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UNIFORM (C,α) $(-1<\alpha<0)$ SUMMABILITY OF FOURIER SERIES WITH RESPECT TO THE WALSH–PALEY SYSTEM

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Abstract. In the present paper we prove a number of statements dealing with the uniform convergence of Cesàro means of negative order of the Fourier–Walsh series.

1. Definitions and Notation

Let $r_0(x)$ be the function defined by

$$r_0(x)=\begin{cases}1 & \text{if } x\in\left[0,\frac{1}{2}\right),\\ -1 & \text{if } x\in\left[\frac{1}{2},1\right),\end{cases}\qquad r_0(x+1)=r_0(x).$$

The Rademacher system is defined by

$$r_n(x)=r_0(2^n x),\quad n\geq 1,\quad \text{and}\quad x\in[0,1).$$

Let $\psi_0(x),\psi_1(x),\psi_2(x),\dots$ represent the Walsh functions, i.e. $\psi_0(x)=1$, and if $k=2^{n_1}+2^{n_2}+\dots+2^{n_s}$ is a positive integer with $n_1>n_2>\dots>n_s$, then

$$\psi_k(x)=r_{n_1}(x)\cdot r_{n_2}(x)\cdots r_{n_s}(x).$$

Denote by $K_n^\alpha(t)$ the kernel of the method (C,α) and call it the Cesàro kernel:

$$K_n^\alpha(t)=\frac{1}{A_n^\alpha}\sum_{\nu=0}^{n}A_{n-\nu}^\alpha\psi_\nu(t),$$

$$A_k^\alpha=\frac{(\alpha+1)(\alpha+2)\cdots(\alpha+k)}{k!}\qquad(\alpha\neq -k).$$

It is well-known ([19, Ch. 3]), that

(I) $\quad A_n^\alpha=\sum_{k=0}^{n}A_{n-k}^{\alpha-1};$

(II) $\quad A_n^\alpha-A_{n-1}^\alpha=A_n^{\alpha-1};$

(III) $\quad A_n^\alpha\sim n^\alpha.$

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By $C(0,1)$ we denote the space of the continuous periodic functions with period 1 and norm

$$\|f\|_C = \max_{0\le x\le 1} |f(x)|.$$

Let $f \in C(0,1)$; the *modulus of continuity* of function f is called the function

$$\omega(\delta, f) = \max_{\substack{|x-y|\le\delta \\ x,y\in[0,1]}} |f(x)-f(y)|, \quad 0\le\delta\le 1.$$

A modulus of continuity is called the nonnegative function ω of the nonnegative argument possessing the following properties:

(1) $\omega(0)=0$;
(2) $\omega(\delta)$ is nondecreasing;
(3) $\omega(\delta)$ is continuous on $[0,1]$;
(4) $\omega(\delta_1+\delta_2)\le\omega(\delta_1)+\omega(\delta_2)$ for $0\le\delta_1\le\delta_2\le\delta_1+\delta_2\le 1$.

Given the modulus of continuity $\omega(\delta)$, by H^ω we denote a set of all those functions $f\in C(0,1)$ for each of which $\omega(\delta,f)=O(\omega(\delta))$ as $\delta\to 0$. If, however, $\omega(\delta)=\delta^\alpha$ $(0<\alpha\le 1)$, then by $\mathrm{Lip}\,\alpha$ we denote a class H^{δ^α}.

Let ϕ be an increasing continuous function on $[0,\infty)$, and $\phi(0)=0$.

By V_ϕ it is denoted the class of bounded on $[0,1]$ functions f for which

$$V_\phi(f) = \sup_\Pi \sum_{k=1}^{n} \phi\left(|f(x_k)-f(x_{k-1})|\right) < \infty,$$

where $\Pi=\{0\le x_0<x_1<x_1,\dots<x_n\le 1\}$ is an arbitrary partitioning of the segment $[0,1]$.

Let $M(0,1)$ denote a class of bounded functions on $[0,1]$. The modulus of variation of the function $f\in M(0,1)$ is called the function of an entire argument $\upsilon(n,f)$ defined as follows: $\upsilon(0,f)=0$, and for $n\ge 1$

$$\upsilon(n,f) = \sup_{\Pi_n} \sum_{k=0}^{n-1} |f(t_{2k+1})-f(t_{2k})| \ ,$$

where Π_n is an arbitrary system of n nonintersecting intervals (t_{2k},t_{2k+1}) $(k=0,1,2,\dots,n-1)$ of the segment $[0,1]$.

The notion of modulus of variation has been introduced by Z. Chanturia [3].

If $\upsilon(n)$ is nondecreasing convex upwards function and $\upsilon(0)=0$, then $\upsilon(n)$ is called the *modulus of variation.* Given the modulus of variation $\upsilon(n)$, by $V[\upsilon]$ we denote the class of those functions which satisfy the relation $\upsilon(n,f)=O(\upsilon(n))$ as $n\to\infty$.

Next, let $f\in C(0,1)$, and $\sigma(f)$ be the Fourier–Walsh–Paley series of that function, i.e.

$$\sigma(f)\sim\sum_{k=0}^{\infty} c_k\psi_k(x), \quad \text{where } c_k=\int_0^1 f(t)\psi_k(t)\,dt, \quad k=0,1,2,\dots.$$

By $\sigma_n^\alpha(f,x)$ we denote Cesàro means, or (C,α) means of the Fourier–Walsh–Paley series of the function f, i.e.

$$\sigma_n^\alpha(f,x)=\frac{1}{A_n^\alpha}\sum_{\nu=0}^{n}A_{n-\nu}^\alpha c_\nu\psi_\nu(t).$$

2. Introduction

N. Fine [5] has proved that for any summable function f the means $\sigma_n^\alpha(x,f)$ converge almost everywhere for all $\alpha>0$, and for any continuous function $\sigma(f)$ is uniformly $(C,1)$ summable to f.

In [18], S. Yano has studied the points (C,α) convergence of the summable function f, namely: if $\lim\limits_{x\to x_0}f(x)=A$, then $\sigma(f)$ is (C,α) summable to A at the point x_0 $(\alpha>0)$. Moreover, Yano has shown that if $f(x)$ satisfies the Lipschitz condition of order α $(0<\alpha<1)$, then for every $\beta>\alpha$

$$|\sigma_n^\beta(f,x)-f(x)|=O(n^{-\alpha}).$$

This result has been somewhat amplified by V. Kokilashvili [12], who found that

$$\|\sigma_{n-1}^\beta(f,x)-f(x)\|_C\le\frac{1}{n}\sum_{\nu=1}^{n}E_\nu(f),\quad \beta\ge1,$$

$E_n(f)$ is the best approximation of $f(x)$ in the metric $C(0,1)$ with the help of polynomials by the Walsh system.

The problems of summability of Cesàro means of positive order for Walsh–Fourier series were studied in [8]–[7].

The questions dealing with the uniform convergence of Cesàro means of negative order were first studied by the author, and the obtained results without proof were published in [17].

Important results in this direction have been obtained by Goginava in [9], [10].

3. Main Results

The main results of the paper are presented in the form of the following propositions.

Theorem 1. *Let $\omega(\delta,f)$ be the modulus of continuity, and let $\upsilon(n,f)$ be the modulus of variation of the function $f\in C(0,1)$.*

If

$$\lim_{n\to\infty}\min_{1\le m\le n}\left\{\omega\left(\frac{1}{n},f\right)\sum_{k=1}^{m}\frac{1}{k^{1-\alpha}}+\sum_{k=m+1}^{n}\frac{\upsilon(k,f)}{k^{2-\alpha}}\right\}=0,\quad 0<\alpha<1,$$

then $\sigma(f)$ is uniformly $(C,-\alpha)$ summable to f.

Theorem 1 can be rewritten in the following equivalent form.

Theorem 2. *Let $\omega(\delta,f)$ be the modulus of continuity, and let $\upsilon(n,f)$ be the modulus of variation of the function f. Then*

$$|\sigma_n^{-\alpha}(f,x)-f(x)| \le c(\alpha)\Bigg\{\omega\left(\frac{1}{n},f\right)\sum_{k=1}^{m_0(n)}\frac{1}{k^{1-\alpha}} + \sum_{k=m_0(n)+1}^{n}\frac{\upsilon(k,f)-\upsilon(k-1,f)}{k^{1-\alpha}}\Bigg\}+o(1),$$

where $m_0(n)=m_0$ is defined by the inequality

$$(1^0)\qquad \frac{\upsilon(m_0+1,f)}{m_0+1}\le\omega\left(\frac{1}{n},f\right)\le\frac{\upsilon(m_0,f)}{m_0}.$$

If $\omega(\frac{1}{n},f)<\frac{\upsilon(n,f)}{n}$, then we take $m_0=n$.

Let $\omega(\delta)$ be an arbitrary modulus of continuity and $\upsilon(n)$ be an arbitrary modulus of variation. Furthermore, let

$$(2^0)\qquad \tau(n)=\omega\left(\frac{1}{n}\right)\sum_{k=1}^{m_0(n)}\frac{1}{k^{1-\alpha}}+\sum_{k=m_0(n)+1}^{n}\frac{\upsilon(k)-\upsilon(k-1)}{k^{1-\alpha}}.$$

where $m_0(n)$ is defined by the relation (1^0) and omitting in it the function f.

