

$$\begin{aligned} f((x_1,x_2)) \in \{&(\rho(\mathrm{Re}(x_1)) + \rho(\mathrm{Im}(x_1))\cdot \boldsymbol{i},\ \rho(\mathrm{Re}(x_2)) + \rho(\mathrm{Im}(x_2))\cdot \boldsymbol{i}),\\ &(\rho(\mathrm{Re}(x_1)) - \rho(\mathrm{Im}(x_1))\cdot \boldsymbol{i},\ \rho(\mathrm{Re}(x_2)) - \rho(\mathrm{Im}(x_2))\cdot \boldsymbol{i})\}. \end{aligned} \tag{2}$$

Proof. Obviously, $g = f_{|\mathbb{R}^2}\colon \mathbb{R}^2 \to \mathbb{C}^2$ preserves unit distance. The author proved in [8] that such a g has a form $I \circ (\rho,\rho)$, where $\rho\colon \mathbb{R} \to \mathbb{C}$ is a field homomorphism and $I\colon \mathbb{C}^2 \to \mathbb{C}^2$ is an affine mapping with orthogonal linear part. Since $f((0,0)) = (0,0)$, $f((1,0)) = (1,0)$, $f((0,1)) = (0,1)$, we conclude that $f_{|\mathbb{R}^2} = (\rho,\rho)$. From this, condition (2) holds true if $(x_1,x_2) \in \mathbb{R}^2$. Assume now that $(x_1,x_2) \in \mathbb{C}^2 \setminus \mathbb{R}^2$.

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Let $x_1 = a_1 + b_1 \cdot \boldsymbol{i}$, $x_2 = a_2 + b_2 \cdot \boldsymbol{i}$, where $a_1, b_1, a_2, b_2 \in \mathbb{R}$, and, for example $b_1 \neq 0$. For each $t \in \mathbb{R}$

$$\varphi((a_1 + b_1 \cdot \boldsymbol{i},\ a_2 + b_2 \cdot \boldsymbol{i}),\ (a_1 + tb_2,\ a_2 - tb_1)) = (t^2 - 1)(b_1^2 + b_2^2).$$

By this and (1): for each $t \in \mathbb{R}$ with rational $(t^2 - 1)(b_1^2 + b_2^2)$ we have:

$$\varphi(f((a_1 + b_1 \cdot \boldsymbol{i},\ a_2 + b_2 \cdot \boldsymbol{i})),\ f((a_1 + tb_2,\ a_2 - tb_1))) = (t^2 - 1)(b_1^2 + b_2^2). \tag{3}$$

Let $f((a_1 + b_1 \cdot \boldsymbol{i},\ a_2 + b_2 \cdot \boldsymbol{i})) = (y_1, y_2)$. From (3) and $f_{|\mathbb{R}^2} = (\rho, \rho)$ we obtain: for each $t \in \mathbb{R}$ with rational $(t^2 - 1)(b_1^2 + b_2^2)$ we have:

$$(y_1 - \rho(a_1) - \rho(t)\rho(b_2))^2 + (y_2 - \rho(a_2) + \rho(t)\rho(b_1))^2 = (t^2 - 1)(b_1^2 + b_2^2). \tag{4}$$

For each $t \in \mathbb{R}$ with rational $(t^2 - 1)(b_1^2 + b_2^2)$ we have:

$$(t^2 - 1)(b_1^2 + b_2^2) = \rho((t^2 - 1)(b_1^2 + b_2^2)) = (\rho(t)^2 - 1)(\rho(b_1)^2 + \rho(b_2)^2).$$

By this and (4): for each $t \in \mathbb{R}$ with rational $(t^2 - 1)(b_1^2 + b_2^2)$ we have:

$$\begin{gathered}(y_1 - \rho(a_1))^2 + (y_2 - \rho(a_2))^2 + \rho(b_1)^2 + \rho(b_2)^2 + 2\rho(t) \\ \times(\rho(b_1)(y_2 - \rho(a_2)) - \rho(b_2)(y_1 - \rho(a_1))) = 0.\end{gathered} \tag{5}$$

