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A POINTWISE APPROXIMATION OF ISOLATED TREES IN A RANDOM GRAPH

K. NEAMMANEE

Abstract. In this paper, we give a pointwise approximation of the number of isolated trees of order k in a random graph by Poisson distribution. The technique we used here is the Stein's method.

1. Introduction

A random graph is a collection of points, or vertices, with lines, or edges, connecting pairs of them at random. The study of random graphs has a long history. Starting with the influential work of Erdős and Rényi in the 1950s and 1960s [7-9], random graph theory has developed into one of the mainstays of modern discrete mathematics, and has produced a prodigious number of results, many of them highly ingenious, describing statistical properties of graphs, such as distribution of component sizes, existence and size of a giant component, and typical vertex-vertex distances.

Random graphs are not merely a mathematical toy; they have been employed extensively as models of real world networks of various types, particularly in epidemiology. The passage of a disease through a community depends strongly on the pattern of contacts between those infected with the disease and those susceptible to it. This pattern can be depicted as a network, with individuals represented by vertices and contacts capable of transmitting the disease by edges. A large class of epidemiological models known as susceptible/infectious/recovered (or SIR) model [4,17,19] makes frequent use of the so-called fully mixed approximation, which is the assumption that contacts are random and uncorrelated, i.e., that they form a random graph.

Random graphs however turn out to have severe shortcomings as models of such real world phenomena. Although it is difficult to determine experimentally the structure of the network of contacts by which a disease is spread [14], studies have been performed of other social networks such as networks of friendships within a variety of communities [5,10,13], networks of telephone calls [1,2], airline timetables [3], the power grid [22], the structure and conformation space of polymers [16], metabolic pathways [11,15], and food webs [23]. There are many situations in which

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the theory tells us that the distribution of a random variable may be approximated by Poisson distribution. In the random graph theory, one application of the approximation by Poisson distribution arises naturally when counting the number of occurrences of individually rare and unrelated events within a large ensemble. In this paper, we choose to count the number of isolated trees of order k in a random graph with n vertices and give a non-uniform bound of the Poisson approximation to this number.

Let $G(n,p)$ be a random graph with n vertices $1,2,\dots,n$, in which each possible edge $\{i,j\}$ is present independently with probability p. A tree is, by definition, a connected graph containing no cycles and a tree in $G(n,p)$ is isolated if there is no edge in $G(n,p)$ with one vertex in the tree and the other outside of the tree. Let

$$D_{n,k} = \{(i_1, i_2, \cdots, i_k) | 1 \le i_1 < i_2 < \ldots < i_k \le n\}$$

be the set of all possible combinations of k vertices. For each $i \in D_{n,k}$, we define

$$X_i = \begin{cases} 1 & \text{if there is an isolated tree in } G(n,p) \text{ that spans the vertices} \\ & i = (i_1, i_2 \ldots, i_k), \\ 0 & \text{otherwise,} \end{cases}$$

and set

$$W_{n,k} = \sum_{i \in D_{n,k}} X_i.$$

Clearly $W_{n,k}$ is the number of isolated trees of order k in $G(n,p)$ and the X_i's are not independent unless $k = 1$. For the small value of probability p, that is when $k^2 p \to 0$ and $\frac{k^2}{n} \to 0$, Stein ([21], chapter 13) proved that the distribution of $W_{n,k}$ can be approximated by Poisson distribution with parameter

$$\lambda = EW_{n,k} = \binom{n}{k} P(X_i = 1) = k^{k-2} p^{k-1} (1-p)^{k(n-k)+\binom{k}{2}-k+1}$$

and the uniform error bound is given by

$$(1.1) \qquad |P(W_{n,k} \in A) - P(Poi_\lambda \in A)| \le \frac{B}{\sqrt{k}} (1+c_n) e^{1-c_n} (c_n e^{1-c_n})^{k-1}$$

for all $A \subseteq \mathbb{N} \cup \{0\}, n \in \mathbb{N}$, and $k \le n$, where Poi_λ is a Poisson random variable with parameter λ, B is a constant independent of A, and

$$c_n = -n \log(1-p).$$

It is evident from (1.1) that the error bound tend to zero as c_n decreases to zero provided $k \ge 2$. Observe that the bound in (1.1) is a uniform bound that works for any possible number of trees in the graph. In this paper, we shall introduce a non-uniform bound of the Poisson approximation, i.e. a better error bound once the number of trees is specified.