Suppose $\overline{\lim\limits_{n\to\infty}}\,\tau(n)=\lim\limits_{i\to\infty}\tau(n_i)=\tau_0>0$. Then the following theorem is valid.

Theorem 3. *In the class $H^\omega\cap V(\upsilon)$ there exists a function f_0, such that*

$$(3^0)\qquad \overline{\lim}\,\frac{|f_0(0)-\sigma_{2^{n_i}}^{-\alpha}(f_0,0)|}{\tau(2^{n_i})}>0.$$

4. Auxiliary Results

We shall need the following

Lemma 1 ([16])**.** *Let*

$$K_n^\alpha(t)=\frac{1}{A_n^\alpha}\sum_{\nu=0}^{n}A_{n-\nu}^\alpha\psi_\nu(t).$$

Then the estimate

$$|K_n^{-\alpha}(t)|\le c(\alpha)\,\frac{1}{A_n^{-\alpha}}\,\frac{1}{t^{1-\alpha}},\quad t\in(0,1),\quad 0<\alpha<1,$$

holds.

Lemma 2 ([16])**.** *For any $\alpha\in(0,1)$ and $p\ge 2^m$ the equality*

$$\operatorname{Sgn}\left(\sum_{\nu=0}^{2^m-1}A_{n-\nu}^{-\alpha}\psi_\nu(t)\right)=\operatorname{Sgn}(\psi_{2^m-1}(t)),\quad t\in[0,1),$$

is valid.

Lemma 3 ([3])**.** *If $f\in C\cap V_\phi$, where $\phi(u)$ is strictly increasing for $u\in[0,\infty)$ and $\phi(0)=0$, then*

$$\upsilon(n,f)\le c(f)n\phi^{-1}\left(\frac{1}{n}\right),\quad n\ge 1.$$

Lemma 4. *If $f \in C \cap V_\phi$, where ϕ satisfies the conditions of Lemma 3, then the following two conditions*

$$\sum_{k=1}^{\infty} \frac{1}{k^{-\alpha}}\,\phi^{-1}\left(\frac{1}{k}\right) < \infty \quad (0<\alpha<1) \tag{1}$$

and

$$\int_0^1 \frac{1}{\phi^{\alpha}(\tau)}\,d\tau < \infty \quad (0<\alpha<1) \tag{2}$$

are equivalent.

Proof. We have

$$\sum_{k=2}^{m} \frac{1}{k^{1-\alpha}}\,\phi^{-1}\left(\frac{1}{k}\right) \le \int_1^m \frac{1}{t^{1-\alpha}}\,\phi^{-1}\left(\frac{1}{t}\right) dt \le \sum_{k=1}^{m-1} \frac{1}{k^{1-\alpha}}\,\phi^{-1}\left(\frac{1}{k}\right). \tag{3}$$

Since

$$\begin{aligned}
\int_1^m \frac{1}{t^{1-\alpha}}\,\phi^{-1}\left(\frac{1}{t}\right) dt &= \int_{\frac{1}{m}}^{1} u^{1-\alpha}\,\phi^{-1}(u)\,\frac{1}{u^2}\,du \\
&= \int_{\frac{1}{m}}^{1} \frac{1}{u^{1+\alpha}}\,\phi^{-1}(u)\,du = \int_{\phi^{-1}(\frac{1}{m})}^{\phi^{-1}(1)} \frac{\tau}{\phi^{1+\alpha}(\tau)}\,\phi'(\tau)\,d\tau \\
&= -\frac{1}{\alpha}\,\frac{\tau}{\phi^{\alpha}(\tau)}\Bigg|_{\phi^{-1}(\frac{1}{m})}^{\phi^{-1}(1)} + \frac{1}{\alpha}\int_{\phi^{-1}(\frac{1}{m})}^{\phi^{-1}(1)} \frac{\tau}{\phi^{1+\alpha}(\tau)}\,\phi'(\tau)\,d\tau \qquad (4) \\
&= -\frac{1}{\alpha}\,\frac{\tau}{\phi^{\alpha}(\tau)}\Bigg|_{\phi^{-1}(\frac{1}{m})}^{\phi^{-1}(1)} + \frac{1}{\alpha}\int_{\phi^{-1}(\frac{1}{m})}^{\phi^{-1}(1)} \frac{\tau}{\phi^{\alpha}(\tau)}\,d\tau \\
&= \frac{1}{\alpha}\,\phi^{-1}\left(\frac{1}{m}\right) m^{\alpha} - \frac{1}{\alpha}\,\phi^{-1}(1) + \frac{1}{\alpha}\int_{\phi^{-1}(\frac{1}{m})}^{\phi^{-1}(1)} \frac{1}{\phi^{\alpha}(\tau)}\,d\tau,
\end{aligned}$$

therefore if the condition (1) is fulfilled, then the condition (2) is likewise fulfilled; and if the condition (2) is fulfilled, then

$$\int_0^1 \frac{1}{\phi^{\alpha}(\tau)}\,d\tau \ge \int_0^{\phi^{-1}(\frac{1}{m})} \frac{d\tau}{\phi^{\alpha}(\tau)} \ge \phi^{-1}\left(\frac{1}{m}\right) m^{\alpha}$$

and by virtue of (3) and (4) the condition (1) is fulfilled. □

Lemma 5. *([15]) Let $f \in C(0,1)$, then for every $\alpha \in (0,1)$ the estimation*

$$\frac{1}{A_n^{-\alpha}}\left|\int_0^1 \sum_{\nu=0}^{2^{k-1}-1} A_{n-\nu}^{-\alpha}\psi_\nu(u)[f(x\dot{+}t) \;-\; f(x)]dt\right| \;\le\; c(\alpha)\sum_{r=0}^{k-1} 2^{r-k}\omega\left(\frac{1}{2^r}, f\right),$$

where $2^k \le n < 2^{k+1}$, holds.

5. Proof of Main Results

Proof of Theorem 1. Represent $n \geq 1$ in the form $n = 2^m + p$, $0 \leq p < 2^m$. As is known (see [4, p. 393]),[1]

$$
\begin{aligned}
\sigma_n^{-\alpha}(f,x) - f(x) &= \int_0^1 K_n^{-\alpha}(t)[f(x\dot{+}t) - f(x)]\,dt \\
&= \frac{1}{A_n^{-\alpha}} \int_0^1 \sum_{\nu=0}^{2^{m-1}-1} A_{n-\nu}^{-\alpha} \psi_\nu(t)[f(x\dot{+}t) - f(x)]\,dt \\
&\quad + \frac{1}{A_n^{-\alpha}} \int_0^1 \sum_{\nu=2^{m-1}}^{2^m-1} A_{n-\nu}^{-\alpha} \psi_\nu(t)[f(x\dot{+}t) - f(x)]\,dt \\
&\quad + \frac{1}{A_n^{-\alpha}} \int_0^1 \sum_{\nu=2^m}^{2^m+p} A_{n-\nu}^{-\alpha} \psi_\nu(t)[f(x\dot{+}t) - f(x)]\,dt \\
&= \frac{1}{A_n^{-\alpha}} \int_0^1 \sum_{\nu=0}^{2^{m-1}-1} A_{n-\nu}^{-\alpha} \psi_\nu(t)[f(x\dot{+}t) - f(x)]\,dt \\
&\quad + \frac{1}{A_n^{-\alpha}} \int_0^1 \sum_{\nu=0}^{2^{m-1}-1} A_{n-2^{m-1}-\nu}^{-\alpha} \psi_\nu(t)[f(x\dot{+}t) - f(x)]\,dt \\
&\quad + \frac{1}{A_n^{-\alpha}} \int_0^1 \sum_{\nu=0}^{p} A_{n-2^m-\nu}^{-\alpha} \psi_{2^m}(t) \psi_\nu(t)[f(x\dot{+}t) - f(x)]\,dt \\
&= A_1 + A_2 + A_3.
\end{aligned} \tag{5}
$$

Estimate A_1. By Lemma 5, we have

$$
\begin{aligned}
|A_1| &= \left| \frac{1}{A_n^{-\alpha}} \int_0^1 \sum_{\nu=0}^{2^{m-1}-1} A_{n-\nu}^{-\alpha} \psi_\nu(t)[f(x\dot{+}t) - f(x)]\,dt \right| \leq \\
&\leq c(\alpha) \sum_{\nu=0}^{m-1} 2^{\nu-m} \omega\left(\frac{1}{2^\nu}, f\right),
\end{aligned}
$$

whence

$$
A_1 = o(1) \quad \text{as} \quad n \to \infty. \tag{6}
$$

[1] $x\dot{+}y$ means that x, y is written as dyadic fractions which are combined pairwise with respect to $[\operatorname{mod} 2]$.

