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HERZ-TYPE BESOV SPACES ON LOCALLY COMPACT VILENKIN GROUPS

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Abstract. Let G be a locally compact Vilenkin group. In this paper the characterizations of the Herz-type Besov space on G are obtained. And some properties of this space are discussed.

1. Introduction

M. Frazier and B. Jawerth give the characterizations of Besov space on $\mathbb{R}^n$ in [3]. For additional results, see [4], the atomic decomposition of Besov spaces on locally compact Vilenkin groups G is obtained by C.W. Onneweer and Su Weiyi. Xu ([7]) introduced the Herz-type Besov spaces on $\mathbb{R}^n$ and give an unified approach for Herz-type Besov spaces and Triebel–Lizorkin spaces. We can also find the associate results in [5], [2], [6]. These papers were the motivation for the present paper in which we consider the characterizations of this Herz-type Space on locally compact Vilenkin groups G.

Throughout this paper, G will denote a bounded locally compact Vilenkin group, that is, G is a locally compact Abelian group containing a strictly decreasing sequence of compact open subgroups $\{G_n\}_{n=-\infty}^{\infty}$ such that (a) $\cup_{n=-\infty}^{\infty} G_n = G$ and $\cap_{n=-\infty}^{\infty} G_n = 0$; (b) $\sup\{$ order $(G_n/G_{n+1}) : n \in \mathrm{Z}\} = B < \infty$. We choose Haar measure dx on G so that $|G_0| = 1$, where $|A|$ denotes the measure of a measurable subset A of G. Let $|G_n| = (m_n)^{-1}$ for each $n \in \mathrm{Z}$. Since $2m_n \le m_{n+1} \le Bm_n$ for each $n \in \mathrm{Z}$, it follows that

$$\sum_{n=k}^{\infty} (m_n)^{-\alpha} \le c(m_k)^{-\alpha}$$

and

$$\sum_{n=-\infty}^{k} (m_n)^{\alpha} \le c(m_k)^{\alpha}$$

for any $\alpha > 0, k \in \mathrm{Z}$, where c is a constant independent of k. For each $n \in \mathrm{Z}$ we choose elements $z_{l,n} \in G(l \in \mathrm{Z}_+)$ so that the subsets $G_{l,n} := z_{l,n} + G_n$ of G satisfy $G_{k,n} \cap G_{l,n} = \emptyset$ if $k \ne l$ and $\cup_{l=0}^{\infty} G_{l,n} = G$; moreover, we choose $z_{0,n}$ so that $G_{0,n} = G_n$. We now define the function $d : G \times G \to R$ by $d(x, y) = 0$ if $x - y = 0$ and $d(x, y) = (m_n)^{-1}$ if $x - y \in G_n \backslash G_{n+1}$, then d defines a metric on $G \times G$ and the topology on G introduced by this metric is the same as the original topology on G. For $x \in G$, we set $|x| = d(x, 0)$. Then $|x| = (m_n)^{-1}$ if and only if $x \in G_n \backslash G_{n+1}$.

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We now briefly recall the definitions of the spaces $S(G)$ of test functions and $S'(G)$ of distributions; for more details, see [6]. A function $\phi : G \to C$ belongs to $S(G)$ if there exist integers k, l, depending on ϕ, so that $\operatorname{supp}\phi \subset G_k$ and ϕ is constant on the cosets of the subgroup G_l of G. And the space of all continuous functionals on $S(G)$ will denoted by $S'(G)$.

In this paper, the authors characterize the Herz-type homogeneous Besov spaces $\dot{K}_q^{\alpha,p}\dot{B}_\beta^s(G)$. In fact, when $\alpha = 0$, $q = p$, $\dot{K}_q^{\alpha,p}\dot{B}_\beta^s(G) = \dot{B}_\beta^{s,p}(G)$. then it turns to be the case which was discussed in [4]. According to the definition of locally compact Vilenkin groups and its topological structure, we can give the atomic decomposition of this Herz-type space on $\mathbb{R}^n$.

