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A GENERALIZED AMMAN'S FIXED POINT THEOREM AND ITS APPLICATION TO NASH EQULIBRIUM

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Abstract. In this paper, we first give a generalization of Amann's fixed point theorem: if $(X, \leq)$ is a nonempty partially ordered set with the property that every nonempty chain has a supremum and $F: X \to 2^X$ is a monotone set-valued map and there is $a \in X$ such that for all $b \in F(a)$ we have $a \leq b$, then F has a least fixed point in the subset $\{x \in X : a \leq x\}$. By using the duality principle, we obtain the existence of the greatest fixed point for monotone set-valued maps. As application we apply our results to show that the set of Nash equilibrium of a subcategory of D'Orey's extended supermodular game has a least and a greatest elements.

1. Introduction and preliminaries

During the last century many authors studied the theory of supermodular game (for example see [2, 4, 5, 6, 7]). In 1996, D'Orey presented in [2] an extended supermodular game and proved the existence of a Nash equilibrium of its.

Let X be a nonempty set. We said that $(X, \leq)$ is a partially ordered set if $\leq$ is a binary relation $\leq$ which is reflexive, antisymmetric, and transitive. Let $x, y \in X$. We write $x < y$ if $x \leq y$ and $x \neq y$. A nonempty subset C of X is called a chain if for every $x, y \in C$ we have either $x \leq y$ or $y \leq x$.

Let $(X, \leq)$ be a nonempty partially ordered set and A a nonempty subset of X. An element $b \in X$ is called an upper bound (lower bound) of A if $a \leq b$ ($b \leq a$) for all $a \in A$. If b is an upper bound (lower bound) of A and $b \in A$, then b is a greatest element (least element) of A. The least upper bound (greatest lower bound) of A when it exist is called the supremum (infimum) of A and will be denoted $\sup_X(A)$ ($\inf_X(A)$). A nonempty partially ordered set $(X, \leq)$ is said to be a complete partially ordered set if every nonempty chain of X has a supremum in X. A map $f: X \to X$ is monotone if $f(x) \leq f(y)$ whenever $x \leq y$. A point x of X is called a fixed point of f is $f(x) = x$. The set of all fixed points of f will be denoted by $\mathrm{Fix}(f)$.

In [3], Knaster started the study of the existence of fixed point for maps with values in nonempty partially ordered sets. Later on, H. Amann [8, Theorem 11.D] established the following: let $(X, \leq)$ be a nonempty partially ordered set with the property that every nonempty chain has a supremum and $f: X \to X$ be a monotone

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map. Assume that there is $a \in X$ such that $a \leq f(a)$. Then, the map f has a least fixed point in the subset $\{x \in X : a \leq x\}$.

Let X be a nonempty set and 2^X be the set of all nonempty subsets of X. A set-valued map on X is any map $F\colon X \to 2^X$. An element x of X is called a fixed point of F if $x \in F(x)$. We denote by $\mathrm{Fix}(F)$ the set of all fixed points of F. In this paper, we shall use the following definition of monotonicity for set-valued maps.

Definition 1.1. Let $(X, \leq)$ be a nonempty partially ordered set. A set-valued map $F\colon X \to 2^X$ is said to be monotone if for any $x, y \in X$ with $x < y$, then for every $a \in F(x)$ and $b \in F(y)$, we have $a \leq b$.

In this work, we first prove the existence of the least fixed point for monotone set-valued maps by using the same hypothesis used by D'Orey in [2, Theorem 3]. In addition, we give a generalization of Amann's fixed point theorem [8, Theorem 11.D] for monotone set-valued maps (see Theorems 2.1 and 2.4). We also establish that, if a partially ordered set $(X, \leq)$ has a greatest element such that every nonempty chain has in X an infimum, then every monotone set-valued map has a greatest fixed point. As application we show that a subcategory of D'Orey's extended supermodular game has a least and a greatest Nash equilibrium.

The remainder of this paper is organized as follows. In section 2, we present our generalization of Amann's result (see Theorems 2.1 and 2.4). In section 3, we apply our result to show the existence of a least and greatest Nash equilibrium for a subcategory of D'Orey's extended supermodular game (see Theorems 3.5).

2. A generalized Amman's fixed point theorem

The key result of this section is the following:

Theorem 2.1. *Let $(X, \leq)$ be a nonempty complete partially ordered set with a least element and let $F\colon X \to 2^X$ be a monotone set-valued map. Then, F has a least fixed point in X.*

Proof. By [2, Theorem 3], the set-valued map F has at least a fixed point in X. Let A be the following subset of X defined by

$$A = \{x \in X : \text{ there exists } y \in X, y \in F(x) \text{ and } x \leq y \leq z \text{ for all } z \in \mathrm{Fix}(F)\}.$$

Let $l = \inf_X(X)$.