Now we proceed to estimation A_2; first we estimate the sum

$$\sum_{\nu=0}^{2^{m-1}-1} A^{-\alpha}_{n-2^{m-1}-\nu}\psi_\nu(t).$$

We have

$$\begin{aligned}&\left|\sum_{\nu=0}^{2^{m-1}-1} A^{-\alpha}_{n-2^{m-1}-\nu}\psi_\nu(t)\right|\\ &=\left|\sum_{\nu=0}^{n-2^{m-1}} A^{-\alpha}_{n-2^{m-1}-\nu}\psi_\nu(t)-\sum_{\nu=2^{m-1}}^{n-2^{m-1}} A^{-\alpha}_{n-2^{m-1}-\nu}\psi_\nu(t)\right|\\ &\le\left|\sum_{\nu=0}^{p} A^{-\alpha}_{p-\nu}\psi_\nu(t)\right|+\left|\sum_{\nu=0}^{q} A^{-\alpha}_{q-\nu}\psi_\nu(t)\right|,\end{aligned}$$

where $p=n-2^{m-1}$, $q=n-2^m$. This by virtue of Lemma 1 implies that

$$\left|\sum_{\nu=0}^{2^{m-1}-1} A^{-\alpha}_{n-2^{m-1}-\nu}\psi_\nu(t)\right|\le c(\alpha)\,\frac{1}{t^{1-\alpha}}\,. \tag{7}$$

For A_2 we have

$$\begin{aligned}|A_2|&=\frac{1}{A_n^{-\alpha}}\left|\int_0^1\sum_{\nu=0}^{2^{m-1}-1} A^{-\alpha}_{n-2^{m-1}-\nu}\psi_{2^{m-1}}(t)\psi_\nu(t)\ \big(f(x\dot{+}t)-f(x)\big)\ dt\right|\\ &=\frac{1}{A_n^{-\alpha}}\Bigg|\sum_{j=0}^{2^{m-1}-1}\Bigg(\int_{\frac{2j}{2^m}}^{\frac{2j+1}{2^m}}\sum_{\nu=0}^{2^{m-1}-1} A^{-\alpha}_{n-2^{m-1}-\nu}\psi_\nu(t)\ \big(f(x\dot{+}t)-f(x)\big)\ dt\\ &\quad-\int_{\frac{2j+1}{2^m}}^{\frac{2j+2}{2^m}}\sum_{\nu=0}^{2^{m-1}-1} A^{-\alpha}_{n-2^{m-1}-\nu}\psi_\nu(t)\ \big(f(x\dot{+}t)-f(x)\big)\ dt\Bigg)\Bigg|.\end{aligned}$$

Taking into account the fact that the functions $\psi_\nu(t)$ $(\nu=0,1,\dots,2^{m-1}-1)$ are constant in the intervals $\big[\frac{j}{2^{m-1}},\frac{j+1}{2^{m-1}}\big)$, $j=0,1,\dots,2^{m-1}-1$, we find that if $t\in\big[\frac{2j}{2^m},\frac{2j+1}{2^m}\big)$, then $\psi_\nu\big(t+\frac{1}{2^m}\big)=\psi_\nu(t)=\psi_\nu\big(\frac{2j}{2^m}\big)$, and hence

$$|A_2| = \frac{1}{A_n^{-\alpha}}\Bigg|\sum_{j=0}^{2^{m-1}-1}\int\limits_{\frac{2j}{2^m}}^{\frac{2j+1}{2^m}}\sum_{\nu=0}^{2^{m-1}-1} A^{-\alpha}_{n-2^{m-1}-\nu}\psi_\nu(t)\,\big(f(x\dot{+}t)-f(x)\big)$$

$$-\sum_{j=0}^{2^{m-1}-1} A^{-\alpha}_{n-2^{m-1}-\nu}\psi_\nu\left(t+\frac{1}{2^m}\right)\left(f\left(x\dot{+}\left(t+\frac{1}{2^m}\right)\right)-f(x)\right)dt\Bigg|$$

$$= \frac{1}{A_n^{-\alpha}}\Bigg|\sum_{j=0}^{2^{m-1}-1}\int\limits_{\frac{2j}{2^m}}^{\frac{2j+1}{2^m}}\sum_{\nu=0}^{2^{m-1}-1} A^{-\alpha}_{n-2^{m-1}-\nu}\psi_\nu\left(\frac{j}{2^{m-1}}\right)$$

$$\times\left(f(x\dot{+}t)-f\left(x\dot{+}\left(t+\frac{1}{2^m}\right)\right)\right)\,dt\Bigg|$$

$$= \frac{1}{A_n^{-\alpha}}\Bigg|\sum_{j=0}^{2^{m-1}-1}\int\limits_{0}^{\frac{1}{2^m}}\sum_{\nu=0}^{2^{m-1}-1} A^{-\alpha}_{n-2^{m-1}-\nu}\psi_\nu\left(\frac{j}{2^{m-1}}\right)$$

$$\times\left(f\left(x\dot{+}\left(t+\frac{2j}{2^m}\right)\right)-f\left(x\dot{+}\left(t+\frac{2j+1}{2^m}\right)\right)\right)\,dt\Bigg| \tag{8}$$

$$= \frac{1}{A_n^{-\alpha}}\Bigg|\sum_{j=0}^{2^{m-1}-1}2^{-m}\int\limits_{0}^{1}\sum_{\nu=0}^{2^{m-1}-1} A^{-\alpha}_{n-2^{m-1}-\nu}\psi_\nu\left(\frac{j}{2^{m-1}}\right)$$

$$\times\left(f\left(x\dot{+}\frac{t+2j}{2^m}\right)-f\left(x\dot{+}\frac{t+2j+1}{2^m}\right)\right)\,dt\Bigg|$$

$$\le \frac{1}{A_n^{-\alpha}}\Bigg|\sum_{j=1}^{2^{m-1}-1}2^{-m}\int\limits_{0}^{1}\sum_{\nu=0}^{2^{m-1}-1} A^{-\alpha}_{n-2^{m-1}-\nu}\psi_\nu\left(\frac{j}{2^{m-1}}\right)$$

$$\times\left(f\left(x\dot{+}\frac{t+2j}{2^m}\right)-f\left(x\dot{+}\frac{t+2j+1}{2^m}\right)\right)\,dt\Bigg|$$

$$+\frac{1}{A_n^{-\alpha}}2^{-m}\Bigg|\int\limits_0^1\sum_{\nu=0}^{2^{m-1}-1}A^{-\alpha}_{n-2^{m-1}-\nu}\left(f\left(x\dot{+}\frac{t}{2^m}\right)-f\left(x\dot{+}\frac{t+1}{2^m}\right)\right)dt\Bigg|$$

$$= A_2^{(1)} + A_2^{(2)}.$$

It can be seen easily that

$$A_2^{(2)} \le c(\alpha)n^\alpha\,2^{-m}\omega\left(\frac{1}{2^m},f\right)\sum_{\nu=0}^{2^{m-1}-1}A^{-\alpha}_{n-2^{m-1}-\nu}$$

$$\le c(\alpha)\omega\left(\frac{1}{2^m},f\right)n^\alpha 2^{-m}2^{m(1-\alpha)} \le c(\alpha)\omega\left(\frac{1}{2^m},f\right). \tag{9}$$

Estimate now $A_2^{(1)}$. Applying (7), we have

$$A_2^{(1)} \le c(\alpha)n^\alpha 2^{-m}\int\limits_0^1\sum_{j=1}^{2^{m-1}-1}\left(\frac{2^{m-1}}{j}\right)^{1-\alpha}$$

$$\times\left|\left(f\left(x\dot{+}\frac{t+2j}{2^m}\right)-f\left(x\dot{+}\frac{t+2j+1}{2^m}\right)\right)\right|dt$$

$$\leq c(\alpha)\int\limits_0^1\left(\sum_{j=1}^{2^{m-1}-1}\frac{1}{j^{1-\alpha}}\left|f\left(x\dot{+}\frac{t+2j}{2^m}\right)-f\left(x\dot{+}\frac{t+2j+1}{2^m}\right)\right|\right)dt.$$

Estimate the sum

$$A=\sum_{j=1}^{2^{m-1}-1}\frac{1}{j^{1-\alpha}}\left|f\left(x\dot{+}\frac{t+2j}{2^m}\right)-f\left(x\dot{+}\frac{t+2j+1}{2^m}\right)\right|.$$

It is evident that for every $t\in[0,1)$ there exists a $y(t)\in[0,1)$, such that $x\dot{+}\frac{t+q}{2^m}=y\dot{+}\frac{q}{2^m}$, $q=1,2,\dots,2^m-1$.

