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A NEW PROOF OF SZABÓ'S THEOREM ON THE RIEMANN-METRIZABILITY OF BERWALD MANIFOLDS

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Abstract. The starting point of the famous *structure theorems on Berwald spaces* due to Z.I. Szabó [4] is an observation on the Riemann-metrizability of positive definite Berwald manifolds. It states that there always exists a Riemannian metric on the underlying manifold such that its Levi-Civita connection is just the canonical connection of the Berwald manifold. In this paper we present a new elementary proof of this theorem. After constructing a Riemannian metric by the help of integration of the canonical Riemann-Finsler metric on the indicatrix hypersurface it is proved that in case of Berwald manifolds the canonical connection and the Levi-Civita connection coincide.

Introduction

Traditionally Berwald manifolds are defined as special Finsler manifolds such that the horizontal part of the canonical Berwald connection depends only on the position. This means that it reduces to the horizontal lift of a linear connection on the underlying manifold. In his paper [4] Z.I. Szabó proved that there always exists a Riemannian metric such that its Levi-Civita connection coincides the canonical (linear) connection of the Berwald manifold. The original reasoning is based on the theory of integration on compact Lie groups with respect to the bi-invariant Haar-measure. We are going to present an elementary proof of this theorem by the help of integration of the canonical Riemann-Finsler metric on the indicatrix hypersurface. The Riemannian metric γ is defined by the formula

$$\gamma(X,Y)(p) := \int_{S_p} g(X^v, Y^v)\, \mu_p, \tag{1}$$

where X and Y are vector fields on the underlying manifold and X^v denotes the vertical lift of the vector field X. The integral is taken with respect to the (oriented) volume form on the indicatrix hypersurface. Our main result states that if the indicatrices of a Finsler manifold are invariant under the parallel transport with respect to a linear connection on the underlying manifold, then it is metrical with respect to the Riemannian metric defined by the formula (1). As a direct consequence we have that in case of Berwald manifolds the canonical (linear) connection of the Finsler manifold and the Levi-Civita connection coincide.

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1. Preliminaries

1.1. **Finsler manifolds.** Let M be a connected differentiable manifold equipped with a function $E\colon TM \to \mathbb{R}$ such that

(i) $\forall v \in TM \setminus \{0\} : E(v) > 0$ and $E(0) = 0$.
(ii) E is homogeneous of degree 2, i.e. $\forall t \in \mathbb{R}^+ : E(tv) = t^2 E(v)$.
(iii) E is of class C^1 on the tangent manifold TM and smooth except the zero section.
(iv) The *fundamental form* $\omega := dd_J E$ is nondegenerate.

The *Riemann-Finsler metric* of the Finsler manifold (M, E) is defined by the formula

$$g(JX, JY) := \omega(JX, Y),$$

where X, Y are vector fields on TM and J is the canonical almost tangent structure on the tangent bundle $\pi\colon TM \to M$; for the details see [2], [3] and [5]. The Finsler manifold is called *positive definite* if g is positive definite.

Remark 1. In what follows we suppose that the Finsler manifold is positive definite without any further comment.

Note that for any point $p \in M$ the restriction $g_p := g|_{\mathcal{T}_pM}$ is a Riemannian metric on the "manifold" $\mathcal{T}_pM := T_pM \setminus \{0\}$ in the usual sense. The indicatrix hypersurface at the point p is defined as follows:

$$\mathrm{S}_p := \{v \in T_pM \mid L(v) = 1, \text{ where } E = \frac{1}{2}L^2\}.$$

1.2. **The gradient operator.** Let a smooth function $\varphi\colon TM \to \mathbb{R}$ be given. Since the fundamental form ω is nondegenerate, there exists a unique vector field $\operatorname{grad}\varphi$ such that

$$\iota_{\operatorname{grad}\varphi}\omega = d\varphi;$$

this vector field is called the gradient of φ. Note that the gradient vector field is smooth only on the splitted tangent manifold

$$\mathcal{T}M := TM \setminus \{0\};$$

in general *differentiability is guaranteed only over $\mathcal{T}M$*, unless otherwise stated.