2. Herz-type Homogeneous and Nonhomogeneous Besov Spaces

We first recall the definitions of the homogeneous (non-homogeneous) Herz spaces and Besov spaces on G.

Definition 1. Let $\alpha \in \mathbb{R}$, $0 < p, q \le \infty$.

(a) A measurable function $f : G \to R$ belongs to the homogeneous Herz space if it satisfies

$$\|f\|_{\dot{K}_q^{\alpha,p}(G)} = \Big\{ \sum_{l=-\infty}^{\infty} m_l^{-\alpha p} \|f\chi_{G_l\setminus G_{l+1}}\|_{L^q(G)}^p \Big\}^{1/p} < \infty,$$

a modification will be done if $p = \infty$ or $q = \infty$ (That is, if $p = \infty$, $\|f\|_{\dot{K}_q^{\alpha,\infty}(G)} = \sup_l m_l^{-\alpha}\|f\chi_{G_l\setminus G_{l+1}}\|_{L^q(G)}$).

(b) A measurable function $f : G \to R$ belongs to the non-homogeneous Herz space if it satisfies

$$\|f\|_{K_q^{\alpha,p}(G)} = \Big\{ \|f\chi_{G_0}\|_{L^q(G)}^p + \sum_{l=-\infty}^{-1} m_l^{-\alpha p} \|f\chi_{G_l\setminus G_{l+1}}\|_{L^q(G)}^p \Big\}^{1/p} < \infty,$$

a modification will be done if $p = \infty$ or $q = \infty$.

Before giving the definition of the Herz-type Besov spaces on G, we give the notes of a second space of test functions and distributions([4]). Let

$$Z(G) = \{\psi \in S(G) : \hat{\psi}(0) = \int_G \psi(t)dt = 0\},$$

and define convergence in $Z(G)$ like in $S(G)$. Let $Z'(G)$ be the space of linear functionals on $Z(G)$ with convergence in $Z'(G)$ defined like in $S'(G)$. If ϱ denotes the set of constant distributions in $S'(G)$ then $Z'(G)$ can be identified with $S'(G)/\varrho$ in the sense that (i) for each $f \in S'(G)$ its restriction to $Z(G)$ belongs to $Z'(G)$; (ii) if $f, g \in S'(G)$ and if $g = f + c$ for some constant $c \in S'(G)$ then the restrictions of f and g to $Z(G)$ determine the same element of $Z'(G)$; (iii) if $\tilde{f} \in Z'(G)$ then there exists an $f \in S'(G)$ so that its restriction to $Z(G)$ equals $\tilde{f}$, moreover, module constants f is determined uniquely by $\tilde{f}$. We usually disregard the difference between $\tilde{f} \in Z'(G)$ and a corresponding $f \in S'(G)$.

Set $\chi_n(x) = \chi_{G_n\setminus G_{n+1}}(x)$, $\Delta_n(x) = m_n\chi_n(x)$, $\varphi_n(x) = \Delta_n(x) - \Delta_{n+1}(x)$, and $*$ denote convolution operator. Now we give the definition of the Herz-type homogeneous Besov spaces on G.

Definition 2. Let $\alpha \in \mathbb{R}$, $0 < p < \infty$, $0 < q \le \infty$. The Besov space $\dot{B}_q^{\alpha,p}(G)$ is defined as

$$\dot{B}_q^{\alpha,p}(G) = \{f \in Z'(G) | \|f\|_{\dot{B}_q^{\alpha,p}(G)} := \Big(\sum_{n=-\infty}^{\infty} (m_n)^{\alpha q} \|f * \varphi_n\|_p^q \Big)^{1/q} < \infty\},$$

with the usual modification if $q = \infty$.