Thus

$$A=\sum_{j=1}^{2^{m-1}-1}\frac{1}{j^{1-\alpha}}\left|f\left(y\dot{+}\frac{2j}{2^m}\right)-f\left(y\dot{+}\frac{2j+1}{2^m}\right)\right|.$$

Using the Abelian transformation and taking into account (see [19, p. 378]) that $|(x+h)-x|\leq h,\ x,h\in[0,1)$ and (see [3, p. 536]) $\upsilon(n,f)\leq c(f)n\omega(\frac{1}{n},f)$, we obtain

$$\begin{aligned}A&=\sum_{j=1}^{s}\frac{1}{j^{1-\alpha}}\left|f\left(y\dot{+}+\frac{2j}{2^m}\right)-f\left(y\dot{+}+\frac{2j+1}{2^m}\right)\right|\\&\quad+\sum_{j=s+1}^{2^{m-1}-1}\frac{1}{j^{1-\alpha}}\left|f\left(y\dot{+}+\frac{2j}{2^m}\right)-f\left(y\dot{+}+\frac{2j+1}{2^m}\right)\right|\\&\leq\omega\left(\frac{1}{2^m},f\right)\sum_{j=1}^{s}\frac{1}{j^{1-\alpha}}\\&\quad+\sum_{j=s+1}^{2^{m-1}-2}\left(\frac{1}{j^{1-\alpha}}-\frac{1}{(j+1)^{1-\alpha}}\right)\sum_{k=1}^{j}\left|f\left(y\dot{+}\frac{2k}{2^m}\right)-f\left(y\dot{+}\frac{2k+1}{2^m}\right)\right|\\&\quad+\frac{1}{(2^{m-1})^{1-\alpha}}\sum_{j=1}^{2^{m-1}-1}\left|f\left(y\dot{+}\frac{2j}{2^m}\right)-f\left(y\dot{+}\frac{2j+1}{2^m}\right)\right|\\&\quad-\frac{1}{(s+1)^{1-\alpha}}\sum_{j=1}^{s}\left|f\left(y\dot{+}\frac{2j}{2^m}\right)-f\left(y\dot{+}+\frac{2j+1}{2^m}\right)\right|\\&\leq\omega\left(\frac{1}{2^m},f\right)\sum_{j=1}^{s}\frac{1}{j^{1-\alpha}}+\sum_{j=s+1}^{2^{m-1}}\frac{\upsilon(j,f)}{j^{2-\alpha}}+\frac{\upsilon(2^{m-1},f)}{(2^{m-1})^{1-\alpha}}.\end{aligned}$$

Since $\upsilon(n,f)\leq cn\omega\big(\frac{1}{n},f\big)$ and $\frac{\upsilon(n,f)}{n}\downarrow 0$ due to the convexity of $\upsilon(n,f)$ [3], therefore

$$\begin{aligned}&\omega\left(\frac{1}{2^{m-1}},f\right)\sum_{j=1}^{s}\frac{1}{j^{1-\alpha}}+\sum_{j=s+1}^{2^{m-1}}\frac{\upsilon(j,f)}{j^{2-\alpha}}\\&\qquad\geq c\left(\frac{\upsilon(2^{m-1},f)}{2^{m-1}}\sum_{j=1}^{s}\frac{1}{j^{1-\alpha}}\frac{\upsilon(2^{m-1},f)}{2^{m-1}}\sum_{j=s+1}^{2^{m-1}}\frac{1}{j^{1-\alpha}}\right)\\&\qquad\geq c\,\frac{\upsilon(2^{m-1},f)}{2^{m-1}}\sum_{j=1}^{s}\frac{1}{j^{1-\alpha}}\geq c\,\frac{\upsilon(2^{m-1},f)}{(2^{m-1})^{1-\alpha}}\end{aligned}$$

and hence

$$A_2^{(1)} \le c(\alpha)\left(\omega\left(\frac{1}{2^m}, f\right)\sum_{j=1}^{s}\frac{1}{j^{1-\alpha}} + \sum_{j=s+1}^{2^{m-1}-1}\frac{\upsilon(j,f)}{j^{2-\alpha}}\right).$$

It can be easily seen that the last relation is valid for all s, $1 \le s \le n$. Thus finally, for $A_2^{(1)}$ we obtain the estimate

$$A_2^{(1)} \le c(\alpha)\left(\omega\left(\frac{1}{n}, f\right)\sum_{k=1}^{s}\frac{1}{k^{1-\alpha}} + \sum_{k=s+1}^{n}\frac{\upsilon(k,f)}{k^{2-\alpha}}\right). \tag{10}$$

Analogous estimate is obtained for A_3,

$$|A_3| \le c(\alpha)\left(\omega\left(\frac{1}{n}, f\right)\sum_{k=1}^{s}\frac{1}{k^{1-\alpha}} + \sum_{k=s+1}^{n}\frac{\upsilon(k,f)}{k^{2-\alpha}}\right). \tag{11}$$

Taking into account (6), (8), (9), (10) and (11), from (5) we get

$$\begin{aligned}&|\sigma_n^{-\alpha}(x,f) - f(x)| \\ &\le c(\alpha)\min_{1\le m\le n}\left(\omega\left(\frac{1}{n}, f\right)\sum_{k=1}^{m}\frac{1}{k^{1-\alpha}} + \sum_{k=m+1}^{n}\frac{\upsilon(k,f)}{k^{2-\alpha}}\right) + o(1),\end{aligned} \tag{12}$$

where $o(1)$ is the value tending to zero as $n \to \infty$.

This implies that Theorem 1 is valid. □

From Theorem 1 we can obtain a number of corollaries.

Corollary 1. *If $\omega(\delta, f) = O(\delta^\alpha)$ $(0 < \alpha < 1)$, then $\sigma(f)$ is uniformly $(C, -\alpha)$ summable to f.*

Corollary 2. *If $f \in C \cap V[\upsilon]$, and*

$$\sum_{k=1}^{\infty}\frac{\upsilon(k)}{k^{2-\alpha}} < \infty, \quad 0 < \alpha < 1,$$

then $\sigma(f)$ is uniformly $(C, -\alpha)$ summable to f.

Corollary 3. *If $f \in C \cap V_\phi$, where $\phi(u)$ is strictly increasing convex for $u \in [0, \infty)$, $\phi(0) = 0$, and*

$$\sum_{k=1}^{\infty}\frac{1}{k^{1-\alpha}}\,\phi^{-1}\left(\frac{1}{k}\right) < \infty, \quad 0 < \alpha < 1,$$

then $\sigma(f)$ is uniformly $(C, -\alpha)$ summable to f.

Indeed, in the conditions of Corollary 3, by Lemma 3, the relation

$$u(n,f) \le c(f)\, n\phi^{-1}\left(\frac{1}{n}\right), \quad n \ge 1,$$

is valid, and hence

$$\frac{\upsilon(n,f)}{n^{2-\alpha}} \le c(f)\,\frac{n\phi^{-1}(\frac{1}{n})}{n^{2-\alpha}} = c(f)\,\frac{1}{n^{1-\alpha}}\,\phi^{-1}\left(\frac{1}{n}\right)$$

from which, by virtue of Corollary 2, follows Corollary 3.

Corollary 4. *If $f \in C \cap V_\phi$, where ϕ satisfies the conditions of Corollary 3, and*

$$\int\limits_0^1 \frac{1}{\phi^\alpha(\tau)}\,d\tau < \infty, \quad 0<\alpha<1, \tag{13}$$

then $\sigma(f)$ is uniformly $(C,-\alpha)$ summable to f.

Corollary 4 follows directly from Corollary 3 by using Lemma 4.

Corollary 5. *If $f \in C(0,1)$, $m = \min\limits_{0\le t\le 1} f(t)$, $M = \max\limits_{0\le t\le 1} f(t)$, and the Banach indicatrix[2] $N(y,f)$ satisfies the condition*

$$\int\limits_m^M N^\alpha(y,f)\,dy < \infty, \quad 0<\alpha<1,$$

then $\sigma(f)$ is uniformly $(C,-\alpha)$ summable to f.

This corollary follows from Theorem 1 by virtue of the results of [1].

Corollary 6. *Let $f \in C\cap V_\phi$, where $\phi(u)$ is a strictly increasing on $[0,\infty)$ function, $\phi(0)=0$, and (13) is fulfilled, then*

$$|\sigma_n^{-\alpha}(f,x)-f(x)| \le c(\alpha) \int\limits_0^{\omega(\frac{1}{n},f)} \frac{V_\phi(f)}{\phi^\alpha(\tau)}\,d\tau + o(1),$$

where $V_\phi(f)$ is a full ϕ variation of the function f on $[0,1]$.

Corollary 6 follows from Theorem 1 by virtue of the results obtained in [3].

Proof of Theorem 2. Let $m_0<n$ such that

$$\begin{aligned}
&\min_{1\le m\le n}\left(\omega\left(\frac{1}{n},f\right)\sum_{k=1}^{m}\frac{1}{k^{1-\alpha}}+\sum_{k=m+1}^{n}\frac{\upsilon(k,f)}{k^{2-\alpha}}\right)\\
&\qquad=\omega\left(\frac{1}{n},f\right)\sum_{k=1}^{m_0}\frac{1}{k^{1-\alpha}}+\sum_{k=m_0+1}^{n}\frac{\upsilon(k,f)}{k^{2-\alpha}},
\end{aligned}\tag{14}$$

Then

$$\omega\left(\frac{1}{n},f\right)\sum_{k=1}^{m_0}\frac{1}{k^{1-\alpha}}+\sum_{k=m_0+1}^{n}\frac{\upsilon(k,f)}{k^{2-\alpha}} \le \omega\left(\frac{1}{n},f\right)\sum_{k=1}^{m_0-1}\frac{1}{k^{1-\alpha}}+\sum_{k=m_0}^{n}\frac{\upsilon(k,f)}{k^{2-\alpha}}$$

and

$$\omega\left(\frac{1}{n},f\right)\sum_{k=1}^{m_0}\frac{1}{k^{1-\alpha}}+\sum_{k=m_0+1}^{n}\frac{\upsilon(k,f)}{k^{2-\alpha}} \le \omega\left(\frac{1}{n},f\right)\sum_{k=1}^{m_0+2}\frac{1}{k^{1-\alpha}}+\sum_{k=m_0+1}^{n}\frac{\upsilon(k,f)}{k^{2-\alpha}}.$$

