Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
21 (2005), 161–167
www.emis.de/journals
ISSN 1786-0091

LOGARITHMIC SUMMABILITY OF FOURIER SERIES

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Abstract. A set of regular summations logarithmic methods is introduced. This set includes Riesz and Nörlund logarithmic methods as limit cases. The application to logarithmic summability of Fourier series of continuous and integrable functions are given. The kernels of these logarithmic methods for trigonometric system are estimated.

1. Introduction

In the literature it is known Riesz and Nörlund logarithmic summation methods (see [H, HR, Z]). Applications of these methods to Fourier analysis are investigated by many authors (see for example [S, Zh, Y, MS, GG, GT]. We construct the set of logarithmic summation methods which in particular contains both mentioned methods. We study the application of these methods to Fourier series. The estimates of kernels and Lebesgue functions in trigonometric case are obtained. Necessary and sufficient condition which guarantee such logarithmic summability of Fourier series for continuous and integrable functions in corresponding metric are established.

2. Construction of logarithmic summation methods

For any integers m and n such that $0 \le m \le n$ we put

$$F_{m,n}(x) \equiv \frac{1}{l(m,n)} \left\{ \sum_{k=0}^{m-1} \frac{D_k(x)}{m-k+1} + D_m(x) + \sum_{k=m+1}^{n} \frac{D_k(x)}{k-m+1} \right\} \tag{1}$$

where

$$D_k(x) \equiv \frac{\sin(k+\frac{1}{2})x}{2\sin\frac{x}{2}} \tag{2}$$

is Dirichlet kernel and

$$l(m,n) \equiv \sum_{k=0}^{m-1} \frac{1}{m-k+1} + 1 + \sum_{k=m+1}^{n} \frac{1}{k-m+1}. \tag{3}$$

2000 *Mathematics Subject Classification.* 42A24, 40G05.
Key words and phrases. Fourier series, logarithmic summability, spaces of continuous and integrable functions.

Here and further we assume that if $q < p$ then $\sum_{k=p}^{q} d_k = 0$ for any d_k. It is clear that $l(m,n) \asymp \ln(n+2)$. We call

$$t_{m,n}f \equiv F_{m,n} * f$$

the logarithmic means of Fourier series of function $f \in L^1[-\pi,\pi]$.

If we replace in (1) the sequence $\{D_n(x)\}_{n=0}^{\infty}$ by arbitrary sequence $\{S_n\}_{n=0}^{\infty}$, then we define such logarithmic means in general case. It is easy to see that for each fixed sequence of integers $\{m_n\}$ under condition $0 \le m_n \le n$ these means $t_{m_n,n}$ generate a regular logarithmic summation method. The set of all such methods in particular includes Riesz (for $m_n = 0$) and Nörland (for $m_n = n$) logarithmic methods. In the sequel C, C_i denote absolute positive constants.

3. Logarithmic means and their kernels

In following proposition we estimate the kernels $F_{m,n}$.

Theorem 1. *There exist positive constants C_i, $i = 1,2$, $0 < C_1 \le C_2$ such that for any integers m and n, $0 \le m \le n$ holds*

$$C_1 \left\{1 + \frac{\ln^2(m+2)}{\ln(n+2)}\right\} \le \|F_{m,n}\|_{L^1[-\pi,\pi]} \le C_2 \left\{1 + \frac{\ln^2(m+2)}{\ln(n+2)}\right\}. \tag{4}$$

In particular from Theorem 1 we obtain well-known results.

Corollary 1 (see [GT, Y])**.** *For any $n \ge 0$ we have*

$$\|F_{n,n}\|_{L^1[-\pi,\pi]} \asymp C\ln(n+2)$$

$$\|F_{0,n}\|_{L^1[-\pi,\pi]} \asymp C.$$

Theorem 2. *The following conditions are equivalent*

a) $m_n = O\left(\exp\sqrt{\ln n}\right)$ *as* $n \to \infty$.

b) $\|t_{m,n}f - f\|_{C[-\pi,\pi]} \to 0$ *as* $n \to \infty \quad \forall f \in C[-\pi,\pi]$,

c) $\|t_{m,n}f - f\|_{L^1[-\pi,\pi]} \to 0$ *as* $n \to \infty \quad \forall f \in L^1[-\pi,\pi]$.