Definition 3. Let $\alpha, s \in \mathbb{R}$, $0 < p, q, \beta \leq \infty$. Then

$$\dot{K}_q^{\alpha,p}\dot{B}_\beta^s(G) = \{f \in Z'(G) | \|f\|_{\dot{K}_q^{\alpha,p}\dot{B}_\beta^s(G)} := (\sum_{n=-\infty}^{\infty}((m_n)^s\|f*\varphi_n\|_{\dot{K}_q^{\alpha,p}})^\beta)^{1/\beta} < \infty\},$$

with the usual modification if $\beta = \infty$ (That is, if $\beta = \infty$, $\|f\|_{\dot{K}_q^{\alpha,p}\dot{B}_\infty^s(G)} = \sup_n (m_n)^s\|f*\varphi_n\|_{\dot{K}_q^{\alpha,p}}$).

Definition 4. Let $\alpha, s \in \mathbb{R}$, $0 < p, q, \beta \leq \infty$. Then

$$\begin{aligned}\dot{K}_q^{\alpha,p}B_\beta^s(G) = \{f \in Z'(G) | &\|f\|_{\dot{K}_q^{\alpha,p}B_\beta^s(G)} \\ &:= \|f*\Delta_0\|_{\dot{K}_q^{\alpha,p}} + \{\sum_{n=1}^{\infty}((m_n)^s\|f*\varphi_n\|_{\dot{K}_q^{\alpha,p}})^\beta\}^{1/\beta} < \infty\},\end{aligned}$$

with the usual modification if $\beta = \infty$.

In this section, we first consider the link between the homogeneous and nonhomogeneous space. In [4, Theorem 3] it was shown that for the Besov space on G we have $B_{p,\beta}^s = L^p \cap \dot{B}_{p,\beta}^s$ when $s > 0$ and $1 \leq p, \beta \leq \infty$. For Herz-type spaces, we have similar results.

Theorem 1. *Let $s > 0$ and $1 \leq \beta, p, q < \infty$, Then*

$$\dot{K}_q^{\alpha,p}B_\beta^s(G) = \dot{K}_q^{\alpha,p}\dot{B}_\beta^s(G) \cap \dot{K}_q^{\alpha,p}(G).$$

Proof. If $f \in \dot{K}_q^{\alpha,p}\dot{B}_\beta^s(G) \cap \dot{K}_q^{\alpha,p}(G)$, then $\sum_{l=-\infty}^{\infty} m_l^{-\alpha p}\|f\chi_l\|_q^p \leq \infty$. Since

$$\|f*\Delta_0\|_{\dot{K}_q^{\alpha,p}}^p = \sum_{l=-\infty}^{\infty} m_l^{-\alpha p}\|f*\Delta_0\chi_l\|_q^p,$$

and

$$\begin{aligned}\|f*\Delta_0\chi_l\|_q^p &= (\int_{G_l\setminus G_{l+1}}|f*\Delta_0(x)|^q dx)^{\frac{p}{q}} \\ &\leq (\int_{G_0}(\int_{G_l\setminus G_{l+1}}|f(x-t)|^q dx)^{\frac{1}{q}}|\Delta_0(t)|dt)^p \\ &\leq \|f\chi_l\|_q^p(\int_{G_0}|\Delta_0(t)|dt)^p \\ &\leq \|f\chi_l\|_q^p,\end{aligned}$$

therefore, $\|f*\Delta_0\|_{\dot{K}_q^{\alpha,p}}^p \leq \sum_{l=-\infty}^{\infty} m_l^{-\alpha p}\|f\chi_l\|_q^p = \|f\|_{\dot{K}_q^{\alpha,p}}^p$, moreover,

$$\begin{aligned}\|f\|_{\dot{K}_q^{\alpha,p}B_\beta^s(G)} &= \|f*\Delta_0\|_{\dot{K}_q^{\alpha,p}} + \{\sum_{n=1}^{\infty}((m_n)^s\|f*\varphi_n\|_{\dot{K}_q^{\alpha,p}})^\beta)\}^{1/\beta} \\ &\leq \|f\|_{\dot{K}_q^{\alpha,p}} + \{\sum_{n=-\infty}^{\infty}((m_n)^s\|f*\varphi_n\|_{\dot{K}_q^{\alpha,p}})^\beta)\}^{1/\beta} \\ &= \|f\|_{\dot{K}_q^{\alpha,p}} + \|f\|_{\dot{K}_q^{\alpha,p}\dot{B}_\beta^s} \\ &< \infty.\end{aligned}$$