There are infinitely many $t \in \mathbb{R}$ with rational $(t^2 - 1)(b_1^2 + b_2^2)$ and ρ is injective. From these two facts and (5), we obtain:

$$\rho(b_1)(y_2 - \rho(a_2)) - \rho(b_2)(y_1 - \rho(a_1)) = 0 \tag{6}$$

and

$$(y_1 - \rho(a_1))^2 + (y_2 - \rho(a_2))^2 + \rho(b_1)^2 + \rho(b_2)^2 = 0. \tag{7}$$

By (6):

$$y_2 - \rho(a_2) = \frac{\rho(b_2)}{\rho(b_1)} \cdot (y_1 - \rho(a_1)). \tag{8}$$

Applying (8) to (7) we get:

$$(y_1 - \rho(a_1))^2 + \frac{\rho(b_2)^2}{\rho(b_1)^2} \cdot (y_1 - \rho(a_1))^2 + \rho(b_1)^2 + \rho(b_2)^2 = 0.$$

It gives $\left(\dfrac{(y_1 - \rho(a_1))^2}{\rho(b_1)^2} + 1\right) \cdot (\rho(b_1)^2 + \rho(b_2)^2) = 0$. Since $\rho(b_1)^2 + \rho(b_2)^2 \neq 0$, we get

$$\underbrace{y_1 = \rho(a_1) + \rho(b_1) \cdot \boldsymbol{i}}_{\text{case 1}} \text{ or } \underbrace{y_1 = \rho(a_1) - \rho(b_1) \cdot \boldsymbol{i}}_{\text{case 2}}.$$

In case 1, by (8)

$$\begin{aligned} y_2 &= \rho(a_2) + \frac{\rho(b_2)}{\rho(b_1)} \cdot (y_1 - \rho(a_1)) \\ &= \rho(a_2) + \frac{\rho(b_2)}{\rho(b_1)} \cdot (\rho(a_1) + \rho(b_1) \cdot \boldsymbol{i} - \rho(a_1)) \\ &= \rho(a_2) + \rho(b_2) \cdot \boldsymbol{i}. \end{aligned}$$

In case 2, by (8)

$$\begin{aligned} y_2 &= \rho(a_2) + \frac{\rho(b_2)}{\rho(b_1)} \cdot (y_1 - \rho(a_1)) \\ &= \rho(a_2) + \frac{\rho(b_2)}{\rho(b_1)} \cdot (\rho(a_1) - \rho(b_1)\cdot \boldsymbol{i} - \rho(a_1)) \\ &= \rho(a_2) - \rho(b_2)\cdot \boldsymbol{i}. \end{aligned}$$

The proof is completed. □

Let $f\colon \mathbb{C}^2 \to \mathbb{C}^2$ preserves unit distance, $f((0,0)) = (0,0)$, $f((1,0)) = (1,0)$ and $f((0,1)) = (0,1)$. Theorem 1 provides a field homomorphism $\rho\colon \mathbb{R} \to \mathbb{C}$ satisfying (2). By Theorem 1 the sets

$$\begin{aligned} \boldsymbol{A} = \{(x_1,x_2) \in \mathbb{C}^2 : f((x_1,x_2)) = \\ (\rho(\mathrm{Re}(x_1)) + \rho(\mathrm{Im}(x_1))\cdot \boldsymbol{i},\ \rho(\mathrm{Re}(x_2)) + \rho(\mathrm{Im}(x_2))\cdot \boldsymbol{i})\} \end{aligned}$$

and

$$\begin{aligned} \boldsymbol{B} = \{(x_1,x_2) \in \mathbb{C}^2 : f((x_1,x_2)) = \\ (\rho(\mathrm{Re}(x_1)) - \rho(\mathrm{Im}(x_1))\cdot \boldsymbol{i},\ \rho(\mathrm{Re}(x_2)) - \rho(\mathrm{Im}(x_2))\cdot \boldsymbol{i})\} \end{aligned}$$

satisfy $\boldsymbol{A} \cup \boldsymbol{B} = \mathbb{C}^2$. The mapping

$$\mathbb{C} \ni x \xrightarrow{\theta} \rho(\mathrm{Re}(x)) + \rho(\mathrm{Im}(x))\cdot \boldsymbol{i} \in \mathbb{C}$$

is a field homomorphism, θ extends ρ,

$$\boldsymbol{A} = \{(x_1,x_2) \in \mathbb{C}^2 : f((x_1,x_2)) = (\theta(x_1), \theta(x_2))\}.$$