Theorem 2 is corollary of Theorem 1 because the condition a) is equivalent to the boundedness of Lebesgue constants for method $t_{m_n,n}$ what is sufficient and necessary condition for *b*) and *c*) (see [HR], Ch. 5.)

We can construct a logarithmic summation method with given possible growth of logarithmic means. In particular we have

Theorem 3. *For any $\tau(n) \in [1, \ln n]$ and $m_n = [\exp\{\sqrt{\tau(n)\ln n}\}]$ a logarithmic means $t_{m_n,n}$ is such that*

$$\|t_{m_n,n}f\|_{C[-\pi,\pi]} \le C\tau(n)\|f\|_{C[-\pi,\pi]}.$$

Another corollaries of Theorem 1 for divergence of Fourier series of continuous function one can obtain by well-known way using the uniform boundedness principle (see [E], Ch. 10.3.2.)

4. Proof of Theorem 1

Proof of Theorem 1. It is sufficient to prove the inequality (4) for $n > 1$. First note that

$$F_{m,n}(x) = \frac{1}{l(m,n)} \left\{ \sum_{j=2}^{m+1} \frac{D_{m+1-j}(x)}{j} + D_m(x) + \sum_{j=2}^{n-m+1} \frac{D_{j+m-1}(x)}{j} \right\} \tag{5}$$

and

$$\|F_{m,n}\|_{L^1} = 2\left[\int_0^{\frac{1}{n+2}} |F_{m,n}(x)|dx + \int_{\frac{1}{n+2}}^{\pi} |F_{m,n}(x)|dx\right] \equiv 2(A_1 + A_2). \tag{6}$$

Since $|D_k(x)| \le k + \frac{1}{2}$ $\forall k$ $\forall x \in [-\pi, \pi]$, then (see(1) , (2), (3) , (6))

$$A_1 = \int_0^{\frac{1}{n+2}} |F_{m,n}(x)|dx \le 1. \tag{7}$$

Further from (5) we get

$$\begin{aligned} F_{m,n}(x) = \frac{1}{l(m,n)} \Bigg\{ & \frac{\sin\left(m+1+\frac{1}{2}\right)x}{2\sin\frac{x}{2}} \sum_{j=2}^{m+1} \frac{\cos jx}{j} \\ & - \frac{\cos\left(m+1+\frac{1}{2}\right)x}{2\sin\frac{x}{2}} \sum_{j=2}^{m+1} \frac{\sin jx}{j} \\ & + D_m(x) + \frac{\sin(m-1+\frac{1}{2})x}{2\sin\frac{x}{2}} \sum_{j=2}^{n-m+1} \frac{\cos jx}{j} \\ & + \frac{\cos(m-1+\frac{1}{2})x}{2\sin\frac{x}{2}} \sum_{j=2}^{n-m+1} \frac{\sin jx}{j} \Bigg\}. \end{aligned} \tag{8}$$

