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COMPACT SUBSETS OF SPACES OF HOLOMORPHIC FUNCTIONS

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Abstract. In [1] compactness criteria in function spaces are investigated for general coorbit spaces. As an application it is shown that a bounded and closed subset

$$\mathcal{A} \subset \mathcal{F} := \{f : f \text{ is entire and } \int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2}\, d\lambda(z) < \infty\}$$

is compact if and only if $\forall \epsilon > 0\ \exists$ a compact subset $C \subset \mathbb{C}^n$ with

$$\int_{C^C} |f(z)|^2 e^{-|z|^2}\, d\lambda(z) < \epsilon\ \forall f \in \mathcal{A}.$$

The methods of [1] cannot be easily adopted for the spaces

$$\mathcal{F}_m := \{f : f \text{ is entire and } \int_{\mathbb{C}} |f(z)|^2 e^{-|z|^m}\, d\lambda(z) < \infty\}$$

and the Bergman-spaces $B^2(\Omega)$. Here Ω is a bounded domain. We will be able to derive a generalization of the above mentioned result to the spaces $\mathcal{F}_m$ and $B^2(\Omega)$. Furthermore we will be able to derive a sufficient compactness conditions for subsets $\mathcal{A}$ of the Fock-space in terms of the Taylor-expansion of the functions $f \in \mathcal{A}$.

We will introduce increasing norm-spaces, that are a natural generalization of the above mentioned spaces. The main result will be proven for increasing norm-spaces and the results for the spaces $\mathcal{F}_m$ and $B^2(\Omega)$ will follow from this result.

1. Preliminaries

In [1] compactness criteria in function spaces are investigated for general coorbit spaces. As an application [1] considers the special case of the Fock-space

$$\mathcal{F} := \{f : f \text{ is entire and } \int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2}\, d\lambda(z) < \infty\}.$$

We will equip $\mathcal{F}$ with the topology induced by the inner product

$$(f, g) = \int_{\mathbb{C}} f(z)\overline{g(z)} e^{-|z|^2}\, d\lambda(z),$$

where $f, g \in \mathcal{F}$. Here λ denotes the Lebesgue-measure on $\mathbb{C}$.

The approach from [1] has the disadvantage that it makes use of unitary representations of the underlying spaces. In the case of the Fock-space the Fock-Bargmann

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representation of the Heisenberg-group is used. This unitary representation cannot be generalized for the spaces

$$\mathcal{F}_m := \{f : f \text{ is entire and } \int_{\mathbb{C}} |f(z)|^2 e^{-|z|^m}\, d\lambda(z) < \infty\}$$

for arbitrary $m \in \mathbb{N}$. We will call the spaces $\mathcal{F}_m$ generalized Fock-spaces and will again consider the topology induced by the inner product

$$(f,g) = \int_{\mathbb{C}} f(z)\overline{g(z)} e^{-|z|^m}\, d\lambda(z),$$

where $f, g \in \mathcal{F}^m$. Furthermore it would be interesting if similar results as the one in [1] are valid in the case of Bergman-spaces $B^2(\Omega)$ of bounded domains Ω.

$$B^2(\Omega) := \{f : f \text{ is holomorphic on } \Omega \text{ and } \int_{\Omega} |f(z)|^2\, d\lambda(z) < \infty\}$$

In this paper we give a proof for a similar compactness criterium in certain normed spaces which we call increasing norm spaces. These spaces are generalizations of the generalized Fock-spaces and the Bergman-spaces.

In section 2 we will give the essential definitions and we will derive a compactness criterium for the increasing norm spaces. In section 3 we will apply the result from section 2 to generalized Fock-spaces and the Bergman-spaces. Therefore we will be able to generalize the Corollary from [1] to these spaces. Furthermore we will be able to derive a sufficient compactness conditions for subsets $\mathcal{A}$ of generalized Fock-spaces in terms of the Taylor-expansion of the functions $f \in \mathcal{A}$.

2. Compactness in increasing norm spaces

Definition 1. Let $(X, \|.\|)$ be a normed space and let $\|.\|_i$ $i \in \mathbb{N}$ maps such that

$$\|.\|^i := \sqrt{\|.\|^2 - \|.\|_i^2}$$

are semi-norms on X. If for every $f \in X$

$$|\|f\|^2 - \|f\|_i^2| \to 0$$

for $i \to \infty$ and

$$\|f\|_i^2 \leq \|f\|_{i+1}^2 \qquad \forall i \in \mathbb{N}$$

we call $(X, \|.\|, \|.\|_i$ $i \in \mathbb{N})$ increasing norm-space.