The mapping

$$\mathbb{C} \ni x \xrightarrow{\zeta} \rho(\mathrm{Re}(x)) - \rho(\mathrm{Im}(x))\cdot \boldsymbol{i} \in \mathbb{C}$$

is a field homomorphism, ζ extends ρ,

$$\boldsymbol{B} = \{(x_1,x_2) \in \mathbb{C}^2 : f((x_1,x_2)) = (\zeta(x_1), \zeta(x_2))\}.$$

We would like to prove $f = (\theta,\theta)$ or $f = (\zeta,\zeta)$; we will prove it later in Theorem 2.

Let $\psi\colon \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{R}$, $\psi((x_1,x_2),(y_1,y_2)) = \mathrm{Im}(x_1)\cdot\mathrm{Im}(y_1) + \mathrm{Im}(x_2)\cdot\mathrm{Im}(y_2)$.

Lemma 1. *If $x_1,x_2,y_1,y_2 \in \mathbb{C}$, $\varphi((x_1,x_2),(y_1,y_2)) \in \mathbb{Q}$ and $\psi((x_1,x_2),(y_1,y_2)) \neq 0$, then*

$$(y_1,y_2) \in \boldsymbol{A} \text{ implies } (x_1,x_2) \in \boldsymbol{A} \tag{9}$$

and

$$(y_1,y_2) \in \boldsymbol{B} \text{ implies } (x_1,x_2) \in \boldsymbol{B}. \tag{10}$$

Proof. We prove only (9), the proof of (10) follows analogically.

Let $\varphi((x_1,x_2),(y_1,y_2)) = r \in \mathbb{Q}$. Assume, on the contrary, that $(y_1,y_2) \in \boldsymbol{A}$ and $(x_1,x_2) \notin \boldsymbol{A}$. Since $\boldsymbol{A}\cup\boldsymbol{B} = \mathbb{C}^2$, $(x_1,x_2) \in \boldsymbol{B}$. Let $x_1 = a_1 + b_1\cdot\boldsymbol{i}$, $x_2 = a_2 + b_2\cdot\boldsymbol{i}$, $y_1 = \widetilde{a_1} + \widetilde{b_1}\cdot\boldsymbol{i}$, $y_2 = \widetilde{a_2} + \widetilde{b_2}\cdot\boldsymbol{i}$, where $a_1,b_1,a_2,b_2,\widetilde{a_1},\widetilde{b_1},\widetilde{a_2},\widetilde{b_2} \in \mathbb{R}$. By (1):

$$\begin{aligned} r &= \varphi((x_1,x_2),(y_1,y_2)) = \varphi(f((x_1,x_2)), f((y_1,y_2))) \\ &= (\rho(a_1) - \rho(b_1)\cdot\boldsymbol{i} - \rho(\widetilde{a_1}) - \rho(\widetilde{b_1})\cdot\boldsymbol{i})^2 \\ &\quad + (\rho(a_2) - \rho(b_2)\cdot\boldsymbol{i} - \rho(\widetilde{a_2}) - \rho(\widetilde{b_2})\cdot\boldsymbol{i})^2. \end{aligned} \tag{11}$$

Since $r \in \mathbb{Q}$,

$$
\begin{aligned}
r = \theta(r) &= \theta((a_1 + b_1 \cdot \boldsymbol{i} - \widetilde{a_1} - \widetilde{b_1} \cdot \boldsymbol{i})^2 + (a_2 + b_2 \cdot \boldsymbol{i} - \widetilde{a_2} - \widetilde{b_2} \cdot \boldsymbol{i})^2) \\
&= (\rho(a_1) + \rho(b_1) \cdot \boldsymbol{i} - \rho(\widetilde{a_1}) - \rho(\widetilde{b_1}) \cdot \boldsymbol{i})^2 \\
&\quad + (\rho(a_2) + \rho(b_2) \cdot \boldsymbol{i} - \rho(\widetilde{a_2}) - \rho(\widetilde{b_2}) \cdot \boldsymbol{i})^2.
\end{aligned} \tag{12}
$$