Throughout the paper, let us fix the number of trees $w_0 \in \{1, 2, \ldots, \binom{n}{k}\}$ and denote for convenience

$$\Delta(n,k,w_0) \equiv \left| P(W_{n,k} = w_0) - \frac{e^{-\lambda} \lambda^{w_0}}{w_0!} \right|$$

where $\lambda = EW_{n,k}$. The following theorems are our main results.

Theorem 1.1. *Suppose $2k < n$ and $w_0 \neq 0$. Then*

1. $\Delta(n,k,w_0) \le \lambda \min\Big\{\frac{1}{w_0}, \frac{1}{\lambda}\Big\} \min\Big\{2, \frac{\lambda k^2}{n}\Big(1 + c_n e^{\frac{k^2}{n}(c_n-1)}\Big)\Big\}$, *and*
2. $\Delta(n,k,0) \le \min\Big\{1,\lambda\Big\} \min\Big\{2,\lambda\Big\}$.

In order to gain a better understanding of these results, some asymptotic behaviors of $\Delta(n,k,w_0)$ and λ as n goes to infinity are summarized in the following two theorems.

Theorem 1.2. *Let $\delta > 1$, $k \le O(n^{\frac{\delta}{2}})$, and $p = O\Big(\frac{1}{n^\delta}\Big)$. Then, for $\delta^* \in (1,\delta)$, we have*

1. $\lambda \le \frac{1}{k^{\frac{5}{2}}} O\Big(\frac{1}{n^{(\delta^*-1)(k-1)-1}}\Big)$,
2. $\Delta(n,k,w_0) \le \frac{1}{w_0 k^3} O\Big(\frac{1}{n^{2(\delta^*-1)(k-1)-1}}\Big)$, *and*
3. $\Delta(n,k,0) \le \frac{1}{k^3} O\Big(\frac{1}{n^{2(\delta^*-1)(k-1)-1}}\Big)$,

where $\lim_{n\to\infty} \frac{O(g(n))}{g(n)} = c$ *for some* $c > 0$.

Theorem 1.3. *Let $\beta \in \Big(0, \frac{1}{2}\Big)$, $k \le O(n^\beta)$, and $p = O(\frac{1}{n^\delta})$ for some $\delta > 0$. Then*

1. *for $w_0 > 2$ and $\delta > 1$, $\frac{\Delta(n,k,w_0)}{\lambda^{w_0}} \to 0$ as $n \to \infty$ and*
2. *for $\delta > 2$, $w_0!\Delta(n,k,w_0) \to 0$ as $w_0 \to \infty$.*

Some remarks are in order.

1. When the probability p is small compared to a positive power of n, i.e., $p = \mathrm{O}\Big(\frac{1}{n^\delta}\Big)$ for $\delta > 1$, both the error bound and λ tend to zero as n approaches infinity.
2. If $p = \mathrm{O}\Big(\frac{1}{n^\delta}\Big)$ for some $\delta > 1$, then we are dealing with a Poisson distribution with parameter λ smaller than $\mathrm{O}\Big(\frac{1}{n^{(\delta^*-1)(k-1)-1}}\Big)$ for all $\delta^* \in (1,\delta)$. Theorem 1.3(1) says that as n increases without limit, the error bound $\Delta(n,k,w_0)$ tend to zero faster than the Poisson probability $\frac{e^{-\lambda}\lambda^{w_0}}{w_0!}$.
3. Theorem 1.3(2) confirms that the Poisson probability, $\frac{e^{-\lambda}\lambda^{w_0}}{w_0!}$, tends to zero slower than the error bound $\Delta(w_0,k,w_0)$ as w_0 goes to infinity.