The above inequalities imply that

$$\omega\left(\frac{1}{n},f\right)\frac{1}{m_0^{1-\alpha}} \le \frac{\upsilon(m_0,f)}{m_0^{2-\alpha}}, \quad \frac{\upsilon(m_0+1,f)}{(m_0+1)^{2-\alpha}} \le \omega\left(\frac{1}{n},f\right)\frac{1}{(m_0+1)^{1-\alpha}},$$

[2]The Banach indicatrix $N(y,f)$ is a number (finite or infinite) of solutions of the equation $f(x)=y$.

whence

$$\frac{\upsilon(m_0+1,f)}{m_0+1} \le \omega\left(\frac{1}{n},f\right) \le \frac{\upsilon(m_0,f)}{m_0}\,. \tag{15}$$

Because of the fact that $\frac{\upsilon(n,f)}{n}$ is strictly decreasing (see [3, p. 544]) starting from some n_0, then for $n \ge n_0$ the $m_0(n)$ from the relation (15) is defined uniquely. If, however, the minimum in the left-hand side of (14) is attained for $m_0 = n$, we have one relation

$$\omega\left(\frac{1}{n},f\right) \le \frac{\upsilon(m_0,f)}{m_0}\,.$$

Using now the Abelian transformation, we get

$$\sum_{k=m_0+1}^{n} \frac{\upsilon(k,f)-\upsilon(k-1,f)}{k^{1-\alpha}} = \sum_{k=m_0+1}^{n-1}\left(\frac{1}{k^{1-\alpha}} - \frac{1}{(k+1)^{1-\alpha}}\right)\upsilon(k,f)$$
$$+\frac{\upsilon(n,f)}{n^{1-\alpha}} - \frac{\upsilon(m_0,f)}{(m_0+1)^{1-\alpha}} \ge c(\alpha)\sum_{k=m_0+1}^{n-1}\frac{\upsilon(k,f)}{k^{2-\alpha}} - \frac{\upsilon(m_0,f)}{(m_0+1)^{1-\alpha}}\,,$$

whence

$$\sum_{k=m_0+1}^{n}\frac{\upsilon(k,f)}{k^{2-\alpha}} \le c(\alpha)\left(\sum_{n=m_0+1}^{n}\frac{\upsilon(k,f)-\upsilon(k-1,f)}{k^{1-\alpha}} + \frac{\upsilon(m_0,f)}{(m_0+1)^{1-\alpha}}\right), \tag{16}$$

and since

$$\frac{\upsilon(m_0,f)}{(m_0+1)^{1-\alpha}} \le \frac{\upsilon(m_0+1,f)}{m_0+1}\,(m_0+1)^{\alpha} \le 2\omega\left(\frac{1}{n},f\right)m_0^{\alpha},$$

taking into account (12) and (16), we obtain

$$\|\sigma_n^{-\alpha}(f)-f\|_C \le c(\alpha)\left\{\omega\left(\frac{1}{n},f\right)\sum_{k=1}^{m}\frac{1}{k^{1-\alpha}}\right.$$
$$\left.+\sum_{k=m_0+1}^{n}\frac{\upsilon(k,f)-\upsilon(k-1,f)}{k^{1-\alpha}}\right\} + o(1),$$

Thus Theorem 2 is proved. □

To prove Theorem 3 we will need some lemmas.

Lemma 6. *If $\upsilon(n) = o(n^{1-\alpha})$, $\upsilon(n) \to \infty$, as $n \to \infty$, and $\upsilon(n)$ is convex, then there exists the sequence of natural numbers $\{\varphi(n)\}$ possessing the following properties:*

(a) $\varphi(n) = o(n)$ *as* $n \to \infty$*;*

(b) $\displaystyle\sum_{k=\varphi(n)+1}^{n}\frac{\upsilon(k)-\upsilon(k-1)}{k^{1-\alpha}} = o(1)$ *as* $n \to \infty$*.*

Proof. Suppose

$$\varphi(n) = \max\left(m : \frac{\upsilon(m)-\upsilon(m-1)}{m^{1-\alpha}} \ge \frac{1}{n}\right)$$

Because $\upsilon(n)$ is convex, $\upsilon(k)-\upsilon(k-1)\downarrow$, and since $\frac{\upsilon(k)-\upsilon(k-1)}{k^{1-\alpha}} \downarrow 0$, therefore $\varphi(n) \uparrow \infty$.

From the definition of $\varphi(n)$ it follows that

$$\frac{\upsilon(\varphi(n))-\upsilon(\varphi(n)-1)}{\varphi^{1-\alpha}(n)}\geq\frac{1}{n}\quad\text{and}\quad\frac{\upsilon(\varphi(n)+1)-\upsilon(\varphi(n))}{(\varphi(n)+1)^{1-\alpha}}<\frac{1}{n}. \tag{17}$$

Therefore

$$\frac{\varphi^{1-\alpha}(n)}{n}\leq\upsilon(\varphi(n))-\upsilon(\varphi(n)-1)\leq\frac{\upsilon(\varphi(n))}{\varphi(n)},$$

whence

$$\frac{\varphi(n)}{n}\leq\frac{\upsilon(\varphi(n))}{\varphi^{1-\alpha}(n)}=o(1).$$

By virtue of (17) we have

$$\sum_{k=\varphi(n)+1}^{n}\frac{\upsilon(k)-\upsilon(k-1)}{k^{1-\alpha}}\leq(\upsilon(\varphi(n)+1)-\upsilon(\varphi(n)))\sum_{k=\varphi(n)+1}^{n}\frac{1}{k^{1-\alpha}}$$
$$\leq c(\alpha)\,\frac{(\varphi(n)+1)^{1-\alpha}}{n}\,n^{\alpha}\leq c(\alpha)\,\frac{\varphi^{1-\alpha}(n)}{n^{1-\alpha}}=o(1).$$

Thus the lemma is proved. □

Lemma 7. *Let*

$$K_{2^m}^{-\alpha}(t)=\frac{1}{A_{2^m}^{-\alpha}}\sum_{\nu=0}^{2^m}A_{2^m-\nu}^{-\alpha}\psi_\nu(t),\quad 0<\alpha<1.$$

There exists a natural number N such that for $i<m-N$ the estimate

$$\int_{\frac{2^{i-1}}{2^m}}^{\frac{2^i}{2^m}}|K_{2^m}^{-\alpha}(t)|\,dt\geq c(\alpha)2^{i\alpha}$$

is valid for sufficiently large m.

Proof. We have

$$\int_{\frac{2^{i-1}}{2^m}}^{\frac{2^i}{2^m}}|K_{2^m}^{-\alpha}(t)|dt\geq\frac{1}{A_{2^m}^{-\alpha}}\int_{\frac{2^{i-1}}{2^m}}^{\frac{2^i}{2^m}}\left|\sum_{\nu=0}^{2^m-1}A_{2^m-\nu}^{-\alpha}(t)\psi_\nu(t)\right|dt$$
$$-\frac{1}{A_{2^m}^{-\alpha}}\int_{\frac{2^{i-1}}{2^m}}^{\frac{2^i}{2^m}}|A_0^{-\alpha}(t)\psi_{2^m}(t)|dt \tag{18}$$
$$=\frac{1}{A_{2^m}^{-\alpha}}\int_{\frac{2^{i-1}}{2^m}}^{\frac{2^i}{2^m}}\left|\sum_{\nu=0}^{2^m-1}A_{2^m-\nu}^{-\alpha}(t)\psi_\nu(t)\right|dt-\frac{1}{A_{2^m}^{-\alpha}}\frac{2^{i-1}}{2^m}=A_1-A_2.$$

Here we present lower bound of A_1. By Lemma 2, we have

$$A_1=\frac{1}{A_{2^m}^{-\alpha}}\int_{\frac{2^{i-1}}{2^m}}^{\frac{2^i}{2^m}}\left|\sum_{\nu=0}^{2^m-1}A_{2^m-\nu}^{-\alpha}(t)\psi_\nu(t)\right|dt$$

$$= \frac{1}{A_{2^m}^{-\alpha}} \int\limits_{\frac{2^{i-1}}{2^m}}^{\frac{2^i}{2^m}} \Big(\sum_{\nu=0}^{2^m-1} A_{2^m-\nu}^{-\alpha}(t)\psi_\nu(t)\psi_{2^m-1}(t)\Big) dt$$

$$= \frac{1}{A_{2^m}^{-\alpha}} \sum_{k=2^{i-1}}^{2^i-1} \int\limits_{\frac{k}{2^m}}^{\frac{k+1}{2^m}} \Big(\sum_{\nu=0}^{2^m-1} A_{2^m-\nu}^{-\alpha}(t)\psi_\nu(t)\psi_{2^m-1}(t)\Big) dt$$

$$= \frac{1}{A_{2^m}^{-\alpha}} \sum_{\nu=0}^{2^m-1} A_{2^m-\nu}^{-\alpha} \Big(\sum_{k=2^{i-1}}^{2^i-1} \int\limits_{\frac{k}{2^m}}^{\frac{k+1}{2^m}} \psi_\nu(t)\psi_{2^m-1}(t)dt\Big)$$

$$= \frac{1}{A_{2^m}^{-\alpha}} \frac{1}{2^m} \sum_{\nu=0}^{2^m-1} A_{2^m-\nu}^{-\alpha} \Big(\sum_{k=2^{i-1}}^{2^i-1} \psi_\nu\Big(\frac{k}{2^m}\Big)\psi_{2^m-1}\Big(\frac{k}{2^m}\Big)\Big),$$