Subtracting (11) and (12) by sides we obtain:

$$2\rho(b_1) \cdot \boldsymbol{i} \cdot (2\rho(\widetilde{b_1}) \cdot \boldsymbol{i} - 2\rho(a_1) + 2\rho(\widetilde{a_1})) + 2\rho(b_2) \cdot \boldsymbol{i} \cdot (2\rho(\widetilde{b_2}) \cdot \boldsymbol{i} - 2\rho(a_2) + 2\rho(\widetilde{a_2})) = 0.$$

Thus

$$-\rho(b_1\widetilde{b_1} + b_2\widetilde{b_2}) = \rho(b_1(a_1 - \widetilde{a_1}) + b_2(a_2 - \widetilde{a_2})) \cdot \boldsymbol{i}. \tag{13}$$

Squaring both sides of (13) we get:

$$\rho((b_1\widetilde{b_1} + b_2\widetilde{b_2})^2 + (b_1(a_1 - \widetilde{a_1}) + b_2(a_2 - \widetilde{a_2}))^2) = 0,$$

so in particular $\psi((x_1, x_2), (y_1, y_2)) = b_1\widetilde{b_1} + b_2\widetilde{b_2} = 0$, a contradiction. $\square$

The next lemma is obvious.

Lemma 2. *For each $S, T \in \mathbb{R}^2$ there exist $n \in \{1, 2, 3, \ldots\}$ and $P_1, \ldots, P_n \in \mathbb{R}^2$ such that $||S - P_1|| = ||P_1 - P_2|| = \ldots = ||P_{n-1} - P_n|| = ||P_n - T|| = 1$.*

Lemma 3. *For each $X \in \mathbb{C}^2 \setminus \mathbb{R}^2$*

$$(\boldsymbol{i}, \boldsymbol{i}) \in \boldsymbol{A} \textit{ implies } X \in \boldsymbol{A}$$

and

$$(\boldsymbol{i}, \boldsymbol{i}) \in \boldsymbol{B} \textit{ implies } X \in \boldsymbol{B}.$$

Proof. Let $X = (a_1 + b_1 \cdot \boldsymbol{i},\ a_2 + b_2 \cdot \boldsymbol{i})$, where $a_1, b_1, a_2, b_2 \in \mathbb{R}$. Since $X \in \mathbb{C}^2 \setminus \mathbb{R}^2$, $b_1 \neq 0$ or $b_2 \neq 0$. Assume that $b_1 \neq 0$, when $b_2 \neq 0$ the proof is analogous. The points $S = \left(a_1 + \sqrt{1 + b_2^2},\ a_2 + \sqrt{1 + (b_1 - 1)^2}\right)$ and $T = \left(\sqrt{2}, 0\right)$ belong to $\mathbb{R}^2$. Applying Lemma 2 we find $P_1, \ldots, P_n \in \mathbb{R}^2$ satisfying $||S - P_1|| = ||P_1 - P_2|| = \ldots = ||P_{n-1} - P_n|| = ||P_n - T|| = 1$. The points

$$
\begin{aligned}
X_1 &= X, \\
X_2 &= \left(a_1 + \sqrt{1 + b_2^2} + b_1 \cdot \boldsymbol{i},\ a_2\right), \\
X_3 &= S + (\boldsymbol{i}, 0) = \left(a_1 + \sqrt{1 + b_2^2} + \boldsymbol{i},\ a_2 + \sqrt{1 + (b_1 - 1)^2}\right), \\
X_4 &= P_1 + (\boldsymbol{i}, 0), \\
X_5 &= P_2 + (\boldsymbol{i}, 0), \\
&\ldots \\
X_{n+3} &= P_n + (\boldsymbol{i}, 0), \\
X_{n+4} &= T + (\boldsymbol{i}, 0) = \left(\sqrt{2} + \boldsymbol{i},\ 0\right), \\
X_{n+5} &= (\boldsymbol{i}, \boldsymbol{i})
\end{aligned}
$$