Throughout the paper, C stands for an absolute constant with possibly different values at different places.

2. Proof of the Main Results

The main result in Theorem 1.1 will be proved by Stein's method for Poisson distribution. Stein [20] introduced a new technique of computing a bound in normal approximation by using differential equation and Chen [6] applied Stein's idea to the Poisson case. The Stein's equation for Poisson distribution with parameter λ is given by

$$\lambda f(w+1) - wf(w) = h(w) - \mathcal{P}_\lambda(h) \tag{2.1}$$

where f and h are real-valued functions defined on $\mathbb{N}\cup\{0\}$ and $\mathcal{P}_\lambda(h) = E[h(Poi_\lambda)]$.

For each subset A of $\mathbb{N}\cup\{0\}$, define $h_A\colon \mathbb{N}\cup\{0\}\to\mathbb{R}$ by

$$h_A(w)=\begin{cases}1 & \text{if } w\in A,\\ 0 & \text{if } w\notin A.\end{cases}$$

For convenience, we shall write h_ω for $h_{\{\omega\}}$ and denote $C_\omega=\{0,1,2,\ldots,\omega\}$. For each $\omega_0\in\mathbb{N}\cup\{0\}$, it is well known [21, p.87] that the solution $U_\lambda h_{\omega_0}$ of (2.1) is of the form

$$U_\lambda h_{w_0}(w)=\begin{cases}\dfrac{(w-1)!}{w_0!}\lambda^{w_0-w}\mathcal{P}_\lambda(1-h_{C_{w-1}}) & \text{if } w_0<w,\\ -\dfrac{(w-1)!}{w_0!}\lambda^{w_0-w}\mathcal{P}_\lambda(h_{C_{w-1}}) & \text{if } 0<w\le w_0,\\ 0 & \text{if } w=0.\end{cases} \tag{2.2}$$

Some properties of $U_\lambda h_{w_0}$ needed to prove Theorem 1.1.

Lemma 2.1. *Let $w_0\in\mathbb{N}$ and $U_\lambda h_{w_0}$ be the solution of (2.1) with $h=h_{w_0}$. Then*

1. $|U_\lambda h_{w_0}|\le\min\left\{\dfrac{1}{w_0},\dfrac{1}{\lambda}\right\}$ *and*
2. $|V_\lambda h_{w_0}|\le\min\left\{\dfrac{1}{w_0},\dfrac{1}{\lambda}\right\}$

where $V_\lambda h_{w_0}(w)=U_\lambda h_{w_0}(w+1)-U_\lambda h_{w_0}(w)$.

Proof. To prove 1., we shall first derive that $|U_\lambda h_{w_0}(w)|\le\dfrac{1}{w_0}$ by splitting w into two cases according to the definition of $U_\lambda h_{w_0}(w)$.

When $w>w_0$, it follows straightforwardly that

$$\begin{aligned}0<U_\lambda h_{w_0}(w)&=\frac{(w-1)!}{w_0!}\lambda^{w_0-w}e^{-\lambda}\sum_{k=w}^{\infty}\frac{\lambda^k}{k!}\\
&=\frac{(w-1)!}{w_0!}e^{-\lambda}\Big(\frac{\lambda^{w_0}}{w!}+\frac{\lambda^{w_0+1}}{(w+1)!}+\frac{\lambda^{w_0+2}}{(w+2)!}+\cdots\Big)\\
&=\frac{(w-1)!}{w_0!}\frac{e^{-\lambda}}{w!}\Big(\lambda^{w_0}+\frac{\lambda^{w_0+1}}{(w+1)}+\frac{\lambda^{w_0+2}}{(w+1)(w+2)}+\cdots\Big)\\
&=\frac{1}{w}e^{-\lambda}\Big(\frac{\lambda^{w_0}}{w_0!}+\frac{\lambda^{w_0+1}}{w_0!(w+1)}+\frac{\lambda^{w_0+2}}{w_0!(w+1)(w+2)}+\cdots\Big)\\
&\le\frac{1}{w_0}e^{-\lambda}\Big(\frac{\lambda^{w_0}}{w_0!}+\frac{\lambda^{w_0+1}}{(w_0+1)!}+\frac{\lambda^{w_0+2}}{(w_0+2)!}+\cdots\Big)\\
&\le\frac{1}{w_0}.\end{aligned}$$