We have for A_2 the following decomposition

$$\begin{aligned} A_2 = \int_{\frac{1}{n+2}}^{\pi} |F_{m,n}(x)|dx \le \frac{1}{l(m,n)} \Bigg\{ & \int_{\frac{1}{n+2}}^{\pi} \left|\frac{\sin(m+1+\frac{1}{2})x}{2\sin\frac{x}{2}}\right| \left|\sum_{j=2}^{m+1} \frac{\cos jx}{j}\right| dx \\ & + \int_{\frac{1}{n+2}}^{\pi} \left|\frac{\cos(m+1+\frac{1}{2})x}{2\sin\frac{x}{2}}\right| \left|\sum_{j=2}^{m+1} \frac{\sin jx}{j}\right| dx \\ & + \int_{\frac{1}{n+2}}^{\pi} \left|\frac{\sin(m-1+\frac{1}{2})x}{2\sin\frac{x}{2}}\right| \left|\sum_{j=2}^{n-m+1} \frac{\cos jx}{j}\right| dx \\ & + \int_{\frac{1}{n+2}}^{\pi} \left|\frac{\cos(m-1+\frac{1}{2})x}{2\sin\frac{x}{2}}\right| \left|\sum_{j=2}^{n-m+1} \frac{\sin jx}{j}\right| dx + \int_{\frac{1}{n+2}}^{\pi} |D_m(x)|\,dx \Bigg\} \\ \equiv \frac{1}{l(m,n)} \{A_{2,1} + A_{2,2} & + A_{2,3} + A_{2,4} + A_{2,5}\}. \end{aligned} \tag{9}$$

By convention if $m = 0$ then $A_{2,1} = A_{2,2} = 0$ and if $m = n$ then $A_{2,3} = A_{2,4} = 0$.

Since for $N \geq 1$ and $-\pi \leq x \leq \pi$ we have the estimate ([Z], Ch.5.)

$$|\sum_{j=1}^{N} \frac{\sin jx}{j}| \leq C, \tag{10}$$

then (see (9))

$$\begin{aligned} A_{2,2} &= \int_{\frac{1}{n+2}}^{\pi} \left| \frac{\cos(m+1+\frac{1}{2})x}{2\sin\frac{x}{2}} \right| \left| \sum_{j=2}^{m+1} \frac{\sin jx}{j} \right| dx \leq C \int_{\frac{1}{n+2}}^{\pi} \left| \frac{1}{2\sin\frac{x}{2}} \right| dx \\ &\leq C \ln(n+2) \end{aligned} \tag{11}$$

and

$$\begin{aligned} A_{2,4} &= \int_{\frac{1}{n+2}}^{\pi} \left| \frac{\cos(m-1+\frac{1}{2})x}{2\sin\frac{x}{2}} \right| \left| \sum_{j=2}^{n-m+1} \frac{\sin jx}{j} \right| dx \leq C \int_{\frac{1}{n+2}}^{\pi} \left| \frac{1}{2\sin\frac{x}{2}} \right| dx \\ &\leq C \ln(n+2). \end{aligned} \tag{12}$$

Moreover

$$\begin{aligned} A_{2,1} &= \int_{\frac{1}{n+2}}^{\pi} \left| \frac{\sin(m+1+\frac{1}{2})x}{2\sin\frac{x}{2}} \right| \left| \sum_{j=2}^{m+1} \frac{\cos jx}{j} \right| dx \leq C \sum_{j=2}^{m+1} \frac{1}{j} \int_{\frac{1}{n+2}}^{\pi} |D_{m+1}(x)| dx \\ &\leq C \ln(m+2) \|D_{m+1}\|_{L^1[-\pi,\pi]} \leq C \ln^2(m+2). \end{aligned} \tag{13}$$

Now we estimate $A_{2,3}$

$$\begin{aligned} A_{2,3} &= \int_{\frac{1}{n+2}}^{\pi} \left| \frac{\sin(m-1+\frac{1}{2})x}{2\sin\frac{x}{2}} \right| \left| \sum_{j=2}^{n-m+1} \frac{\cos jx}{j} \right| dx \\ &= \int_{\frac{1}{n+2}}^{\frac{1}{m+2}} \left| \frac{\sin(m-1+\frac{1}{2})x}{2\sin\frac{x}{2}} \right| \left| \sum_{j=2}^{n-m+1} \frac{\cos jx}{j} \right| dx \\ &\quad + \int_{\frac{1}{m+2}}^{\pi} \left| \frac{\sin(m-1+\frac{1}{2})x}{2\sin\frac{x}{2}} \right| \left| \sum_{j=2}^{n-m+1} \frac{\cos jx}{j} \right| dx = A_{2,3,1} + A_{2,3,2}. \end{aligned} \tag{14}$$