Conversely, take any $f \in \dot{K}_q^{\alpha,p}B_\beta^s$, since $f \in S'(G)$ we have

$$f = f*\Delta_0 + \sum_{n=1}^{\infty} f*\varphi_n$$

with convergence in $S'(G)$. If $1 \le \beta < \infty$, using the inequalities of Minkowski and Hölder, we have

$$\begin{aligned}\|\sum_{n=1}^{\infty} f*\varphi_n\|_{\dot{K}_q^{\alpha,p}} &\le \sum_{n=1}^{\infty}(m_n)^{-s}(m_n)^s\|f*\varphi_n\|_{\dot{K}_q^{\alpha,p}}\\ &= (\sum_{n=1}^{\infty}(m_n)^{-s\beta'})^{\frac{1}{\beta'}}(\sum_{n=1}^{\infty}(m_n^s\|f*\varphi_n\|_{\dot{K}_q^{\alpha,p}})^{\beta})^{\frac{1}{\beta}}\\ &\le c\|f\|_{\dot{K}_q^{\alpha,p}B_\beta^s},\end{aligned}$$

here β' is the conjugate index of β. Thus, we can conclude that $f \in \dot{K}_q^{\alpha,p}$.

Moreover, as the proof of the first part, through the simple calculation, we have $\|f*\varphi_n\chi_l\|_q^p \le c\|f\chi_l\|_q^p$, consequently,

$$\begin{aligned}\|f\|^{\beta}_{\dot{K}_q^{\alpha,p}\dot{B}_\beta^s} &= c(\sum_{n=-\infty}^{0} m_n^{s\beta}\|f\|^{\beta}_{\dot{K}_q^{\alpha,p}} + \sum_{n=1}^{\infty} m_n^{s\beta}\|f*\varphi_n\|^{\beta}_{\dot{K}_q^{\alpha,p}})\\ &\le c\|f\|^{\beta}_{\dot{K}_q^{\alpha,p}B_\beta^s} < \infty\end{aligned}$$

This completes the proof of Theorem 1. □

In the following, $A_1 \subset A_2$ always means that the topological space A_1 is continuously embedded in the topological space A_2.

Using the proposition 2.2.1 in [8] and the embedding properties of Herz space(see [1]), Theorem 2 is easy to be proved. Here we omit the proof.

Theorem 2. *Let $-\infty < s < \infty$, $0 < p, q < \infty$ and $\alpha > -\frac{1}{q}$.*

(i) If $0 < \beta_1 \le \beta_2 \le \infty$, then

$$K_q^{\alpha,p}B^s_{\beta_1}(G) \subset K_q^{\alpha,p}B^s_{\beta_2}(G) \text{ and } \dot{K}_q^{\alpha,p}B^s_{\beta_1}(G) \subset \dot{K}_q^{\alpha,p}B^s_{\beta_2}(G).$$

(ii) If $0 < \beta_1, \beta_2 \le \infty$, $\varepsilon > 0$, then

$$K_q^{\alpha,p}B^{s+\varepsilon}_{\beta_1}(G) \subset K_q^{\alpha,p}B^s_{\beta_2}(G) \text{ and } \dot{K}_q^{\alpha,p}B^{s+\varepsilon}_{\beta_1}(G) \subset \dot{K}_q^{\alpha,p}B^s_{\beta_2}(G).$$

(iii) If $\alpha_1 < \alpha_2$, then $K_q^{\alpha_2,p}B^s_\beta(G) \subset K_q^{\alpha_1,p}B^s_\beta(G)$.