For $w\le w_0$, the bound of $U_\lambda h_{w_0}(w)$ is obtained as follows:

$$\begin{aligned}0&<\frac{(w-1)!}{w_0!}\lambda^{w_0-w}\mathcal{P}_\lambda(h_{C_{w-1}})\\
&=\frac{(w-1)!}{w_0!}e^{-\lambda}\left(\frac{\lambda^{w_0-w}}{0!}+\frac{\lambda^{w_0-w+1}}{1!}+\cdots+\frac{\lambda^{w_0-1}}{(w-1)!}\right)\\
&=\frac{(w-1)!}{w_0!}e^{-\lambda}\Bigg\{\frac{[(w_0-1)-w+1]!\lambda^{(w_0-1)-w+1}}{[(w_0-1)-w+1]!}\end{aligned}$$

$$
\begin{aligned}
&+\frac{2\cdot 3\cdots[(w_0-1)-w+2]!\lambda^{(w_0-1)-w+2}}{[(w_0-1)-w+2]!}\\
&+\cdots+\frac{w(w+1)\cdots(w_0-1)\lambda^{(w_0-1)}}{(w_0-1)!}\Bigg\}\\
&\le \frac{(w-1)!}{w_0!}e^{-\lambda}\frac{(w_0-1)!}{(w-1)!}\Bigg\{\frac{\lambda^{(w_0-1)-w+1}}{[(w_0-1)-w+1]!}\\
&+\frac{\lambda^{(w_0-1)-w+2}}{[(w_0-1)-w+2]!}+\cdots+\frac{\lambda^{(w_0-1)}}{(w_0-1)!}\Bigg\}\\
&=\frac{1}{w_0}e^{-\lambda}\Bigg\{\frac{\lambda^{(w_0-1)-w+1}}{[(w_0-1)-w+1]!}+\frac{\lambda^{(w_0-1)-w+2}}{[(w_0-1)-w+2]!}\\
&+\cdots+\frac{\lambda^{(w_0-1)}}{(w_0-1)!}\Bigg\}\\
&\le \frac{1}{w_0}.
\end{aligned}
$$

Combining the two cases gives

$$
|U_\lambda h_{w_0}(w)| \le \frac{1}{w_0}. \tag{2.3}
$$

Similarly, we show that $|U_\lambda h_{w_0}| \le \dfrac{1}{\lambda}$ by considering two cases. If $w > w_0$,

$$
\begin{aligned}
0 < U_\lambda h_{w_0}(w) &= \frac{1}{\lambda}\frac{(w-1)!}{w_0!}e^{-\lambda}\left(\frac{\lambda^{w_0+1}}{w!}+\frac{\lambda^{w_0+2}}{(w+1)!}+\frac{\lambda^{w_0+3}}{(w+2)!}+\cdots\right)\\
&=\frac{1}{\lambda}e^{-\lambda}\left(\frac{\lambda^{w_0+1}}{w_0!w}+\frac{\lambda^{w_0+2}}{w_0!w(w+1)}+\frac{\lambda^{w_0+3}}{w_0!w(w+1)(w+2)}+\cdots\right)\\
&\le \frac{1}{\lambda}e^{-\lambda}\left(\frac{\lambda^{w_0+1}}{(w_0+1)!}+\frac{\lambda^{w_0+2}}{(w_0+2)!}+\frac{\lambda^{w_0+3}}{(w_0+3)!}+\cdots\right)\\
&\le \frac{1}{\lambda}.
\end{aligned}
$$