Remark. It is easily seen that the spaces $\mathcal{F}_m$ are increasing norm-spaces. Let $(K_i)_{i\in\mathbb{N}}$ be a family of compact sets satisfying $K_i \subset K_{i+1}$ and $\cup_{i=1}^{\infty} K_i = \mathbb{C}$. We define

$$\|f\|_i = \sqrt{\int_{K_i} |f(z)|^2 e^{-|z|^m}\, d\lambda(z)} \qquad \forall i \in \mathbb{N}.$$

It is clear that the maps given by

$$\|f\|^i := \sqrt{\|f\|^2 - \|f\|_i^2} = \sqrt{\int_{K_i^C} |f(z)|^2 e^{-|z|^m}\, d\lambda(z)}$$

are semi-norms. Moreover $\forall f \in \mathcal{F}_m$

$$|\|f\|^2 - \|f\|_i^2| \to 0 \text{ as } i \to \infty.$$

The condition
$$\|f\|_i^2 \le \|f\|_{i+1}^2$$
$\forall i \in \mathbb{N}$ and $\forall f \in \mathcal{F}_m$ is obvious. Therefore $\mathcal{F}_m$ is an increasing norm-space.

Remark. Similarly one can show that the Bergman-spaces of bounded domains Ω are increasing norm-spaces.

Remark. Note that $\forall f \in X$ we must have $\|f\| \ge \|f\|^i \quad \forall i \in \mathbb{N}$ and $\|f\| \ge \|f\|_i \quad \forall i \in \mathbb{N}$, where $(X, \|.\|, \|.\|_i \ i \in \mathbb{N})$ is an increasing norm-space.

Now we want to establish a necessary condition for compactness in increasing norm-spaces.

Theorem 1. *Let $(X, \|.\|, \|.\|_i \ i \in \mathbb{N})$ be an increasing norm-space. Then the following condition is necessary for a subset Y of X to be compact:*
For every $\epsilon > 0$ there exists a $i \in \mathbb{N}$ such that $\|y\|^2 - \|y\|_i^2 < \epsilon$ for all $y \in Y$.

Proof. Let us assume that Y is compact and that the condition fails. This means, that there is a $\epsilon > 0$ such that there exists no i such that
$$\|y\|^2 - \|y\|_i^2 < \epsilon^2 \text{ for all } y \in Y.$$
So for every i there is a $y_i \in Y$ with
$$\|y_i\|^2 - \|y_i\|_i^2 > \epsilon^2.$$

We construct a sequence that has no subsequence that is a Cauchy-sequence. We have
$$\|y_i\|^i = \sqrt{\|y_i\|^2 - \|y_i\|_i^2} > \epsilon$$
and
$$\|y_i\|^{i+1} = \sqrt{\|y_i\|^2 - \|y_i\|_{i+1}^2} < \frac{\epsilon}{2}.$$
The last statement can be assumed without restriction of generality. So we have
$$\begin{aligned}\|y_{i+1} - y_i\| &\ge \|y_{i+1} - y_i\|^{i+1} \\ &\ge |\|y_{i+1}\|^{i+1} - \|y_i\|^{i+1}| > \frac{\epsilon}{2}\end{aligned}$$
Since
$$\|f\|_i \le \|f\|_{i+1} \qquad \forall i \in \mathbb{N} \text{ and } f \in X,$$
there is no subsequence of $(y_i)_{i\in\mathbb{N}}$ that is a Cauchy-sequence. □

We can even derive a sufficient condition in certain increasing norm-spaces.

Theorem 2. *Let $(X, \|.\|, \|.\|_i \ i \in \mathbb{N})$ be an increasing norm-space. In addition we assume that the functions $\|.\|_i$ are semi-norms as well. Furthermore let $(X, \|.\|)$ be reflexive and let each weakly convergent bounded sequence in $\|.\|$ have a convergent subsequence in $\|.\|_i \ \forall i \in \mathbb{N}$, with the understanding that the limit is always the same element in X. Then a closed and bounded subset $Y \subseteq X$ is compact if and only if $\forall \epsilon > 0 \, \exists i \in \mathbb{N}$ with $\|y\|^i < \epsilon$ for all $y \in Y$.*

Proof. It remains to be shown that the condition is sufficient.

Let $\{x_n\}$ be a sequence in Y. We have to show that this sequence has a convergent subsequence $\{x_{n_k}\}$. It follows from the fact that X is reflexive that there is a subsequence $\{x_{n_k}\}$ that converges weekly to a $x \in X$. According to our assumption we have $\|x_{n_k} - x\|_i \to 0$ for all $i \in \mathbb{N}$ as $k \to \infty$ (without loss of generality).