(iv) If $b \le c$, then $K_q^{\alpha,b}B^s_\beta(G) \subset K_q^{\alpha,c}B^s_\beta(G)$ and $\dot{K}_q^{\alpha,b}B^s_\beta(G) \subset \dot{K}_q^{\alpha,c}B^s_\beta(G)$.

(v) If $q_1 \le q_2$, then $K_{q_2}^{\alpha,p}B^s_\beta(G) \subset K_{q_1}^{r,p}B^s_\beta(G)$ and $\dot{K}_{q_2}^{\alpha,p}B^s_\beta(G) \subset \dot{K}_{q_1}^{r,p}B^s_\beta(G)$ where $r = \alpha - (\frac{1}{q_1} - \frac{1}{q_2})$.

Theorem 3. *Let $\alpha = \frac{1}{p} - \frac{1}{q}$, $0 < p \le q < \infty$, $0 < \beta \le \infty$, $s \in \mathbb{R}$, then $\dot{K}_q^{\alpha,p}B^s_\beta(G) \subset B^s_{p,\beta}(G)$.*

Proof. Let $D_j = G_j \backslash G_{j+1}$. Using the Hölder inequality, we have

$$\begin{aligned}\|f\|^p_{L^p(G)} &= \sum_{j=-\infty}^{\infty}\int_{D_j}|f(x)|^p dx\\ &\le C\sum_{j=-\infty}^{\infty}|D_j|^{1-\frac{p}{q}}\left(\int_{D_j}|f(x)|^q dx\right)^{\frac{p}{q}}\\ &= C\sum_{j=-\infty}^{\infty}m_j^{\frac{p}{q}-1}\left(\int_{D_j}|f(x)|^q dx\right)^{\frac{p}{q}}\\ &= C\sum_{j=-\infty}^{\infty}m_j^{-\alpha p}\left(\int_{D_j}|f(x)|^q dx\right)^{\frac{p}{q}}\\ &= C\|f\|^p_{\dot{K}_q^{\alpha,p}(G)}.\end{aligned}$$

Therefore, $\dot{K}_q^{\alpha,p}B_\beta^s(G) \subset B_{p,\beta}^s(G)$. □

Theorem 4. *Let $0 < p < \infty$, $0 < q < \infty$, $0 < r \le q$, $0 < \beta \le \infty$, $s \in \mathbb{R}$, and $0 < r < p < \infty$, $\alpha > \frac{1}{r} - \frac{1}{q}$, or $0 < p \le r < \infty$ and $\alpha \ge \frac{1}{r} - \frac{1}{q}$. Then $K_q^{\alpha,p}B_\beta^s(G) \subset B_{r,\beta}^s(G)$.*

Proof. Suppose $D_0 = G_0$, $D_j = G_j \backslash G_{j+1}$. By the Hölder inequality, we can obtain

$$\begin{aligned}
\|f\|_{L^r(G)}^r &= \sum_{j=-\infty}^{0} \int_{D_j} |f(x)|^p dx \\
&\le C \sum_{j=-\infty}^{0} |D_j|^{1-\frac{r}{q}} \Big(\int_{D_j} |f(x)|^q dx \Big)^{\frac{r}{q}} \\
&= C \sum_{j=-\infty}^{0} m_j^{\frac{r}{q}-1} \Big(\int_{D_j} |f(x)|^q dx \Big)^{\frac{r}{q}}
\end{aligned}$$