Claim 1. We have: $l \in A$. Assume on the contrary that $l \notin A$. As $\mathrm{Fix}(F) \subseteq A$, then $l \notin \mathrm{Fix}(F)$. So, $l < z$ for all $z \in \mathrm{Fix}(F)$. Let $k \in F(l)$. Hence, by monotonicity of F, we get $k \leq z$ for all $z \in \mathrm{Fix}(F)$. Thus, $l \in A$. That is a contradiction and our claim is proved.

Claim 2. $(A, \leq)$ is a nonempty complete partially ordered set. Indeed, let C be a nonempty chain in A and let $s = \sup_X(C)$. By absurd, assume that $s \notin A$. Then, $x < s$ for all $x \in C$. Since $C \subseteq A$, hence for $x \in C$ there is $y_x \in X$ such that $y_x \in F(x)$ and

$$x \leq y_x \leq z \text{ for all } z \in \mathrm{Fix}(F). \tag{2.1}$$

On the other hand, by (2.1), we get

$$x \leq z \text{ for all } z \in \mathrm{Fix}(F). \tag{2.2}$$

Then, every element z of $\mathrm{Fix}(F)$ is an upper bound of C. Since $s = \sup_X(C)$, we have

$$s \leq z \text{ for all } z \in \mathrm{Fix}(F). \tag{2.3}$$

As $s \notin A$ and $\mathrm{Fix}(F) \subset A$, then $s \notin \mathrm{Fix}(F)$. From this and by (2.3), we obtain

$$s < z \text{ for all } z \in \mathrm{Fix}(F). \tag{2.4}$$

Let t be a given element of $F(s)$. Then, by monotonicity of F and (2.4), we get

$$t \leq z \text{ for all } z \in \mathrm{Fix}(F). \tag{2.5}$$

On the other hand we know that $x < s$ for every $x \in C$. Using this and the monotonicity of F, we obtain

$$y_x \leq t \text{ for all } x \in C. \tag{2.6}$$

Combining (2.1) and (2.6), we get

$$x \leq t \text{ for all } x \in C. \tag{2.7}$$

From this and as x is a general element of C, we deduce that t is an upper bound of C. As $s = \sup_X(C)$, we get $s \leq t$. By using (2.4) and the monotonicity of F, we obtain $t \leq z$ for all $z \in \mathrm{Fix}(F)$. Therefore, we deduce that $s \in A$. That is a contradiction and our claim is proved.

Claim 3. The subset A has a maximal element. Indeed, by Claim 2, every nonempty chain of A has a supremum in A. Then, from Zorn's Lemma, the set A has a maximal element, m, say.

Claim 4. The element m is a fixed point of F. On the contrary assume that $m \notin \mathrm{Fix}(F)$. So, $m < z$ for all $z \in \mathrm{Fix}(F)$. On the other hand, by Claim 3, we know that $m \in A$. Then, there is $n \in F(m)$ with

$$m < n \leq z \text{ for all } z \in \mathrm{Fix}(F). \tag{2.8}$$

As m is a maximal element of A and $m < n$, then we deduce that $n \notin A$. So $n \notin \mathrm{Fix}(F)$. Hence, we get

$$m < n < z \text{ for all } z \in \mathrm{Fix}(F). \tag{2.9}$$

Now, let p be a given element of $F(n)$. Then, by monotonicity of F and (2.9), we have

$$n \leq p \leq z \text{ for all } z \in \mathrm{Fix}(F). \tag{2.10}$$

From (2.10), we deduce that $n \in A$. That is a contradiction and our claim is proved.

Claim 5. The element m is the least fixed point of F. Indeed, by Claim 3, $m \in A$. Then, m is a lower bound of $\mathrm{Fix}(F)$. On the other hand, from Claim 4, we know that $m \in \mathrm{Fix}(F)$. Therefore, m is the least fixed point of F. □

Remark. In Theorem 2.1 we have proved the existence of the least fixed point under the same hypothesis used by D'Orey in [2, Theorem 3] to establish only the existence of a fixed point for monotone set-valued maps.

Definition 2.2. Let X be a nonempty set. Let $F\colon X \to 2^X$ be a set-valued map and A be a nonempty subset of X. The restriction of F on A is the set-valued map $F_A\colon A \to 2^X$ defined by $F_A(x) = F(x)$, for every $x \in A$.