satisfy $\varphi(X_{k-1}, X_k) = 1$ for $k \in \{2, 3, \ldots, n+5\}$; $\psi(X_1, X_2) = b_1^2 \neq 0,\ \psi(X_2, X_3) = b_1 \neq 0$ and $\psi(X_{k-1}, X_k) = 1$ for $k \in \{4, 5, \ldots, n+5\}$.

By Lemma 1 for each $k \in \{2, 3, \ldots, n+5\}$

$$X_k \in \boldsymbol{A} \text{ implies } X_{k-1} \in \boldsymbol{A}$$

and

$$X_k \in \boldsymbol{B} \text{ implies } X_{k-1} \in \boldsymbol{B}.$$

Therefore, $(\boldsymbol{i}, \boldsymbol{i}) = X_{n+5} \in \boldsymbol{A}$ implies $X = X_1 \in \boldsymbol{A}$, and also, $(\boldsymbol{i}, \boldsymbol{i}) = X_{n+5} \in \boldsymbol{B}$ implies $X = X_1 \in \boldsymbol{B}$. □

Theorem 2. *If $f\colon \mathbb{C}^2 \to \mathbb{C}^2$ preserves unit distance, $f((0,0)) = (0,0)$, $f((1,0)) = (1,0)$ and $f((0,1)) = (0,1)$, then there exists a field homomorphism $\gamma\colon \mathbb{C} \to \mathbb{C}$ satisfying $f = (\gamma, \gamma)$.*

Proof. By Lemma 3

$$(\boldsymbol{i}, \boldsymbol{i}) \in \boldsymbol{A} \text{ implies } \mathbb{C}^2 \setminus \mathbb{R}^2 \subseteq \boldsymbol{A}$$

and

$$(\boldsymbol{i}, \boldsymbol{i}) \in \boldsymbol{B} \text{ implies } \mathbb{C}^2 \setminus \mathbb{R}^2 \subseteq \boldsymbol{B}.$$

Obviously, $\mathbb{R}^2 \subseteq \boldsymbol{A}$ and $\mathbb{R}^2 \subseteq \boldsymbol{B}$. Therefore,

$$\boldsymbol{A} = \mathbb{C}^2 \text{ and } f = (\theta, \theta), \text{ if } (\boldsymbol{i}, \boldsymbol{i}) \in \boldsymbol{A},$$

and also,

$$\boldsymbol{B} = \mathbb{C}^2 \text{ and } f = (\zeta, \zeta), \text{ if } (\boldsymbol{i}, \boldsymbol{i}) \in \boldsymbol{B}.$$

□

As a corollary of Theorem 2 we get:

Theorem 3. *Each unit-distance preserving mapping $f\colon \mathbb{C}^2 \to \mathbb{C}^2$ has a form $I \circ (\gamma, \gamma)$, where $\gamma\colon \mathbb{C} \to \mathbb{C}$ is a field homomorphism and $I\colon \mathbb{C}^2 \to \mathbb{C}^2$ is an affine mapping with orthogonal linear part.*

Proof. By (1):

$$\begin{aligned} 1 &= \varphi((0,0),(1,0)) = \varphi(f((0,0)), f((1,0))), \\ 1 &= \varphi((0,0),(0,1)) = \varphi(f((0,0)), f((0,1))), \\ 2 &= \varphi((1,0),(0,1)) = \varphi(f((1,0)), f((0,1))). \end{aligned}$$

By the above equalities there exists an affine mapping $J\colon \mathbb{C}^2 \to \mathbb{C}^2$ with orthogonal linear part such that

$$J(f((0,0))) = (0,0), \ J(f((1,0))) = (1,0), \ J(f((0,1))) = (0,1).$$

By Theorem 2 there exists a field homomorphism $\gamma\colon \mathbb{C} \to \mathbb{C}$ satisfying $J \circ f = (\gamma, \gamma)$, so $f = J^{-1} \circ (\gamma, \gamma)$. □