For $w \le w_0$,

$$
\begin{aligned}
0 &< \frac{(w-1)!}{w_0!}\lambda^{w_0-w}\mathcal{P}_\lambda(h_{C_{w-1}})\\
&= \frac{1}{\lambda}\frac{(w-1)!}{w_0!}e^{-\lambda}\Bigg\{\frac{[w_0-w+1]!\lambda^{w_0-w+1}}{[w_0-w+1]!}+\frac{2\cdot 3\cdots[w_0-w+2]\lambda^{w_0-w+2}}{[w_0-w+2]!}\\
&+\cdots+\frac{w(w+1)\cdots w_0\lambda^{w_0}}{w_0!}\Bigg\}\\
&\le \frac{1}{\lambda}\frac{(w-1)!}{w_0!}e^{-\lambda}\frac{w_0!}{(w-1)!}\Bigg\{\frac{\lambda^{w_0-w+1}}{[w_0-w+1]!}+\frac{\lambda^{w_0-w+2}}{[w_0-w+2]!}+\cdots+\frac{\lambda^{w_0}}{w_0!}\Bigg\}
\end{aligned}
$$

$$= \frac{1}{\lambda} e^{-\lambda} \left\{ \frac{\lambda^{w_0-w+1}}{[w_0-w+1]!} + \frac{\lambda^{w_0-w+2}}{[w_0-w+2]!} + \cdots + \frac{\lambda^{w_0}}{w_0!} \right\}$$
$$\leq \frac{1}{\lambda}.$$

The above two inequalities demonstrate that

$$|U_\lambda h_{w_0}(w)| \leq \frac{1}{\lambda}. \tag{2.4}$$

Therefore, by (2.3) and (2.4), 1. is proved.

Formula of $V_\lambda h_{w_0}$ is easily derived from that of $U_\lambda h_{w_0}$ in (2.2), that is

$$V_\lambda h_{w_0}(w) = \begin{cases} \lambda^{w_0-w-1} \dfrac{(w-1)!}{w_0!} [w\mathcal{P}_\lambda(1-h_{C_w}) - \lambda \mathcal{P}_\lambda(1-h_{C_{w-1}})] & \text{if } w \geq w_0+1, \\ \lambda^{w_0-w-1} \dfrac{(w-1)!}{w_0!} [w\mathcal{P}_\lambda(1-h_{C_w}) + \lambda \mathcal{P}_\lambda(1-h_{C_{w-1}})] & \text{if } w = w_0, \\ -\lambda^{w_0-w-1} \dfrac{(w-1)!}{w_0!} [w\mathcal{P}_\lambda(h_{C_w}) - \lambda \mathcal{P}_\lambda(h_{C_{w-1}})] & \text{if } w \leq w_0+1. \end{cases}$$

Similar arguments as in Neammanee [18] produce the desired bound for $V_\lambda h_{w_0}$ in 2. □

Proof of Theorem 1.1. Proof of 1. is divided into two steps.
Step 1. We claim that

$$\Delta(n,k,w_0) \leq \lambda \min\left\{\frac{1}{w_0}, \frac{1}{\lambda}\right\} \min\{2, E|W_{n,k} - W_{n-k,k}|\}.$$

In fact, for each $i \in D_{n,k}$,

$$\begin{aligned} E[X_i f(W_{n,k})] = &\ E\{E[X_i f(W_{n,k})|X_i]\} \\ &= E[X_i f(W_{n,k})|X_i = 0] P(X_i = 0) \\ &+ E[X_i f(W_{n,k})|X_i = 1] P(X_i = 1) \\ &= E[f(W_{n,k})|X_i = 1] P(X_i = 1) \\ &= P(X_i = 1) E[f(W^*_{n-k,k} + 1)], \end{aligned}$$

where $W^*_{n-k,k} \sim (W_{n,k} - X_i)|X_i = 1$ is the number of isolated trees of order k in the graph $G(n-k,p)$ obtained from $G(n,p)$ by dropping the vertices $i_1, i_2 \ldots i_k$ and all the edges containing any of these vertices. By the fact that $W^*_{n-k,k}$ has identical distribution as $W_{n-k,k}$, we easily deduce

$$\begin{aligned} E[W_{n,k} f(W_{n,k})] &= \sum_{i \in D_{n,k}} E[X_i f(W_{n,k})] \\ &= \sum_{i \in D_{n,k}} P(X_i = 1) E[f(W_{n-k,k}+1)] \\ &= \lambda E[f(W_{n-k,k}+1)]. \end{aligned} \tag{2.5}$$