Let $(X, \leq)$ be a nonempty partially ordered set and let $F\colon X \to 2^X$ be a set-valued map. Let A_a the following subset of X defined by $A_a = \{x \in X : a \leq x\}$ and let F_{A_a} be the restriction of F on A_a. The range of F_{A_a} is the subset of X defined by $F_{A_a}(A_a) = \bigcup_{x \in A_a} F(x)$.

In what follows, we shall need the following lemma.

Lemma 2.3. *Let $(X, \leq)$ be a nonempty partially ordered set and let $F\colon X \to 2^X$ be a set-valued map. Let us suppose that A_a is defined as above. Then,*

(i) $F_{A_a}(A_a) \subset A_a$.

(ii) if F is monotone, then F_{A_a} is also monotone;

(iii) if $(X, \leq)$ is a complete partially ordered set, so $(A_a, \leq)$ is also a complete partially ordered set;

(iv) $\mathrm{Fix}(F_{A_a}) = \mathrm{Fix}(F) \cap A_a$.

Proof. Let $(X, \leq)$ be a nonempty partially ordered set and let $F\colon X \to 2^X$ be a set-valued map. Let $A_a = \{x \in X : a \leq x\}$.

(i) Let $x \in A_a$. If $x = a$, then by our hypothesis, we have $F(a) \subset A_a$. Otherwise, assume that $a < x$ and let $b \in F(a)$ and $y \in F(x)$. Then, by monotonicity of F, we get $b \leq y$. On the other hand, by our hypothesis, we know that $a \leq b$. So, we obtain $a \leq y$. Thus, $F(x) \subset A_a$ for every $x \in A_a$. Therefore, $F_{A_a}(A_a) \subset A_a$.

(ii) Let $x, y \in A_a$ such that $x < y$. Now, let $a \in F_{A_a}(x)$ and $b \in F_{A_a}(y)$. Since $F_{A_a}(x) = F(x)$, $F_{A_a}(y) = F(y)$ and F is monotone, so $a \leq b$. Thus, F_{A_a} is a set-valued monotone map.

(iii) Let C be a nonempty fuzzy chain of A_a and let $s = \sup_X(C)$. Let $c \in C$ be a given element. Then, $c \leq s$. As $C \subseteq A_a$, so $a \leq c$. Hence, we get $a \leq s$. Thus, $s \in A_a$.

(iv) Let $x \in \mathrm{Fix}(F_{A_a})$. Then, $x \in A_a$ and $x \in F_{A_a}(x)$. On the other hand, by our definition, $F_{A_a}(x) = F(x)$. Hence, $x \in F(x)$. So, $x \in A_a$ and $x \in \mathrm{Fix}(F)$. Conversely, if $x \in \mathrm{Fix}(F) \cap A_a$, then $x \in A_a$ and $x \in F(x)$. Hence, $x \in F_{A_a}(x)$. Thus, $x \in \mathrm{Fix}(F_{A_a})$. □

Next, we shall show the main result in this section.

Theorem 2.4. *Let $(X, \leq)$ be a nonempty complete partially ordered set and let $F\colon X \to 2^X$ be a monotone set-valued map. Assume that there is $a \in X$ such that for all $b \in F(a)$, we have $a \leq b$. Then, F has a least fixed point in the subset $\{x \in X : a \leq x\}$.*

Proof. Let $(X, \leq)$ be a nonempty complete partially ordered set. Let $F\colon X \to 2^X$ be a monotone set-valued map. Let $a \in X$ such that for every $b \in F(a)$, we have $a \leq b$. Recall that $A_a = \{x \in X : a \leq x\}$. Then, by Lemma 2.3, $(A_a, \leq)$ is a nonempty complete partially ordered set and F_{A_a} is a monotone set-valued map. On the other hand, we have $F_{A_a}(a) = F(a)$. Then, all hypotheses of Theorem 2.1 are fulfilled for the monotone set-valued map $F_{A_a}\colon A_a \to 2^{A_a}$. Hence, from Theorem 2.1, F_{A_a} has a least fixed point in A_a. Since by Lemma 2.3, $\mathrm{Fix}(F_{A_a}) = \mathrm{Fix}(F) \cap A_a$, therefore F has a least fixed point in A_a. □

As a consequence of Theorem 2.4, we reobtain Amann's result [8, Theorem 11.D].