We have

$$\begin{aligned}\|x_{n_k}-x\|^2 &= (\|x_{n_k}-x\|^i)^2+(\|x_{n_k}-x\|_i)^2\\ &\le \|x_{n_k}-x\|_i^2+(\|x_{n_k}\|^i)^2+(\|x\|^i)^2+2\|x_{n_k}\|^i\|x\|^i.\end{aligned}$$

We know that

$$\|x_{n_k}\|^i \to 0$$

uniformly as $i \to \infty$. So we can chose i so large, that

$$(\|x_{n_k}\|^i)^2+(\|x\|^i)^2+2\|x_{n_k}\|^i\|x\|^i \le \frac{\epsilon}{2} \qquad \forall k.$$

For this i we can choose K − without restriction of generality − such that

$$\|x_{n_k}-x\|_i^2 \le \frac{\epsilon}{2} \quad \forall k \ge K.$$

So finally

$$\|x_{n_k}-x\| \le \sqrt{\epsilon} \quad \forall k \ge K.$$

□

Remark. We will give a concrete example that shows that the condition $\forall\, \epsilon > 0\, \exists i \in \mathbb{N}$ with $\|y\|^i < \epsilon$ for all $y \in Y$ is important. We will refer to this condition as tightness.

Consider a family $(K_i)_{i\in\mathbb{N}}$ of disjoint compact sets in $\mathbb{R}^n$ such that $\forall R \in \mathbb{R}$ only finitely many of the K_i's have a nonempty intersection with $K_R(0) = \{x \in \mathbb{R}^n \text{ with } \|x\| \le R\}$. Let $\phi_i : \mathbb{R}^n \to \mathbb{R}$ have support K_i and $\int_{\mathbb{R}^n} |\phi_i(x)|^2\, dx = K$ $\forall i \in \mathbb{N}$. It follows that the ϕ_i are pairwise orthogonal with respect to the inner product

$$(f,g) = \int_{\mathbb{R}^n} f(x)g(x)\, dx.$$

We denote the corresponding norm by $\|.\|$. Let $X = << \{\phi_i, i \in \mathbb{N}\} >>$, that is the closure of the linear span of the ϕ_i. It is clear that X is reflexive. Let

$$\|f\|_i = \sqrt{\int_{K_{R_i}(0)} |f(x)|^2\, dx}$$

where $R_i \to \infty$ as $i \to \infty$. It is clear that every bounded sequence with respect to $\|.\|$ has a convergent subsequence with respect to $\|.\|_i$. However, the conclusion of the above theorem clearly does not hold for the set $\{\phi_i, i \in \mathbb{N}\}$.

3. Applications to spaces of holomorphic functions

In this section we will show that the Fock-space, the generalized Fock-spaces and and Bergman-spaces with arbitrary bounded domains are special increasing norm spaces fulfilling the properties needed in Theorem 2. The following Theorem only considers the Fock-space but actually the argument also applies to generalized Fock-spaces and and Bergman-spaces with arbitrary bounded domains as mentioned in the following remark.

Theorem 3. *Let $\mathcal{A} \subset \mathcal{F}$. Then the following is true: $\mathcal{A}$ is compact if and only if $\mathcal{A}$ is closed, bounded and for all $\epsilon > 0$ there exists a compact subset $C \subset \mathbb{C}$ with*

$$\int_{C^C} |f(z)|^2 e^{-|z|^2}\, d\lambda(z) < \epsilon \qquad \forall f \in \mathcal{A}.$$

Proof. Remember
$$\|f\|^2 = \int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2}\, d\lambda(z).$$
Let $\{C_i : i \in \mathbb{N}\}$ be a set of compact subsets of $\mathbb{C}$ satisfying $C_i \subseteq C_{i+1}\, \forall i \in \mathbb{N}$ and $C_i \uparrow \mathbb{C}$. Choosing
$$\|f\|_i^2 = \int_{C_i} |f(z)|^2 e^{-|z|^2}\, d\lambda(z)$$
we have seen above that $\mathcal{F}$ is an increasing norm space. Therefore it is enough to show that for every bounded sequence x_k in $\mathcal{F}$ there exists an element $x \in \mathcal{F}$ and a subsequence x_{k_l} that satisfies
$$\|x - x_{k_l}\|_i^2 = \int_{C_i} |x(z) - x_{k_l}(z)|^2 e^{-|z|^2}\, d\lambda(z) \to 0$$
for $l \to \infty$ and $\forall i \in \mathbb{N}$. To show this we first note that every norm-bounded set in $\mathcal{F}$ is uniformly bounded on compact subsets of $\mathbb{C}$. This follows from a routine calculation. We have - basically using Cauchy's theorem - for every compact $K \subset \mathbb{C}$
$$|f(z)| \leq C_K \|f\|.$$