Once we set $h = h_{w_0}$ in (2.1) and apply (2.5) to the left hand side of (2.1), it follows immediately that

$$\begin{aligned} \left| P(W_{n,k} = w_0) - e^{-\lambda} \frac{\lambda^{w_0}}{w_0!} \right| &= |E[\lambda U_\lambda h_{w_0}(W_{n,k}+1) - W_{n,k} U_\lambda h_{w_0}(W_{n,k})]| \\ &\leq \lambda E|U_\lambda h_{w_0}(W_{n,k}+1) - U_\lambda h_{w_0}(W_{n-k,k}+1)| \end{aligned}$$

$$\leq \lambda[2 \sup_w |U_\lambda h_{w_0}(w+1)|]$$
$$\leq 2\lambda \min\left\{\frac{1}{w_0}, \frac{1}{\lambda}\right\}$$

where Lemma 2.1(1) was used in the last inequality.

By writing $U_\lambda h_{w_0}(W_{n,k}+1) - U_\lambda h_{w_0}(W_{n-k,k}+1)$ as the sum of 1-step increments and applying Lemma 2.1(2),

$$\begin{aligned}\left|P(W_{n,k}=w_0) - e^{-\lambda}\frac{\lambda^{w_0}}{w_0!}\right| &\leq \lambda E|U_\lambda h_{w_0}(W_{n,k}+1) - U_\lambda h_{w_0}(W_{n-k,k}+1)| \\ &\leq \lambda E|\sup_w [U_\lambda h_{w_0}(w+1) - U_\lambda h_{w_0}(w)] \\ &\quad \times [(W_{n,k}+1) - (W_{n-k,k}+1)]| \\ &\leq \lambda \min\{\frac{1}{w_0}, \frac{1}{\lambda}\} E|W_{n,k} - W_{n-k,k}|.\end{aligned}$$

Hence $\Delta(n,k,w_0) \leq \lambda \min\{\frac{1}{w_0}, \frac{1}{\lambda}\} \min\{2, E|W_{n,k} - W_{n-k,k}|\}$.

Step 2. It now suffices to find a bound of $E|W_{n,k} - W_{n-k,k}|$ for $n \geq 2k$. By [21] (p. 140, 142), this expectation can be estimated by

$$\begin{aligned}E|W_{n,k} - W_{n-k,k}| &= E(W_{n,k} - W_{n-k,k})^+ + E(W_{n-k,k} - W_{n,k})^+ \\ &\leq \frac{k^2}{n} EW_{n,k} + \left[1-(1-p)^{k^2}\right] EW_{n-k,k} \\ &= \left(\frac{k^2}{n} + \left[1-(1-p)^{k^2}\right]\frac{EW_{n-k,k}}{\lambda}\right)\lambda\end{aligned}$$

and for $n > 2k$, we have

$$\frac{EW_{n-k,k}}{\lambda} < e^{\frac{k^2}{n}(c_n-1)}.$$

Therefore

$$\begin{aligned}E|W_{n,k} - W_{n-k,k}| &\leq \left(\frac{k^2}{n} + \left[1-(1-p)^{k^2}\right] e^{\frac{k^2}{n}(c_n-1)}\right)\lambda \\ &= \left(\frac{k^2}{n} + e^{\frac{k^2}{n}(c_n-1)} - e^{-\frac{k^2}{n}}\right)\lambda \\ &= \left(\frac{k^2}{n} + (e^{\frac{k^2 c_n}{n}} - 1)e^{-\frac{k^2}{n}}\right)\lambda \\ &\leq \frac{\lambda k^2}{n}\left(1 + c_n e^{\frac{k^2}{n}(c_n-1)}\right)\end{aligned}$$

where we have used the fact that $e^x - 1 \leq xe^x$ for $x \geq 0$ in the last inequality.