1.3. **Further formulas.** [2], [3]. Let h be the canonical horizontal endomorphism (the so-called *Barthel endomorphism*) associated with the *canonical spray* $S := -\operatorname{grad} E$; we have

$$\iota_S\omega = -dE, \quad h := \frac{1}{2}\big([J, S] + 1\big).$$

The horizontal endomorphism h determines an almost complex structure F such that

$$F \circ J = h, \quad F \circ h = -J.$$

Using the standard technical tools of tangent bundle differential geometry such as the vertical and complete lifts X^v and X^c of a vector field $X \in \mathfrak{X}(M)$ we define the horizontal lift X^h as follows:

$$X^h := h(X^c) \Rightarrow \; FX^v = X^h, \quad FX^h = -X^v.$$

As it is well-known, *any* horizontal endomorphism induces a (in general non-linear) covariant derivative operator ∇ on the underlying manifold and vice-versa.

Lemma 1. *Let X be a vector field on the manifold M and consider its integral curve*

$$c\colon \mathcal{I} \subset \mathbb{R} \to M$$

starting from a point $c(0) = p \in M$. If a vector field W along c is parallel with respect to the induced operator ∇, then it is an integral curve of the horizontal lift X^h starting from the point $W(0) = v \in T_pM$.

Proof. In terms of local coordinates we have that

$$X^h|_{\pi^{-1}(U)} = X^i \circ \pi\big(\frac{\partial}{\partial x^i} - \Gamma_i^j \frac{\partial}{\partial y^j}\big),$$

where the functions Γ_i^j are the parameters of the horizontal endomorphism with respect to the coordinate system $(U, (u^i)_{i=1}^n)$ on the underlying manifold M- as usual $(\pi^{-1}(U), (x^i, y^i)_{i=1}^n)$ denotes the induced coordinate system on the tangent manifold. Since W is parallel it follows that for any indeces $j \in \{1, \dots, n\}$

$$W^{j\prime} + c^{i\prime}\Gamma_i^j \circ W = 0.$$

Therefore

$$X^h \circ W = c^{i\prime}\big(\frac{\partial}{\partial x^i} \circ W - \Gamma_i^j \circ W \frac{\partial}{\partial y^j} \circ W\big) = c^{i\prime} \frac{\partial}{\partial x^i} \circ W + W^{j\prime} \frac{\partial}{\partial y^j} \circ W =$$
$$= (x^i \circ W)' \frac{\partial}{\partial x^i} \circ W + (y^j \circ W)' \frac{\partial}{\partial y^j} \circ W = \dot{W}$$

as was to be stated. □

1.4. **Berwald manifolds.** [1], [4] and [5]. If the induced covariant derivative operator is linear, then the Finsler manifold is called a *Berwald manifold*. In other words we have a unique linear connection ∇ on the underlying manifold M such that the canonical Barthel endomorphism h coincides the horizontal endomorphism induced by ∇. It is conservative, i.e. the h-covariant derivatives of the energy function E vanish. This means that any linear isomorphism induced by the parallel transport along a curve preserves the Finslerian norm $L(v)$ of the tangent vectors. Therefore the indicatrices are invariant under these isomorphisms. On the other hand, as an easy calculation shows,

$$\tau^*(g|_{\mathcal{T}_qM}) = g|_{\mathcal{T}_pM}, \tag{2}$$

where $\mathcal{T}_pM := T_pM \setminus \{0\}$ and $\tau\colon T_pM \to T_qM$ is the corresponding linear isomorphism induced by the parallel transport with respect to ∇ along a curve joining p and q. Roughly speaking, any *linear* transformation preserving the (Finslerian) norm is an isometry.