Since (see [Z], Ch.5)

$$\left| \sum_{j=1}^{N} \frac{\cos jx}{j} \right| \leq C + \ln\frac{1}{x} \qquad \forall x \in (0,\pi] \qquad \forall N \geq 1$$

then

$$\left| \sum_{j=2}^{s} \frac{\cos jx}{j} \right| \leq C + \ln(m+2) \qquad x \in [\frac{1}{m+2}, \pi] \tag{15}$$

and we obtain the estimates

$$A_{2,3,1} \leq (m+2) \int_{\frac{1}{n+2}}^{\frac{1}{m+2}} \left(C + \ln\frac{1}{x} \right) dx \leq C + \ln(m+2) \tag{16}$$

and

$$A_{2,3,2} \le (C + \ln(m+2)) \int_{\frac{1}{m+2}}^{\pi} \frac{1}{2\sin\frac{x}{2}} dx \le C\left(\ln^2(m+2) + \ln(m+2)\right). \tag{17}$$

Since

$$A_{2,5} = \int_{\frac{1}{n+2}}^{\pi} |D_m(x)|\, dx \le \|D_m\|_{L^1[-\pi,\pi]} \le C\ln(m+2), \tag{18}$$

then we have from (9), (11), (12), (13), (14), (16), (17), (18)

$$A_2 \le C\frac{\ln^2(m+2)}{\ln(n+2)} + C. \tag{19}$$

Finally (see (6), (7), (19)) we obtain the right estimate of (4).

Now we prove the left side of (4) From (8) it is evident that

$$\begin{aligned} l(m,n)F_{m,n}(x) &= D_m(x) + \frac{\cos(m-1+\frac{1}{2})x}{2\sin\frac{x}{2}} \sum_{j=2}^{n-m+1} \frac{\sin jx}{j} \\ &- \frac{\cos(m+1+\frac{1}{2})x}{2\sin\frac{x}{2}} \sum_{j=2}^{m+1} \frac{\sin jx}{j} + \frac{\cos(m+\frac{1}{2})x\sin x}{2\sin\frac{x}{2}} \sum_{j=2}^{m+1} \frac{\cos jx}{j} \\ &- \frac{\cos(m+\frac{1}{2})x\sin x}{2\sin\frac{x}{2}} \sum_{j=2}^{n-m+1} \frac{\cos jx}{j} + D_m(x)\cos x \sum_{j=2}^{m+1} \frac{\cos jx}{j} \\ &+ D_m(x)\cos x \sum_{j=2}^{n-m+1} \frac{\cos jx}{j} = \sum_{k=1}^{7} B_k. \end{aligned} \tag{20}$$

By virtue of (10) for all $x \in [0, \frac{\pi}{2}]$ we obtain the following estimate

$$|B_k| \le \frac{C}{x} \qquad k = 1,2,3. \tag{21}$$

By virtue of estimate (15) we get for $x \in [\frac{1}{m+2}, \pi]$

$$|B_k| \le C\ln(m+2) \qquad k = 4,5. \tag{22}$$

Since for $s \ge 2$

$$\begin{aligned} \sum_{j=2}^{s} \frac{\cos jx}{j} &= \sum_{j=1}^{s-2} \frac{2}{j(j+1)(j+2)} \frac{\sin^2\frac{j+1}{2}x}{2\sin^2\frac{x}{2}} \\ &+ \frac{1}{s(s-1)} \frac{\sin^2\frac{sx}{2}}{2\sin^2\frac{x}{2}} + \frac{1}{s}D_s(x) - \frac{3}{4} - \cos x \end{aligned} \tag{23}$$