It follows readily from step 1. and step 2. that

$$\Delta(n,k,w_0) \leq \lambda \min\left\{\frac{1}{w_0}, \frac{1}{\lambda}\right\} \min\left\{2, \frac{\lambda k^2}{n}\left(1 + c_n e^{\frac{k^2}{n}(c_n-1)}\right)\right\}.$$

The bound of $\Delta(n,k,0)$ in 2. is obtained in the same fashion as that of $\Delta(n,k,w_0)$ except that the inequalities

1. $|U_\lambda h_0| \leq \min\left\{1, \frac{1}{\lambda}\right\}$
2. $|V_\lambda h_0| \leq \min\left\{1, \frac{1}{\lambda}\right\}$

are used instead of Lemma 2.1. With a few obvious adjustments, proof of Lemma 2.1 work equally well for $U_\lambda h_0$ and $V_\lambda h_0$. □

Proof of Theorem 1.2. From Theorem 1.1 and the fact that

$$c_n = \mathrm{O}\Big(\frac{1}{n^{\delta-1}}\Big) \text{ when } p = \mathrm{O}\Big(\frac{1}{n^{\delta}}\Big) \tag{2.6}$$

we see that

$$\Delta(n,k,w_0) \le C\frac{\lambda^2 k^2}{n w_0} \tag{2.7}$$

and

$$\Delta(n,k,0) \le C\frac{\lambda^2 k^2}{n}. \tag{2.8}$$

Under the settings, we would naturally like an estimate of λ in terms of n and k. By (19)–(22) in p. 141 of [21], λ can be factored as

$$\lambda = \alpha(k)\beta(k,p)\gamma(n,k,p) \tag{2.9}$$

where $\alpha(k) = \dfrac{k^{k+\frac{1}{2}}e^{-k}}{k!}, \beta(k,p) = k^{-\frac{5}{2}}e^k p^{k-1}(1-p)^{-(\frac{k^2+3k}{2})+1}$ and

$$\gamma(n,k,p) = n^k e^{-kc_n}\prod_{j=1}^{k-1}\Big(1-\frac{j}{n}\Big).$$

The Stirling's formula ([12] p.54),

$$\sqrt{2\pi}k^{k+\frac{1}{2}}e^{-k}e^{\frac{1}{12k+1}} < k! < \sqrt{2\pi}k^{k+\frac{1}{2}}e^{-k}e^{\frac{1}{12k}},$$

easily derives the inequalities

$$\frac{1}{\sqrt{2\pi}e^{\frac{1}{12k}}} < \alpha(k) < \frac{1}{\sqrt{2\pi}e^{\frac{1}{12k+1}}}. \tag{2.10}$$

Finally, by (2.6), (2.9)–(2.10) and the fact that $k \le \mathrm{O}(n^{\frac{\delta}{2}})$,

$$\begin{aligned}
\lambda &\le \frac{Cn^k}{k^{\frac{5}{2}}}e^{k(1-c_n)}p^{k-1}\\
&= \frac{Cn}{k^{\frac{5}{2}}}(c_n e^{1-c_n})^{k-1}e^{1-c_n}\Big(\frac{p}{-\log(1-p)}\Big)^{k-1}\\
&\le \frac{Cn}{k^{\frac{5}{2}}}(c_n e^{1-c_n})^{k-1}e^{1-c_n}\\
&\le \frac{n}{k^{\frac{5}{2}}}\mathrm{O}\Big(\frac{e}{n^{\delta-1}}\Big)^{k-1}\\
&\le \frac{1}{k^{\frac{5}{2}}}\mathrm{O}\Big(\frac{1}{n^{(\delta^*-1)(k-1)-1}}\Big)
\end{aligned} \tag{2.11}$$

where we have used the facts that $\lim\limits_{n\to\infty}\Big(\dfrac{p}{-\log(1-p)}\Big)^{k-1} = 1$ in the second inequality and $\lim\limits_{n\to\infty}\dfrac{e}{n^{\beta}} = 0$ for all $\beta > 0$ in the last inequality. And the theorem follows from (2.7)–(2.8) and (2.11). □