from which it follows that since (see [2, p. 17])

$$\psi_k\left(\frac{\nu}{2^m}\right) = \psi_\nu\left(\frac{k}{2^m}\right), \quad \nu,k = 0,1,2,\dots,$$

and ([4, p. 379])

$$\psi_k\left(\frac{\nu}{2^m} \dot{+} \frac{2^m-1}{2^m}\right) = \psi_k\left(\frac{\nu}{2^m}\right)\psi_k\left(\frac{2^m-1}{2^m}\right), \quad \nu,k = 1,2,\dots,2^m-1,$$

we get

$$A_1 = \frac{1}{A_{2^m}^{-\alpha}} \frac{1}{2^m} \sum_{\nu=0}^{2^m-1} A_{2^m-\nu}^{-\alpha} \left(\sum_{k=2^{i-1}}^{2^i-1} \psi_k\left(\frac{\nu}{2^m}\right)\psi_k\left(\frac{2^m-1}{2^m}\right)\right)$$

$$= \frac{1}{A_{2^m}^{-\alpha}} \frac{1}{2^m} \sum_{\nu=0}^{2^m-1} A_{2^m-\nu}^{-\alpha} \left(\sum_{k=2^{i-1}}^{2^i-1} \psi_k\left(\frac{\nu}{2^m} \dot{+} \frac{2^m-1}{2^m}\right)\right)$$

$$= \frac{1}{A_{2^m}^{-\alpha}} \frac{1}{2^m} \sum_{\nu=0}^{2^m-1} A_{2^m-\nu}^{-\alpha} \left[D_{2^i}\left(\frac{\nu}{2^m} \dot{+} \frac{2^m-1}{2^m}\right) - D_{2^{i-1}}\left(\frac{\nu}{2^m} \dot{+} \frac{2^m-1}{2^m}\right)\right].$$

Since (see [4])

$$D_{2^m}(t) = \begin{cases} 2^m, & 0 \le t < 2^{-m}, \\ 0, & 2^{-m} \le t < 1, \end{cases} \quad \text{where} \quad D_n(t) = \sum_{k=0}^{n-1} \psi_k(t), \tag{19}$$

therefore for $\nu < 2^{m-1} + \dots + 2^{m-i+1}$ we will have

$$D_{2^i}\left(\frac{\nu}{2^m} \dot{+} \frac{2^m-1}{2^m}\right) = 0, \quad D_{2^{i-1}}\left(\frac{\nu}{2^m} \dot{+} \frac{2^m-1}{2^m}\right) = 0$$

and hence

$$A_1 = \frac{1}{A_{2^m}^{-\alpha}} \frac{1}{2^m} \Bigg\{ \sum_{\nu=p_i+q_i+1}^{2^m-1} A_{2^m-\nu}^{-\alpha} \Bigg(D_{2^i}\left(\frac{\nu}{2^m} \dot{+} \frac{2^m-1}{2^m}\right) - D_{2^{i-1}}\left(\frac{\nu}{2^m} \dot{+} \frac{2^m-1}{2^m}\right)\Bigg) + \sum_{\nu=p_i}^{p_i+q_i} A_{2^m-\nu}^{-\alpha} \left(D_{2^i}\left(\frac{\nu}{2^m} \dot{+} \frac{2^m-1}{2^m}\right) - D_{2^{i-1}}\left(\frac{\nu}{2^m} \dot{+} \frac{2^m-1}{2^m}\right)\right)\Bigg\},$$

where $p_i = 2^{m-1} + 2^{m-2} + \dots + 2^{m-i+1} = 2^m - 2^{m-i+1}$, $q_i = 2^{m-i+1} + 2^{m-i+2} + \dots + 2^1 + 2^0 = 2^{m-i} - 1$.

Taking again into account (19) and equalities $A_k^{\alpha} - A_{k-1}^{\alpha} = A_k^{\alpha-1}$, we obtain

$$
\begin{aligned}
A_1 &= \frac{1}{A_{2^m}^{-\alpha}} \frac{2^{i-1}}{2^m} \left(\sum_{\nu=2^m-2^{m-i}}^{2^m-1} A_{2^m-\nu}^{-\alpha} - \sum_{\nu=2^m-2^{m-i+1}}^{2^m-2^{m-i}} A_{2^m-\nu}^{-\alpha} \right) \\
&= \frac{1}{A_{2^m}^{-\alpha}} \frac{2^{i-1}}{2^m} \left(\sum_{\nu=0}^{2^{m-i}-1} A_{2^{m-i}-\nu}^{-\alpha} - \sum_{\nu=0}^{2^{m-i}} A_{2^{m-i+1}-\nu}^{-\alpha} \right) \\
&= \frac{-1}{A_{2^m}^{-\alpha}} \frac{2^{i-1}}{2^m} \Bigg\{ \sum_{\nu=0}^{2^{m-i}-1} \left(A_{2^{m-i}-\nu}^{-\alpha-1} A_{2^{m-i}+1-\nu}^{-\alpha-1} + \dots + A_{2^{m-i+1}-\nu}^{-\alpha-1} \right) \\
&\quad + A_{2^{m-i}}^{-\alpha} \Bigg\} = -\frac{1}{A_{2^m}^{-\alpha}} \frac{2^{i-1}}{2^m} \sum_{\nu=0}^{2^{m-i}-1} \sum_{k=0}^{2^{m-i}} A_{2^{m-i}+k-\nu}^{-\alpha-1} \\
&\quad - \frac{1}{A_{2^m}^{-\alpha}} \frac{2^{i-1}}{2^m} A_{2^{m-i}}^{-\alpha} = B_1 + B_2.
\end{aligned} \tag{20}
$$

Estimate now B_1. By virtue of (III) we have

$$
\begin{aligned}
|B_1| &\geq c(\alpha)(2^m)^{\alpha} \frac{2^{i-1}}{2^m} \sum_{\nu=0}^{2^{m-i}-1} (2^{m-i} - \nu)^{-\alpha} \\
&\geq c(\alpha) 2^{m\alpha} (2^{m-i})^{1-\alpha} \geq c_0(\alpha) 2^{i\alpha}.
\end{aligned} \tag{21}
$$

For B_2 we have

$$
|B_2| \leq c(\alpha) 2^{m\alpha} \frac{2^{i-1}}{2^m} \leq c(\alpha) \frac{2^i}{2^{m(1-\alpha)}} . \tag{22}
$$

Similar estimate is obtained for A_2. That is,

$$
A_2 \leq c(\alpha) \frac{2^i}{2^{m(1-\alpha)}} . \tag{23}
$$

Taking into account (20), (21), (22) and (23), from (18) we obtain

$$
\int_{\frac{2^{i-1}}{2^m}}^{\frac{2^i}{2^m}} |K_{2^m}^{-\alpha}(t)|\, dt \geq c_0(\alpha) 2^{i\alpha} - c_1(\alpha) \frac{2^i}{2^{m(1-\alpha)}} .
$$

Since $2^{-(m-i)(1-\alpha)} \to 0$ as $(m-i) \to \infty$, there exists a natural number N such that

$$
c_1(\alpha) \frac{2^i}{2^{m(1-\alpha)}} < \frac{c_0(\alpha)}{2} 2^{i\alpha} \text{ for } i < m - N,
$$

and therefore

$$
\int_{\frac{2^{i-1}}{2^m}}^{\frac{2^i}{2^m}} |K_{2^m}^{-\alpha}(t)|\, dt \geq \frac{c_0(\alpha)}{2} 2^{i\alpha}, \quad i < m - N.
$$

Thus the lemma is proved. $\square$

Proof of Theorem 3. Let $\tau(n)$ be defined by the relation (2^0), where $m_0(n)$ for $n > n_0$ is defined uniquely by inequality (15) by omitting the function f.

Next, let $\{n_i\} \subset \mathbb{N}$ such that $\lim\limits_{i\to\infty} \tau(2^{n_i}) = \tau_0 > 0$.

Without restriction of generality we can assume (see [3, p. 1545]) that $\upsilon(n)$ is convex, and $\upsilon(n) \le cn\omega(\frac{1}{n})$. There can take place two cases: (1) $\upsilon(n) = o(n^{1-\alpha})$; (2) $\upsilon(n) \ne o(n^{1-\alpha})$.