1.5. **Integral formulas.** Suppose that the manifold M is orientable and consider a volume form $\eta \in \wedge^n(M)$. Then for any point $p \in M$ we have an orientation represented by η_p on the tangent space T_pM. Let us define the mapping

$$d\mu\colon p \in M \to d\mu_p \in \wedge^n(T_pM)$$

as follows:

$$d\mu_p(X_1^v, \dots, X_n^v) := \begin{cases} \sqrt{\det g(X_i^v, X_j^v)} & \text{if } \eta(X_1, \dots, X_n)(p) > 0 \\ -\sqrt{\det g(X_i^v, X_j^v)} & \text{otherwise;} \end{cases}$$

$d\mu_p$ is called the *(oriented) volume form* on the tangent space T_pM. Let

$$\mu_p := \iota_C d\mu_p$$

be the induced volume form on the indicatrix hypersurface which provides an orientation for the manifold S_p. The integral of a (continuous) function f over S_p is defined as the integral of an $(n-1)$-form on an oriented manifold of dimension $n-1$ as usual:

$$\int_{S_p} f := \int_{S_p} f\,\mu_p.$$

Actually, the orientation was convenient but not necessary in the definition..., for the citation see [7], p. 150. Indeed, if we change the orientation on the manifold M, then the orientation changes on the indicatrix hypersurface. For a moment, let us denote by S_p^+ and S_p^- the manifold S_p equipped with different orientations; we have that

$$\int_{S_p} f := \int_{S_p^+} f\,\mu_p = -\int_{S_p^-} f\,\mu_p = \int_{S_p^-} f\,(-\mu_p).$$

This means that the mapping

$$p \in M \to \int_{S_p} f$$

is well-defined even if there couldn't be nowhere-vanishing n-form on the manifold M.

Lemma 2. *Let f be a (smooth) function on the splitted tangent manifold $\mathcal{T}M$ which is homogeneous of degree* 0. *Then*

$$\int_{B_p} f = \frac{1}{n}\int_{S_p} f,$$

where $B_p := \{v \in T_pM \mid L(v) \le 1\}$ is the "unit ball" at the point $p \in M$.

Proof. Since the form $d\mu_p$ has the homogeneity property

$$\mathcal{L}_C\ d\mu_p = n\ d\mu_p$$

and, by our assumption, $\mathcal{L}_C\ f = 0$, the Stokes' theorem shows that

$$\int_{B_p} f := \int_{B_p} f\,d\mu_p = \frac{1}{n}\int_{B_p} \mathcal{L}_C\ (f d\mu_p) = \frac{1}{n}\int_{B_p} d\ \iota_C\ (f d\mu_p) =$$
$$= \frac{1}{n}\int_{S_p} \iota_C\ (f d\mu_p) = \frac{1}{n}\int_{S_p} f\,\mu_p = \frac{1}{n}\int_{S_p} f$$

as was to be stated. □

2. The proof of Szabó's theorem

Definition 1. Let (M, E) be a positive definite Finsler manifold; the *associated Riemannian metric* is defined by the formula

$$\gamma(X, Y)(p) := \int_{S_p} g(X^v, Y^v);$$

for a similar construction see e.g. [6]. The Levi-Civita connection of this metric is called the *associated linear connection* of the Finsler manifold.

Lemma 3. *Let (M, E) be a positive definite Finsler manifold and suppose that ∇ is a linear connection on the underlying manifold M such that the horizontal endomorphism induced by ∇ is conservative. Then ∇ is metrical with respect to the associated Riemannian metric of the Finsler manifold.*

Proof. As it is well-known, the linear connection ∇ induces a horizontal endomorphism h on the manifold M. In this case for any vector fields $X,\ Z \in \mathfrak{X}(M)$:

$$(\nabla_X Z)^v = [X^h, Z^v]. \tag{3}$$

Since h is conservative, i.e. $d_h E = 0$ we have that the horizontal lift of the linear connection ∇ is h-metrical with respect to the Riemann-Finsler metric. Indeed, for any vector fields $X,\ Y$ and $Z \in \mathfrak{X}(M)$

$$(\mathcal{L}_{X^h} g)(Y^v, Z^v) = [Y^v, [X^h, Z^v]]E + Y^v\big(Z^v(X^h E)\big) \overset{(3)}{=} 0;$$

for the details see [5]. On the other hand

$$(\mathcal{L}_{X^h} g)(Y^v, Z^v) \overset{(3)}{=} X^h g(Y^v, Z^v) - g((\nabla_X Y)^v, Z^v) - g((\nabla_X Z)^v, Y^v)$$

and, consequently,

$$X^h g(Y^v, Z^v) - g((\nabla_X Y)^v, Z^v) - g((\nabla_X Z)^v, Y^v) = 0. \tag{4}$$