Proof of Theorem 1.3. Let us first observe that

$$\prod_{j=1}^{k-1}\Big(1-\frac{j}{n}\Big)=\exp\Big[-\frac{k(k-1)}{2n}-\frac{\theta k^3}{3n^2}\Big] \tag{2.12}$$

for $k<\frac{n}{2}$ with $0<\theta<1$. So, after substituting (2.12) into (2.9) and noting the fact that $\lim_{n\to\infty}\frac{k^2}{n}=0$, we obtain

$$\begin{aligned}\lambda &\geq \frac{Cn}{k^{\frac{5}{2}}}(c_n e^{1-c_n})^{k-1}e^{1-c_n}\\ &= \frac{C}{k^{\frac{5}{2}}}e^{k-1}\mathrm{O}\Big(\frac{1}{n^{(\delta-1)(k-1)-1}}\Big).\end{aligned}$$

With this lower bound of λ, (2.7) becomes

$$\begin{aligned}0\leq\frac{\Delta(n,k,w_0)}{\lambda^{w_0}} &\leq \frac{1}{w_0e^{(w_0-2)(k-1)}}\mathrm{O}\Big(\frac{1}{n^{(w_0-2)[(\delta-1)(k-1)-1+\frac{5\beta}{2}]+1-2\beta}}\Big)\\ &= \Big(\frac{n^{(w_0-2)-1+2\beta}}{w_0e^{(w_0-2)(k-1)}}\Big)\mathrm{O}\Big(\frac{1}{n^{(w_0-2)[(\delta-1)(k-1)+\frac{5\beta}{2}]}}\Big)\\ &\leq \frac{1}{w_0}\mathrm{O}\Big(\frac{1}{n^{(w_0-2)[(\delta-1)(k-1)+\frac{5\beta}{2}]}}\Big).\end{aligned} \tag{2.13}$$

Obviously, the right hand side converges to 0 as n goes to infinity. Again by Stirling's formula ([12],p.52), $k!\sim\sqrt{2\pi}k^{k+\frac{1}{2}}e^{-k}$, an upper bound of the number of isolated trees can be computed. That is,

$$\begin{aligned}w_0\leq\binom{n}{k} &= \frac{n!}{(n-k)!k!}\\ &\sim \sqrt{2\pi}\Big(\frac{n}{n-k}\Big)^{n+\frac{1}{2}}\Big(\frac{n-k}{k}\Big)^k\\ &\leq Cn^{(1-\beta)k}\\ &\leq Cn^{(\delta-1)(k-1)}\end{aligned}$$

for k sufficiently large. This immediately implies that, for k large enough,

$$(\omega_0-1)!\leq Cn^{(\omega_0-2)(\delta-1)(k-1)}.$$

Thus, we conclude from this bound and (2.13) that

$$\begin{aligned}0\leq w_0!\Delta(n,k,w_0) &\leq w_0!\frac{\Delta(n,k,w_0)}{\lambda^{w_0}}\\ &\leq (w_0-1)!\mathrm{O}\Big(\frac{1}{n^{(w_0-2)[(\delta-1)(k-1)+\frac{5\beta}{2}]}}\Big)\\ &\leq \frac{1}{n^{\frac{5}{2}\beta(w_0-2)}}\end{aligned}$$

which converges to zero as ω_0 increases to infinity. □

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Department of Mathematics,
Faculty of Science,
Chulalongkorn University,
Bangkok 10330 Thailand
E-mail address: Kritsana.N@chula.ac.th
E-mail address: K_Neammanee@hotmail.com