Corollary 2.5. *Let $(X, \leq)$ be a nonempty complete partially ordered set and $f\colon X \to X$ be a monotone map. Assume that there is $a \in X$ such that $a \leq f(a)$. Then, f has a least fixed point in the subset $\{x \in X : a \leq x\}$.*

By using Theorem 2.4 and the duality principle, we obtain:

Theorem 2.6. *Let $(X, \leq)$ be a nonempty partially ordered set with the property that every nonempty chain of X has an infimum. Let $F\colon X \to 2^X$ be a monotone set-valued map. Assume that there is $a \in X$ such that for every $b \in F(a)$, we have $b \leq a$. Then, F has a greatest fixed point in the subset $\{x \in X : x \leq a\}$.*

Corollary 2.7. *Let $(X, \leq)$ be a nonempty partially ordered set with a greatest element and in which every nonempty chain has an infimum. Let $F\colon X \to 2^X$ be a monotone set-valued map. Then, F has a greatest fixed point.*

3. Least and greatest Nash equilibrium for an extended supermodular game

In this section, we first present a subcategory of D'Orey's extended supermodular game [2]. Secondly, we apply Theorem 2.1 and Corollary 2.7 to show that the set of all Nash equilibrium of this game has a least and a greatest elements. First, we recall the following definitions.

Definition 3.1. Let $(X, \leq)$ be a partially ordered set and $x \in X$.
(i) The down set $x \downarrow$ is defined by $x \downarrow = \{y \in X : y \leq x\}$.
(ii) The up set $x \uparrow$ is defined by $x \uparrow = \{y \in X : x \leq y\}$.

Definition 3.2 ([2, Definition 4])**.** Let $(X, \leq)$ be a partially ordered set and $x, y \in X$.
(i) The element x meets y if $x \downarrow \cap y \downarrow \neq \emptyset$.
(ii) The element x joins y if $x \uparrow \cap y \uparrow \neq \emptyset$.
(iii) $x, y \in X$ are in a regular position, $xRPy$, if x and y meet and join.
(iv) If every pairs of X are in a regular position, $(X, \leq)$ is said to be a quasi-lattice.

Definition 3.3 ([2, Definition 5])**.** Let $(X, \leq)$ be a nonempty partially ordered set and $f\colon X \to \mathbb{R}$ be a real function defined over X. The function f is said to be supermodular if $\forall x, y \in X : xRPy$,

$$\exists a \in x \downarrow \cap y \downarrow, \exists b \in x \uparrow \cap y \uparrow : f(a) + f(b) \geq f(x) + f(y).$$

f is strictly supermodular if the inequality in the previous definition is strict when x and y are in a regular position and unrelated by the $\leq$ relation.

Definition 3.4 ([2, Definition 6])**.** Let X and T be two partially ordered sets and $f\colon X \times T \to \mathbb{R}$ be a real function defined over $X \times T$. Then f has (strictly) increasing differences in (x, t) if $f(x, t) - f(x, t^{'})$ is (strictly) increasing in x for all $t^{'} \leq t$ $(t^{'} \leq t, t \neq t^{'})$.

Next, we define a subcategory H of D'Orey's extended supermodular game (see [2]):

it is a symmetric game with two players, where X, is the strategy common to both players is a quasi-lattice with a least and a greatest element and in which every nonempty chain has a supremum and an infimum. Each player's payoff is a best reply function, f, defined over $X \times X$, that is strictly supermodular and has strictly differences. By using the properties of X and f, we deduce that this game can be considered as an extension of the supermodular game studied by Vives [8].

Now, let $F\colon X \to \mathcal{P}(X)$ be the set-valued map defined by setting:

$$\forall x^{'} \in X, F(x^{'}) = \operatorname{argmax}_{x \in X} f(x, x^{'}) = \{y \in X : f(y, x^{'}) = \max_{x \in X} f(x, x^{'})\}.$$

A Nash equilibrium for the game H is an element $x_0 \in X$ such that

$$x_0 \in \operatorname{argmax}_{x \in X} f(x, x_0).$$

Next, we shall prove the main result in this section.

Theorem 3.5. *The set of Nash equilibrium of the extended supermodular game H is nonempty and has a least and a greatest elements.*

Proof. By using [2, Theorem 4], we deduce that the set-valued map F is monotone. Then, from Theorem 2.1 and Corollary 2.7, we deduce that the set $\mathrm{Fix}(F)$ is nonempty and has a least and a greatest elements. On the other hand, we know that

$$\mathrm{Fix}(F) = \{x_0 \in X : x_0 \in \mathrm{argmax}_{x \in X} f(x, x_0)\}.$$

Therefore, the set of Nash equilibrium for the extended supermodular game H has a least and a greatest elements. □

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