Let $p \in M$ be an arbitrary point and consider the integral curve

$$c\colon \mathcal{I} \subset \mathbb{R} \to M, \quad c(0) = p$$

of the vector field X. Then

$$\begin{aligned} X_p \gamma(Y, Z) = \big(\gamma(Y, Z) \circ c\big)'(0) = \lim_{t \to 0} \frac{\gamma(Y, Z)(c(t)) - \gamma(Y, Z)(p)}{t} = \\ = \lim_{t \to 0} \frac{\int_{S_{c(t)}} g(Y^v, Z^v) - \int_{S_p} g(Y^v, Z^v)}{t}. \end{aligned} \tag{5}$$

Let

$$\tau_t \colon T_p M \to T_{c(t)} M$$

is the linear isomorphism induced by the parallel transport with respect to ∇ along the curve c. Since the h-covariant derivative of the energy function vanish it follows that τ_t preserves the Finslerian norm of the tangent vectors. On the other hand it is a linear transformation and, consequently, for any $t \in \mathcal{I}$ the mapping τ_t is an isomorphism, i.e.

$$(\tau_t)^*(g|_{\mathcal{T}_{c(t)} M}) = g|_{\mathcal{T}_p M}. \tag{6}$$

As we have seen above the integral of a function on the indicatrix hypersurface is independent of the orientation around the point p on the underlying manifold. After choosing one such that the collection $(\tau_t)_{t \in \mathcal{I}}$ consists of orientation preserving transformations we have by Lemma 2 that

$$\begin{aligned} &\frac{1}{n} \int_{S_{c(t)}} g(Y^v, Z^v) = \\ &\quad = \int_{B_{c(t)}} g(Y^v, Z^v)\, d\mu_{c(t)} = \int_{(\tau_t)^{-1}(B_{c(t)})} g(Y^v, Z^v) \circ \tau_t\, (\tau_t)^*(d\mu_{c(t)}) \overset{(6)}{=} \\ &\quad = \int_{B_p} g(Y^v, Z^v) \circ \tau_t\, d\mu_p = \frac{1}{n} \int_{S_p} g(Y^v, Z^v) \circ \tau_t. \end{aligned}$$

Substituting this into the equation (5)

$$X_p\ \gamma(Y,Z) = \int_{S_p} \lim_{t\to 0} \frac{g(Y^v,Z^v)\circ\tau_t - g(Y^v,Z^v)}{t}.$$

If W is a parallel vector field along c such that $W(0)=v\in T_pM$, then

$$\lim_{t\to 0} \frac{g(Y^v,Z^v)\circ\tau_t - g(Y^v,Z^v)}{t}(v) = \big(g(Y^v,Z^v)\circ W\big)'(0)$$

and Lemma 1 shows that

$$\lim_{t\to 0} \frac{g(Y^v,Z^v)\circ\tau_t - g(Y^v,Z^v)}{t}(v) = X^h_v g(Y^v,Z^v).$$

Therefore

$$X_p\gamma(Y,Z) - \gamma(\nabla_{X_p}Y,Z) - \gamma(\nabla_{X_p}Z,Y) = \int_{S_p} X^h g(Y^v,Z^v) - g((\nabla_X Y)^v,Z^v) - g((\nabla_X Z)^v,Y^v) \stackrel{(4)}{=} 0$$

as was to be stated. □

Theorem 1. *The canonical connection of a positive definite Berwald manifold is Riemann-metrizable; it is just the Levi-Civita connection of the associated Riemannian metric.*

Proof. Since the canonical connection is conservative and torsion-free, the theorem is a direct consequence of Lemma 3. □

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