If $0 < r < p < \infty$, $\alpha > \dfrac{1}{r} - \dfrac{1}{q}$, then

$$\begin{aligned}
\|f\|_{L^r(G)}^r &\le C \bigg\{ \sum_{j=-\infty}^{0} m_j^{-\alpha p} \Big(\int_{D_j} |f(x)|^q dx \Big)^{\frac{p}{q}} \bigg\}^{r/p} \\
&\quad \times \bigg\{ \sum_{j=-\infty}^{0} m_j^{(\frac{1}{q}-\frac{1}{r}+\alpha)rp/(p-r)} \bigg\}^{1-r/p} \\
&\le C \bigg\{ \sum_{j=-\infty}^{0} m_j^{-\alpha p} \Big(\int_{D_j} |f(x)|^q dx \Big)^{\frac{p}{q}} \bigg\}^{r/p} \\
&\le C \|f\|_{K_q^{\alpha,p}(G)}^r.
\end{aligned}$$

We can deduce that $m_j \le m_0 = 1$ since $j \le 0$, then if $0 < p \le r < \infty$, $\alpha \ge \frac{1}{r} - \frac{1}{q}$, we have

$$\begin{aligned}
\|f\|_{L^r(G)}^r &\le C \bigg\{ \sum_{j=-\infty}^{0} m_j^{(\frac{1}{q}-\frac{1}{r})p} \Big(\int_{D_j} |f(x)|^q dx \Big)^{\frac{p}{q}} \bigg\}^{r/p} \\
&\le C \bigg\{ \sum_{j=-\infty}^{0} m_j^{-\alpha p} \Big(\int_{D_j} |f(x)|^q dx \Big)^{\frac{p}{q}} \bigg\}^{r/p} \\
&\le C \|f\|_{K_q^{\alpha,p}(G)}^r.
\end{aligned}$$

Furthermore, $K_q^{\alpha,p}B_\beta^s(G) \subset B_{r,\beta}^s(G)$. □

3. Atomic Decomposition of the Herz-type Homogeneous Spaces

The atomic decomposition of the homogeneous Besov space on G were obtained by Onneweer and Su [4, Theorem 6]. Motivated by their work, we will give the atomic decomposition of the Herz-type Homogeneous Besov space.

Definition 5. A function $a : G \to C$ is an (s, ∞) atom, $s \in \mathbb{R}$, if
(i) a is supported on a set $z + G_n$ for some $z \in G$ and $n \in \mathrm{Z}$,
(ii) $|a(x)| \le (m_n)^s$,
(iii) $\int_G a(x)dx = 0$.

We have the following result.

Theorem 5. *Let $0 < p, \beta \le \infty$, $\alpha > -\frac{1}{q}$, $s \in \mathbb{R}$. Then the following two facts are equivalent.*

(a) $f \in \dot{K}_q^{\alpha,p}\dot{B}_\beta^s(G)$,

(b) there exist constants $\lambda_{l,j}$, $l \in Z_+$ and $j \in Z$, and $(-(s-\alpha)+1/q,\infty)$ atoms $a_{i,j}$ with $\operatorname{supp} a_{l,j} \subset z_{l,j} + G_{j-1}$ such that

$$f = \sum_{j=-\infty}^{\infty} \sum_{l=0}^{\infty} \lambda_{l,j} a_{l,j} \quad in \quad S'(G)/\varrho$$

Moreover,

$$\|\lambda\|_{p,\beta} := (\sum_{j=-\infty}^{\infty} (\sum_{l=0}^{\infty} |\lambda_{l,j}|^p)^{\frac{\beta}{p}})^{1/\beta} \le c\|f\|_{\dot{K}_q^{\alpha,p}\dot{B}_\beta^s}.$$

Proof. $(a) \Rightarrow (b)$ For each $f \in S'/\varrho$ we have

$$\begin{aligned} f &= \sum_{n=-\infty}^{\infty} f * \varphi_n(x) \\ &= \sum_{n=-\infty}^{\infty} f * \varphi_n * \varphi_n(x) \\ &= \sum_{n=-\infty}^{\infty} \sum_{l=0}^{\infty} (m_n)^{-1} f * \varphi_n(z_{l,n})\varphi_n(x - z_{l,n}) \\ &= \sum_{n=-\infty}^{\infty} \sum_{l=0}^{\infty} \lambda_{l,n} a_{l,n} \end{aligned}$$

where $\lambda_{l,n} = (m_{n-1})^{(s-\alpha)-1/q} f * \varphi_n(z_{l,n})$ and

$$a_{l,n} = (m_{n-1})^{-(s-\alpha)+1/q} m_n^{-1} \varphi_n(x - z_{l,n}).$$

For each $a_{l,n}$ we have

(i) $\operatorname{supp} a_{l,n} \subset z_{l,n} + G_{n-1}$;