Obviously, Theorem 3 implies (1). The author proved in [10]:

(14) if $n \geq 2$ and a continuous $f\colon \mathbb{C}^n \to \mathbb{C}^n$ preserves unit distance, then f has a form $I \circ (\rho, \ldots, \rho)$, where $I\colon \mathbb{C}^n \to \mathbb{C}^n$ is an affine mapping with orthogonal linear part and $\rho\colon \mathbb{C} \to \mathbb{C}$ is the identity or the complex conjugation.

The only continuous endomorphisms of $\mathbb{C}$ are the identity and the complex conjugation, see [6, Lemma 1, p. 356]. Therefore, Theorem 3 implies (14) restricted to $n = 2$.

Let $\boldsymbol{K}$ be a field, $\operatorname{char}(\boldsymbol{K}) \notin \{2, 3, 5\}$. Let $d\colon \boldsymbol{K}^2 \times \boldsymbol{K}^2 \to \boldsymbol{K}$ denote the Lorentz-Minkowski distance defined by $d((x_1, x_2), (y_1, y_2)) = (x_1 - y_1) \cdot (x_2 - y_2)$. H. Schaeffer proved in [7, Satz 1, Satz 2, Satz 3]:

(15) if $f\colon \boldsymbol{K}^2 \to \boldsymbol{K}^2$ preserves the Lorentz-Minkowski distance 1, $f((0,0)) = (0,0)$ and $f((1,1)) = (1,1)$, then there exists a field homomorphism $\sigma\colon \boldsymbol{K} \to \boldsymbol{K}$ satisfying $\forall x_1, x_2 \in \boldsymbol{K}\ f((x_1,x_2)) = (\sigma(x_1), \sigma(x_2))$or $\forall x_1, x_2 \in \boldsymbol{K}\ f((x_1,x_2)) = (\sigma(x_2), \sigma(x_1))$.

Unfortunately, the proof of Satz 3 in [7] is complicated, the main part of this proof was constructed using computer software.

Let $\varphi_{\boldsymbol{K}}\colon \boldsymbol{K}^2 \times \boldsymbol{K}^2 \to \boldsymbol{K}$, $\varphi_{\boldsymbol{K}}((x_1,x_2),(y_1,y_2)) = (x_1-y_1)^2 + (x_2-y_2)^2$. Theorem 4 generalizes Theorem 3.

Theorem 4. *Let there exists $i \in \boldsymbol{K}$ such that $i^2+1=0$. Let $f\colon \boldsymbol{K}^2 \to \boldsymbol{K}^2$ preserves unit distance defined by $\varphi_{\boldsymbol{K}}$. We claim that f has a form $I \circ (\sigma, \sigma)$, where $\sigma\colon \boldsymbol{K} \to \boldsymbol{K}$ is a field homomorphism and $I\colon \boldsymbol{K}^2 \to \boldsymbol{K}^2$ is an affine mapping with orthogonal linear part.*

Proof. Assume that $f((0,0)) = (0,0)$. The mappings

$$\boldsymbol{K}^2 \ni (x_1,x_2) \xrightarrow{\xi} (x_1 + i\cdot x_2,\ x_1 - i\cdot x_2) \in \boldsymbol{K}^2$$

and

$$\boldsymbol{K}^2 \ni (x_1,x_2) \xrightarrow{\eta} \left(\frac{1}{2}x_1 + \frac{1}{2}x_2,\ -\frac{i}{2}x_1 + \frac{i}{2}x_2\right) \in \boldsymbol{K}^2$$

satisfy:

$$\begin{gathered}\eta \circ \xi = \xi \circ \eta = \mathrm{id}(\boldsymbol{K}^2),\\ \forall x_1,x_2,y_1,y_2 \in \boldsymbol{K}\, \varphi_{\boldsymbol{K}}((x_1,x_2),(y_1,y_2)) = d(\xi((x_1,x_2)), \xi((y_1,y_2))),\\ \forall x_1,x_2,y_1,y_2 \in \boldsymbol{K}\, d((x_1,x_2),(y_1,y_2)) = \varphi_{\boldsymbol{K}}(\eta((x_1,x_2)), \eta((y_1,y_2))).\end{gathered}$$