then for $m \ge 3$

$$\begin{aligned} B_6 &= D_m(x)\cos x \sum_{j=1}^{m-1} \frac{1}{j(j+1)(j+2)} \frac{\sin^2\frac{j+1}{2}x}{\sin^2\frac{x}{2}} + \frac{D_m(x)\cos x}{m(m+1)} \frac{\sin^2\frac{m+1}{2}x}{2\sin^2\frac{x}{2}} \\ &+ D_m(x)\cos x \frac{1}{m+1} D_{m+1}(x) - \frac{3}{4}D_m(x)\cos x - D_m(x)\cos^2 x = \sum_{i=1}^{5} B_{6,i}. \end{aligned} \tag{24}$$

It is clear that for $x \in [0,\pi]$ by analogical assertions it follows for $j = 2,\dots,5$

$$|B_{6,j}| \le \frac{C}{x}. \tag{25}$$

Since for $0 \le k \le \sqrt{m+1}$ and

$$x \in J_m = \left\{ x : \sin(m+\frac{1}{2})x \ge \frac{1}{2}, \quad \frac{\pi}{2}\frac{1}{m+1} < x < \frac{\pi}{2}\frac{1}{\sqrt{m+1}} \right\}$$

follows $0 \le kx \le \frac{\pi}{2}$ and $\sin kx \ge \frac{2}{\pi}kx$, then we obtain

$$\begin{aligned} B_{6,1} &\ge D_m(x)\cos x \sum_{1\le k\le\sqrt{m+1}} \frac{1}{k(k+1)(k+2)} \frac{\sin^2\frac{k+1}{2}x}{\sin^2\frac{x}{2}} \\ &\ge C\frac{\sin(m+\frac{1}{2})x}{2\sin\frac{x}{2}} \sum_{1\le k\le\sqrt{m+1}} \frac{1}{k} \ge \frac{C}{x}\ln(m+2), \qquad x \in J_m. \end{aligned} \tag{26}$$

We note that if $m = n$ then $B_7 = 0$ and if $m = n-1$ then $B_7 \le \frac{C}{x}$ $\forall x \in [-\pi,\pi]$. For $m < n-1$ we have (see (23))

$$\begin{aligned} B_7 = D_m(x)\cos x &\sum_{j=1}^{n-m-1} \frac{2}{j(j+1)(j+2)} \frac{\sin^2\frac{j+1}{2}x}{2\sin^2\frac{x}{2}} \\ &+ \frac{D_m(x)\cos x}{(n-m+1)(n-m)} \frac{\sin^2\frac{n-m+1}{2}x}{2\sin^2\frac{x}{2}} \\ &+ D_m(x)\cos x \frac{1}{n-m+1} D_{m-n+1}(x) \\ &- \frac{3}{4}D_m(x)\cos x - D_m(x)\cos^2 x = \sum_{i=1}^{5} B_{7,i} \end{aligned} \tag{27}$$

and we obtain the same estimates for $B_{7,i}$, $i = 2,\dots,5$ as for $B_{6,i}$, $i = 2,\dots,5$. We note also that $B_{7,1} > 0$. Thus for $x \in J_m$ and m big enough we have (see (21-22), (24-27))

$$|F_{m,n}(x)| \ge \frac{1}{l(m,n)} \left\{ C\frac{\ln(m+2)}{x} - \frac{C}{x} - C\ln(m+2) \right\} \ge C\frac{\ln(m+2)}{xl(m,n)}.$$

It imply that for m big enough

$$\|F_{m,n}\|_{L^1[-\pi,\pi]} \ge \int_{J_m} |F_{m,n}(x)|dx \ge C\frac{\ln^2(m+2)}{\ln(n+2)} \tag{28}$$

and since for all m and n holds

$$\|F_{m,n}\|_{L^1[-\pi,\pi]} \ge \left| \int_0^{2\pi} F_{m,n}(x)dx \right| = 1 \tag{29}$$

then from (28) and (29) follows the left side estimate in (4). $\square$

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Received March 16, 2005.

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