(ii) $|a_{l,n}(x)| \le (m_{n-1})^{-(s-\alpha)+1/q}$;

(iii) $\int_G a_{l,n}(x)dx = 0$.

Thus $a_{l,n}$ is an $(-(s-\alpha)+1/q,\infty)$ atom on G with support in $z_{l,n} + G_{n-1}$. Moreover, for the $\lambda_{l,n}$ we have

$$\begin{aligned} \|\lambda\|_{p,\beta} &= (\sum_{n=-\infty}^{\infty} (\sum_{l=0}^{\infty} |(m_{n-1})^{(s-\alpha)-1/q} f * \varphi_n(z_{l,n})|^p)^{\frac{\beta}{p}})^{\frac{1}{\beta}} \\ &\le c(\sum_{n=-\infty}^{\infty} m_n^{s\beta} (\sum_{k=-\infty}^{\infty} m_k^{-\alpha p} \|f * \varphi_n \chi_k\|_q^p)^{\frac{\beta}{p}})^{\frac{1}{\beta}} \\ &= c\|f\|_{\dot{K}_q^{\alpha,p}\dot{B}_\beta^s} < \infty. \end{aligned}$$

This completes the proof of (a).

$(b) \Rightarrow (a)$ Let $a_{l,j}$ be an $(-(s-\alpha)+1/q,\infty)$ atom on G with support in $z_{l,j}+G_j$. Similar to the proof in [4], we have $a_{l,j} * \varphi_n = 0$ when $j \ge n$ for each $x \in G$. So we only consider the case of $j < n$.

If $k+1 \le j$, then $\|(a_{l,j} * \varphi_n)\chi_k\|_q = 0$ since $\operatorname{supp} a_{l,j} * \varphi_n \subset G_j$. If $k > j$, we have $\|(a_{l,j} * \varphi_n)\chi_k\|_q \le c m_j^{-(s-\alpha)+\frac{1}{q}} m_k^{-\frac{1}{q}}$.

In the following, we estimate $\|f\|_{\dot{K}_q^{\alpha,p}\dot{B}_\beta^s}$. If $f = \sum_{j=-\infty}^{\infty} \sum_{l=0}^{\infty} \lambda_{l,j} a_{l,j}$ and $\|\lambda\|_{p,\beta} < \infty$,

(i) If $0 < p \le 1$, then for each $n \in \mathrm{Z}$,

$$\|f*\varphi_n\|^p_{\dot K^{\alpha,p}_q} \le \sum_{j=-\infty}^{n-1}\sum_{l=0}^{\infty}|\lambda_{l,j}|^p\|a_{l,j}*\varphi_n\|^p_{\dot K^{\alpha,p}_q}$$

Therefore, for each β with $0 < \beta < \infty$ we have

$$\begin{aligned}\|f\|_{\dot K^{\alpha,p}_q\dot B^s_\beta} &= \{\sum_{n=-\infty}^{\infty} m_n^{s\beta}(\sum_{j=-\infty}^{n-1}\sum_{l=0}^{\infty}|\lambda_{l,j}|^p\|a_{l,j}*\varphi_n\|^p_{\dot K^{\alpha,p}_q})^{\frac{\beta}{p}}\}^{\frac{1}{\beta}}\\ &= \{\sum_{n=-\infty}^{\infty} m_n^{s\beta}(\sum_{j=-\infty}^{n-1}\sum_{l=0}^{\infty}|\lambda_{l,j}|^p\sum_{k=j}^{\infty}m_k^{-\alpha p}\|a_{l,j}*\varphi_n\chi_k\|^p_q)^{\frac{\beta}{p}}\}^{\frac{1}{\beta}}\\ &\le c\{\sum_{n=-\infty}^{\infty} m_n^{s\beta}(\sum_{j=-\infty}^{n-1}\sum_{l=0}^{\infty}|\lambda_{l,j}|^p m_j^{-sp})^{\frac{\beta}{p}}\}^{\frac{1}{\beta}}\end{aligned}$$

since $\alpha + 1/q > 0$.