Therefore, $\xi \circ f \circ \eta\colon \boldsymbol{K}^2 \to \boldsymbol{K}^2$ preserves the Lorentz-Minkowski distance 1. Obviously, $(\xi \circ f \circ \eta)((0,0)) = (0,0)$. Let $(\xi \circ f \circ \eta)((1,1)) = (a,b) \in \boldsymbol{K}^2$. We have: $1 = d((1,1),(0,0)) = d((\xi \circ f \circ \eta)((1,1)), (\xi \circ f \circ \eta)((0,0))) = d((a,b),(0,0)) = a \cdot b$. Hence $b = \frac{1}{a}$. For each $z \in \boldsymbol{K} \setminus \{0\}$ the mapping

$$\boldsymbol{K}^2 \ni (x,y) \xrightarrow{\lambda(z)} (\frac{x}{z}, z \cdot y) \in \boldsymbol{K}^2$$

preserves all Lorentz-Minkowski distances, $\lambda(\frac{1}{z}) \circ \lambda(z) = \lambda(z) \circ \lambda(\frac{1}{z}) = \mathrm{id}(\boldsymbol{K}^2)$. The mapping $\lambda(a) \circ \xi \circ f \circ \eta\colon \boldsymbol{K}^2 \to \boldsymbol{K}^2$ preserves the Lorentz-Minkowski distance 1, $(\lambda(a) \circ \xi \circ f \circ \eta)((0,0)) = (0,0)$ and $(\lambda(a) \circ \xi \circ f \circ \eta)((1,1)) = (1,1)$. By (15) there exists a field homomorphism $\sigma\colon \boldsymbol{K} \to \boldsymbol{K}$ satisfying

$$\underbrace{\lambda(a) \circ \xi \circ f \circ \eta = (\sigma,\sigma)}_{\text{case 1}} \text{ or } \underbrace{\lambda(a) \circ \xi \circ f \circ \eta = h \circ (\sigma,\sigma)}_{\text{case 2}},$$

where $h\colon \boldsymbol{K}^2 \to \boldsymbol{K}^2$, $h((x_1,x_2)) = (x_2,x_1)$.

In case 1: $f = \eta \circ \lambda(\frac{1}{a}) \circ (\sigma,\sigma) \circ \xi = f_1 \circ (\sigma,\sigma)$, where $f_1\colon \boldsymbol{K}^2 \to \boldsymbol{K}^2$,

$$\begin{aligned}f_1((x_1,x_2)) = \Bigg(&\left(\frac{a}{2} + \frac{1}{2a}\right)\cdot x_1 + \left(\frac{a}{2} - \frac{1}{2a}\right)\sigma(i)\cdot x_2,\\ &-\left(\frac{a}{2} - \frac{1}{2a}\right) i \cdot x_1 - \left(\frac{a}{2} + \frac{1}{2a}\right) i\sigma(i)\cdot x_2\Bigg).\end{aligned}$$

In case 2: $f = \eta \circ \lambda(\frac{1}{a}) \circ h \circ (\sigma, \sigma) \circ \xi = f_2 \circ (\sigma, \sigma)$, where $f_2 \colon \boldsymbol{K}^2 \to \boldsymbol{K}^2$,

$$f_2((x_1, x_2)) = \left(\left(\frac{a}{2} + \frac{1}{2a}\right) \cdot x_1 - \left(\frac{a}{2} - \frac{1}{2a}\right) \sigma(i) \cdot x_2, \right.$$
$$\left. - \left(\frac{a}{2} - \frac{1}{2a}\right) i \cdot x_1 + \left(\frac{a}{2} + \frac{1}{2a}\right) i\sigma(i) \cdot x_2 \right).$$

The mappings f_1 and f_2 are linear and orthogonal. The proof is completed. □

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