Let us consider the case $\upsilon(n) = o(n^{1-\alpha})$. Suppose that $\upsilon(n) \to \infty$ because $\tau(n) \to 0$ as $n \to \infty$, otherwise. By Lemma 6, there exists the sequence of natural numbers $\{\varphi(n)\}$ with the properties indicated in the lemma. Note that if $\upsilon(n) = o(n^{1-\alpha})$ as $n \to \infty$, then

$$\omega\left(\frac{1}{n}\right)\sum_{k=1}^{m_0(n)} \frac{1}{k^{1-\alpha}} \to 0, \quad n \to \infty. \tag{24}$$

Indeed,

$$\omega\left(\frac{1}{n}\right)\sum_{k=1}^{m_0(n)} \frac{1}{k^{1-\alpha}} \le c(\alpha)\,\frac{\upsilon(m_0(n))}{m_0(n)}\,m_0^{\alpha}(n)$$
$$= c(\alpha)\,\frac{\upsilon(m_0(n))}{m_0^{1-\alpha}(n)} = o(1), \quad n \to \infty.$$

Therefore by virtue of the fact that $\lim \tau(2^{n_i}) = \tau_0 > 0$, starting from some i_0

$$\varphi(2^{n_i}) > m_0(2^{n_i}).$$

Taking into account (24) and also the properties of the sequence $\{\varphi(n)\}$, we find that

$$\begin{aligned}
&\lim_{i\to\infty}\left\{\omega\left(\frac{1}{2^{n_i}}\right)\sum_{k=1}^{m_0(2^{n_i})}\frac{1}{k^{1-\alpha}} + \sum_{k=m_0(2^{n_i})+1}^{2^{n_i}}\frac{\upsilon(k)-\upsilon(k-1)}{k^{1-\alpha}}\right\}\\
&= \lim_{i\to\infty}\omega\left(\frac{1}{2^{n_i}}\right)\sum_{k=1}^{m_0(2^{n_i})}\frac{1}{k^{1-\alpha}} + \lim_{i\to\infty}\sum_{k=m_0(2^{n_i})+1}^{\varphi(2^{n_i})}\frac{\upsilon(k)-\upsilon(k-1)}{k^{1-\alpha}}\\
&\quad + \lim_{i\to\infty}\sum_{k=\varphi(2^{n_i})+1}^{2^{n_i}}\frac{\upsilon(k)-\upsilon(k-1)}{k^{1-\alpha}}\\
&= \lim_{i\to\infty}\sum_{k=m_0(2^{n_i})+1}^{\varphi(2^{n_i})}\frac{\upsilon(k)-\upsilon(k-1)}{k^{1-\alpha}} = \tau_0.
\end{aligned} \tag{25}$$

Suppose

$$\xi(n) = \sum_{k=2^{r(n)+3}}^{2^{q(n)+4}} \frac{\upsilon(k)-\upsilon(k-1)}{k^{1-\alpha}}\,,$$

where $r(n) = [\log_2 m_0(n)]$, $q(n) = [\log_2 \varphi(n)]$.

Now we construct the sequence of natural numbers $\{\ell_k\}$ and the sequence of functions $\{f_k\}$.

Let $\ell_1 \in \{n_i\}$ such that $\ell_1 > n_{i_0}$, and $\ell_1 > n_0$ (see the definition of n_0 and i_0), $2^{\ell_1} - \varphi(2^{\ell_1}) > N$ (N appears in Lemma 7) and $\frac{4\varphi(2^{\ell_1})}{2^{\ell_1}} \le 1$.

The function $\varphi_1(x)$ is defined as follows:

$$\varphi_1(x)=\begin{cases} \upsilon(r+1)-\upsilon(r) & \text{for } x=\frac{2r+1}{2^{\ell_1+2}},\ \ r=m_0(2^{\ell_1}),\dots,4\varphi(2^{\ell_1})-1,\\ 0 & \text{for } x=\frac{r}{2^{\ell_1+1}},\ \ r=m_0(2^{\ell_1}),\dots,4\varphi(2^{\ell_1}),\\ 0 & \text{for } x\in\big[0,\frac{m_0(2^{\ell_1})}{2^{\ell_1}}\big]\cup\big[\frac{4\varphi(2^{\ell_1})}{2^{\ell_1}},1\big]\\ \text{is linear and continuous for the rest } x \text{ from } [0,1]. \end{cases}$$

Assume

$$f_1(x)=\varphi_1(x)\operatorname{Sgn}K_{2^{\ell_1}}^{-\alpha}(x).$$

Let the numbers $\ell_1,\ell_2,\dots,\ell_{k-1}$ and periodic with period 1 functions $f_1,f_2,\dots,f_{k-1}$ be already constructed. Then ℓ_k and f_k can be constructed as follows: we choose ℓ_k in such a way that the following conditions be fulfilled:

(1) $\ell_k>\ell_{k-1}$;

(2) $\ell_k\in\{n_i\}$;

(3) $\frac{4\varphi(2^{\ell_k})}{2^{\ell_k}}<\frac{m_0(2^{\ell_k-1})}{2^{\ell_k-1}}$ $(\varphi(n)=o(n))$;

(4) $\omega\left(\frac{1}{2^{\ell_k}}\right)\le\upsilon(4\varphi(2^{\ell_k-1}))-\upsilon(4\varphi(2^{\ell_k-1})-1)$;

(5) $m_0(2^{\ell_k})>\varphi(2^{\ell_k-1})$;

(6) $\omega\left(\frac{1}{2^{\ell_k}}\right)m_0^{\alpha}(2^{\ell_{k-1}})\le c_1(\alpha)\xi(\ell_{k-1})$;

(7) $\sum\limits_{i=1}^{k-1}\omega\left(\frac{1}{2^{\ell_i}}\right)\varphi(2^{\ell_i})\int\limits_0^{\omega(\frac{1}{2^{\ell_k}})}\frac{1}{\tau^{\alpha}}\,d\tau<c_2(\alpha)\xi(2^{\ell_k})$

(the constants $c_1(\alpha)$ and $c_2(\alpha)$ will be chosen below).

Let

$$\varphi_k(x)=\begin{cases} \upsilon(r+1)-\upsilon(r) \ \ \text{for } x=\frac{2r+1}{2^{\ell_k+2}},\ \ r=m_0(2^{\ell_k}),\dots,4\varphi(2^{\ell_k})-1,\\ 0 \ \ \text{for } x=\frac{r}{2^{\ell_k+1}},\ \ r=m_0(2^{\ell_k}),\dots,4\varphi(2^{\ell_k}),\\ 0 \ \ \text{for } x\in\big[0,\frac{m_0(2^{\ell_k})}{2^{\ell_k}}\big]\cup\big[\frac{4\varphi(2^{\ell_k})}{2^{\ell_k}},1\big]\\ \text{is linear and continuous for the rest } x \text{ from } [0,1]. \end{cases}$$

Assume

$$f_k(x)=\varphi_k(x)\operatorname{Sgn}K_{2^{\ell_k}}^{-\alpha}(x).$$

Let now

$$f_0(x)=\sum_{k=1}^{\infty}f_k(x)$$

Reasoning just as in [3, p. 548], we can show that $f_0\in H^{\omega}\cap V[\upsilon]$. It remains to prove the relation (3^0).

Suppose

$$F_k(x)=\sum_{i=k+1}^{\infty}f_i(x).$$

Taking into account the definition of the function $f_k(x)$ and Lemma 1, we obtain

$$|\sigma_{2^{\ell_k}}^{-\alpha}(F_k,0)-F_k(0)|=\left|\int_0^1 F_k(t)K_{2^{\ell_k}}^{-\alpha}(t)dt\right|$$

$$\leq \|F_k\|_C \frac{1}{A^{-\alpha}_{2^{\ell_k}}} \int\limits_0^{\frac{m_0(2^{\ell_k})}{2^{\ell_k}}} \frac{1}{t^{1-\alpha}}\, dt \leq c(\alpha)\, \omega\left(\frac{1}{2^{\ell_k+1}}\right) 2^{\ell_k\alpha} \left(\frac{m_0(2^{\ell_k})}{2^{\ell_k}}\right)^{\alpha}$$

$$\leq c_0(\alpha)\omega\left(\frac{1}{2^{\ell_k+1}}\right) m_0^{\alpha}(2^{\ell_k}).$$

By virtue of the property (6) of the sequence $\{\ell_k\}$, we get

$$|\sigma^{-\alpha}_{2^{\ell_k}}(F_k, 0)| \leq c_0(\alpha)c_1(\alpha)\xi(2^{\ell_k}), \tag{26}$$

Further, using Lemma 7 and the definition of $f_k(x)$, we can write

$$|\sigma^{-\alpha}_{2^{\ell_k}}(f_k, 0) - f_k(0)| = \left| \int\limits_{\frac{m_0(2^{\ell_k})}{2^{\ell_k}}}^{\frac{4\varphi(2^{\ell_k})}{2^{\ell_k}}} f_k(t) K^{-\alpha}_{2^{\ell_k}}(t)dt \right|$$

$$= \int\limits_{\frac{m_0(2^{\ell_k})}{2^{\ell_k}}}^{\frac{4\varphi(2^{\ell_k})}{2^{\ell_k}}} \varphi_k(t)|K^{-\alpha}_{2^{\ell_k}}(t)|dt \geq \int\limits_{\frac{2^{r(2^{\ell_k})+1}}{2^{\ell_k}}}^{\frac{2^{q(2^{\ell_k})+2}}{2^{\ell_k}}} \varphi_k(t)|K^{-\alpha}_{2^{\ell_k}}(t)|dt$$

$$\geq \sum_{i=r(2^{\ell_k})+1}^{q(2^{\ell_k})+1} \int\limits_{\frac{2^i}{2^{\ell_k}}}^{\frac{2^{i+1}}{2^{\ell_k}}} \varphi_k(t)|K^{-\alpha}_{2^{\ell_k}}(t)|dt \tag{27}$$

$$\geq c(\alpha) \sum_{i=r(2^{\ell_k})+1}^{q(2^{\ell_k})+1} [\upsilon(2^{i+2}+1) - \upsilon(2^{i+2})]2^{i\alpha}$$

$$\geq c_3(\alpha) \sum_{i=2^{r(2^{\ell_k})+3}+1}^{2^{q(2^{\ell_k})+4}} \frac{\upsilon(i) - \upsilon(i-1)}{i^{1-\alpha}} = c_3(\alpha)\xi(2^{\ell_k})$$

since

$$\sum_{i=2^{r(2^{\ell_k})+3}+1}^{2^{q(2^{\ell_k})+4}} \frac{\upsilon(i) - \upsilon(i-1)}{i^{1-\alpha}} \leq c(\alpha) \sum_{i=2^{r(2^{\ell_k})+3}}^{2^{q(2^{\ell_k})+4}} \frac{\upsilon(i+1) - \upsilon(i)}{i^{1-\alpha}}$$

$$= c(\alpha) \sum_{i=r(2^{\ell_k})+1}^{q(2^{\ell_k})+1} \sum_{j=2^{i+2}}^{2^{i+3}} \frac{\upsilon(j+1) - \upsilon(j)}{j^{1-\alpha}}$$

$$\leq c(\alpha) \sum_{i=r(2^{\ell_k})+1}^{q(2^{\ell_k})+1} [\upsilon(2^{i+2}+1) - \upsilon(2^{i+2})]2^{i\alpha}.$$