Therefore using the fact that the sequence x_k is norm-bounded it follows easily that the sequence is uniformly bounded on every compact $K \subset \mathbb{C}$. Similarly one can show that the derivatives of the sequence x_k are uniformly bounded as well. Therefore it is clear that for every $i \in \mathbb{N}$ there exists a function $\tilde{x_i} : K_i \to \mathbb{C}$ and a subsequence x_{k_l} that does not depend on i such that
$$\|\tilde{x_i} - x_{k_l}|_{K_i}\|_i^2 = \int_{C_i} |x(z) - x_{k_l}(z)|^2 e^{-|z|^2}\, d\lambda(z) \to 0$$
for $l \to \infty$ and $\forall i \in \mathbb{N}$. Here $x_{k_l}|_{K_i}$ denotes the restriction of x_{k_l} to K_i. It is easily checked that there exists a function $x : \mathbb{C} \to \mathbb{C}$ such that $x|_{K_i} = \tilde{x_i}$ and that x is furthermore entire. This finishes the proof. □

Remark. The proof of Theorem 3 can easily seen to be valid for generalized Fock-spaces and and Bergman-spaces with arbitrary bounded domains. Therefore similar results are valid for these spaces.

Remark. Let $f_n \in \mathcal{F}$. Consider the measures ν_n on $\mathbb{C}$ given by
$$\nu_n(C) = \int_C |f_n(z)|^2 e^{-|z|^2}\, d\lambda(z).$$
It was shown by Prohorov [4] that a family of probability measures is sequentially precompact - for definitions see [4] - if and only if the set is tight. For the special case of measures of the above form the tightness definition from [4] is exactly the one as the one considered here. So the natural question arises if there is a connection between the theorem of Prohorov and Theorem 3. If we only consider f_n that have a norm of 1 then it follows that the ν_n are probability measures. Prohorov's theorem would imply that the sequence ν_n converges weakly against a probability measure γ. However, it is not completely obvious that γ is of the form
$$\gamma(C) = \int_C |f(z)|^2 e^{-|z|^2}\, d\lambda(z)$$
for some $f \in \mathcal{F}$. Especially it is not even obvious that γ is absolutely continuous. Therefore it may be difficult to derive Theorem 3 by applying Prohorov's Theorem.

We can formulate sufficient conditions for compactness in terms of of the Taylor-expansion in the case of the Fock-space.

Theorem 4. *Let $\mathcal{A} \subset \mathcal{F}$. If $\mathcal{A}$ is closed, bounded and there exists a l^2-sequence c_n such that $\forall \epsilon > 0$ there exists $N(\epsilon)$ such that $|\alpha_n|^2 n! \leq c_n^2 \epsilon$ for all $n > N(\epsilon)$.*

Here the sequence $\{\alpha_n\}$ belongs to the Taylor-expansion of $f \in \mathcal{A}$ with

$$f(z) = \sum_n \alpha_n z^n.$$

Proof. We just have to show that for every $\epsilon > 0$ there exists a compact set $C \subseteq \mathbb{C}$ such that $\int_{C^C} |f(z)|^2 e^{-|z|^2} \, d\lambda(z) < \epsilon \; \forall f \in \mathcal{A}$. Let $K_n(0) = \{z \in \mathbb{C} : |z| \leq n\}$. We have

$$\begin{aligned}
&\int_{K_n(0)^C} |f(z)|^2 e^{-|z|^2} \, d\lambda(z) \\
&\leq \lim_{m\to\infty} \sum_{k=0}^{j} \int_0^{2\pi} \int_n^m |\alpha_k|^2 r^{2k+1} e^{-r^2} \, d\lambda(r) \, d\lambda(\phi) \\
&+ \sum_{k=j+1}^{\infty} \int_0^{2\pi} \int_0^{\infty} |\alpha_k|^2 r^{2k+1} e^{-r^2} \, d\lambda(r) \, d\lambda(\phi) \\
&= \sum_{k=0}^{j} \int_0^{2\pi} \int_n^{\infty} |\alpha_k|^2 r^{2k+1} e^{-r^2} \, d\lambda(r) \, d\lambda(\phi) + \lim_{m\to\infty} 2\pi \sum_{k=j+1}^{\infty} \phi_{k,m} k! |\alpha_k|^2
\end{aligned}$$

Here $\phi_{k,m} := \frac{\int_0^m |\alpha_k|^2 r^{2k+1} e^{-r^2} \, d\lambda(r)}{k!}$ and $\lim_{m\to\infty} 2\phi_{k,m} = 1$ for all $k \in \mathbb{N}$.

Since $\lim_{m\to\infty} \sum_{k=j+1}^{\infty} \phi_{k,m} k! |\alpha_k|^2 = \sum_{k=j+1}^{\infty} k! |\alpha_k|^2$ we have

$$\int_{K_n(0)^C} |f(z)|^2 e^{-|z|^2} \, d\lambda(z) \leq \sum_{k=0}^{j} \int_0^{2\pi} \int_n^{\infty} |\alpha_k|^2 r^{2k+1} e^{-r^2} \, d\lambda(r) \, d\lambda(\phi) + \epsilon C'.$$

Now the Theorem follows easily. □

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