Estimate now $\sigma^{-\alpha}_{2^{\ell_k}}(g_k, 0)$, where $g_k(t) = \sum\limits_{i=1}^{k} f_i(t)$.

Since $g_k(t) \in H^{\omega} \cap V$, using Corollary 6, we find that

$$|\sigma^{-\alpha}_{2^{\ell_k}}(g_k, 0)| \leq c(\alpha) \int\limits_0^{\omega(\frac{1}{2^{\ell_k}})} \frac{V(g_k)}{\tau^{\alpha}}\, d\tau + o(1).$$

It is easy to see that

$$V(g_k)\le 2\sum_{i=1}^{k-1}\omega\left(\frac{1}{2^{\ell_i}}\right)\varphi(2^{\ell_i}),$$

and therefore

$$|\sigma_{2^{\ell_k}}^{-\alpha}(g_k,0)|\le c_1(\alpha)\sum_{i=1}^{k-1}\omega\left(\frac{1}{2^{\ell_i}}\right)\varphi(2^{\ell_i})\int\limits_0^{\omega(\frac{1}{2^{\ell_k}})}\frac{1}{\tau^\alpha}\,d\tau+o(1).$$

Taking into account the property (7) of the sequence $\{\varphi(n)\}$, we have

$$|\sigma_{2^{\ell_k}}^{-\alpha}(g_k,0)|\le c_4(\alpha)c_2(\alpha)\xi(2^{\ell_k})+o(1). \tag{28}$$

It follows from (26), (27) and (28) that

$$\begin{aligned}|\sigma_{2^{\ell_k}}^{-\alpha}(f_0,0)-f_0(0)|&=|\sigma_{2^{\ell_k}}^{-\alpha}(f_k,0)+\sigma_{2^{\ell_k}}^{-\alpha}(g_k,0)+\sigma_{2^{\ell_k}}^{-\alpha}(F_k,0)|\\&\ge|\sigma_{2^{\ell_k}}^{-\alpha}(f_k,0)|-|\sigma_{2^{\ell_k}}^{-\alpha}(F_k,0)|-|\sigma_{2^{\ell_k}}^{-\alpha}(g_k,0)|\\&\ge c_3(\alpha)\xi(2^{\ell_k})-x_0(\alpha)c_1(\alpha)\xi(2^{\ell_k})-c_4(\alpha)c_2(\alpha)\xi(2^{\ell_k})-o(1).\end{aligned}$$

Choosing now $c_1(\alpha)=\frac{c_3(\alpha)}{3c_0(\alpha)}$, $c_2(\alpha)=\frac{c_3(\alpha)}{3c_4(\alpha)}$, we get

$$|\sigma_{2^{\ell_k}}^{-\alpha}(f_0,0)-f_0(0)|\ge\frac{c_3(\alpha)\xi(2^{\ell_k})}{3}-o(1),$$

whence

$$\frac{|\sigma_{2^{\ell_k}}^{-\alpha}(f_0,0)-f_0(0)|}{\xi(2^{\ell_k})}\ge\frac{c_3(\alpha)}{3}-\frac{o(1)}{\xi(2^{\ell_k})}$$

and since $\xi(2^{\ell_k})\sim\tau(2^{\ell_k})$ by virtue of (25), from the last relation we obtain (3^0).

Consider now case 2), i.e. $\upsilon(n)\ne o(n^{1-\alpha})$. We define the function $\upsilon_1(x)$ as follows:

$$\upsilon_1(x)=\begin{cases}\upsilon(n)\ \text{ for }\ x=n,\ \ n\in\mathbb{N},\\ 0\ \text{ for }\ x=0,\\ \text{is linear and continuous for the rest } x \text{ from } [0,\infty).\end{cases}$$

Let

$$\omega_1(\delta)=\delta\upsilon_1\left(\frac{1}{\delta}\right).$$

It is easily seen that $\omega_1(\delta)$ is the modulus of continuity, and since $\upsilon(n)\le cn\omega(\frac{1}{n})$, therefore $\omega_1(\delta)\le c\omega(\delta)$. By S. Stechkin's lemma, there exists the convex modulus of continuity $\omega_0(\delta)$ satisfying the condition

$$c_1\omega_0(\delta)<\omega_1(\delta)<c_2\omega_0(\delta),$$

and since $\upsilon_1(n)=\upsilon(n)$, we have $H^{\omega_0}\subset H^\omega\cap V[\upsilon]$.

By virtue of the fact that $\upsilon(n)\ne o(n^{1-\alpha})$, it follows that $\omega_0(\delta)\ne o(\delta^\alpha)$ $(0<\alpha<1)$. Let us now show that in the class H^{ω_0} there exists the function f_0 whose Fourier–Walsh–Paley series fails to be summable by the method $(C,-\alpha)$ to f_0 at the point $x=0$.

By the condition $\omega_0(\delta) \neq o(\delta^\alpha)$, there exists the sequence $\{n_i\} \subset \mathbb{N}$, such that
$$\omega\left(\frac{1}{2^{n_i}}\right) 2^{n_i\alpha} \geq c > 0, \quad i = 1, 2, \dots .$$
We choose from $\{n_i\}$ the sequence with the following properties:

1) $\sum\limits_{i=k+1}^{\infty} \omega\left(\frac{1}{2^{\ell_i}}\right) \leq 2\omega\left(\frac{1}{2^{\ell_k}}\right)$;

2) $2^{\ell_{k+1}}\omega\left(\frac{1}{2^{\ell_{k+1}}}\right) > 2 \cdot 2^{\ell_k}\omega\left(\frac{1}{2^{\ell_k}}\right)$ since $2^{\ell_k}\omega\left(\frac{1}{2^{\ell_k}}\right) \to \infty$;

3) $\left(\sum\limits_{i=1}^{k-1} \omega\left(\frac{1}{2^{\ell_i}}\right) 2^{\ell_i}\right)^{\alpha} \omega\left(\frac{1}{2^{\ell_k}}\right) \leq c_0(\alpha)\omega\left(\frac{1}{2^{\ell_k}}\right) 2^{\ell_k\alpha}$ ($c_0(\alpha)$ depends on our choice).

Let
$$\varphi_k(t) = \begin{cases} \omega\left(\frac{1}{2^{\ell_k}}\right) \text{ for } x = \frac{2r+1}{2^{\ell_k+2}}, \ r = 0, 1, , \dots, 2^{\ell_k+1} - 1, \\ 0 \text{ for } x = \frac{r}{2^{\ell_k+1}}, \ r = 0, 1, , \dots, 2^{\ell_k+1}, \\ \text{is linear and continuous for the rest } x \text{ from } [0, 1]. \end{cases}$$
Suppose
$$f_0(x) = \sum_{k=1}^{\infty} \varphi_k(t) \operatorname{Sgn} K^{-\alpha}_{2^{\ell_k}}(t).$$
Reasoning analogously as in case 1), we can show that f_0 is the unknown function. Thus Theorem 3 is proved. □

From Theorems 1 and 3 follows

Theorem 4. *For the Fourier–Walsh–Paley series of all functions of the class* $H^\omega \cap V[\upsilon]$ *to be* $(C, -\alpha)$ *uniformly summable, it is necessary and sufficient that the condition*
$$\lim_{n\to\infty} \min_{1\leq m\leq n} \left\{\omega\left(\frac{1}{n}\right) \sum_{k=1}^{m} \frac{1}{k^{1-\alpha}} + \sum_{k=m+1}^{n} \frac{\upsilon(k)}{k^{2-\alpha}}\right\} = 0$$
be fulfilled.

Theorem 5. *For the Fourier–Walsh–Paley series of all functions of the class* $C \cap V[\upsilon]$ *to be uniformly* $(C, -\alpha)$ *summable, it is necessary and sufficient that the condition*
$$\sum_{k=1}^{\infty} \frac{\upsilon(k)}{k^{2-\alpha}} < \infty. \quad 0 < \alpha < 1, \tag{29}$$
be fulfilled.

Proof. The sufficiency of the condition (29) is contained in Theorem 4, and the necessity can be proved by using [3] (see [3, p. 552]). □

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