If $0 < \beta \le p$, so that $0 < \beta/p \le 1$, then

$$\begin{aligned}\|f\|_{\dot K^{\alpha,p}_q\dot B^s_\beta} &\le c\{\sum_{n=-\infty}^{\infty} m_n^{s\beta}\sum_{j=-\infty}^{n-1}m_j^{-s\beta}(\sum_{l=0}^{\infty}|\lambda_{l,j}|^p)^{\frac{\beta}{p}}\}^{\frac{1}{\beta}}\\ &\le c\|\lambda\|_{p,\beta} < \infty\end{aligned}$$

here $s < 0$.

If $p < \beta < \infty$, let $r = \frac{\beta}{p}$. Using the Hölder inequality, we can obtain

$$\begin{aligned}\|f\|_{\dot K^{\alpha,p}_q\dot B^s_\beta} &\le c\{\sum_{n=-\infty}^{\infty} m_n^{s\beta}(\sum_{j=-\infty}^{n-1}\sum_{l=0}^{\infty}|\lambda_{l,j}|^p m_j^{-sp})^{\frac{\beta}{p}}\}^{\frac{1}{\beta}}\\ &\le c\{\sum_{n=-\infty}^{\infty} m_n^{s\beta}(\sum_{j=-\infty}^{n-1}m_j^{-sp})^{\frac{r}{r'}}\sum_{j=-\infty}^{n-1}m_j^{-sp}(\sum_{l=0}^{\infty}|\lambda_{l,j}|^p)^{\frac{\beta}{p}}\}^{\frac{1}{\beta}}\\ &\le c\{\sum_{n=-\infty}^{\infty} m_{n-1}^{sp}\sum_{j=-\infty}^{n-1}m_j^{-sp}(\sum_{l=0}^{\infty}|\lambda_{l,j}|^p)^{\frac{\beta}{p}}\}^{\frac{1}{\beta}}\\ &\le c\|\lambda\|_{p,\beta} < \infty.\end{aligned}$$

Here r' is the conjugate index of r.

If $\beta = \infty$, since $s < 0$ and

$$\begin{aligned}\|f\|_{\dot K^{\alpha,p}_q\dot B^s_\beta} &= \sup_n m_n^s\|f*\varphi_n\|_{\dot K^{\alpha,p}_q}\\ &\le \sup_n m_n^s(\sum_{j=-\infty}^{n-1}\sum_{l=0}^{\infty}|\lambda_{l,j}|^p m_j^{-sp})^{\frac{1}{p}}\end{aligned}$$

moreover,

$$\begin{aligned}\sum_{j=-\infty}^{n-1}m_j^{-sp}\sum_{l=0}^{\infty}|\lambda_{l,j}|^p &\le \sup_j(\sum_{l=0}^{\infty}|\lambda_{l,j}|^p)\sum_{j=-\infty}^{n-1}m_j^{-sp}\\ &\le cm_n^{-sp}\sup_j(\sum_{l=0}^{\infty}|\lambda_{l,j}|^p).\end{aligned}$$

Hence,

$$\|f\|_{\dot K^{\alpha,p}_q\dot B^s_\beta} \le c\sup_n m_n^s m_n^{-s}\sup_j(\sum_{l=0}^{\infty}|\lambda_{l,j}|^p)^{1/p} = c\|\lambda\|_{p,\beta} < \infty.$$

(ii) If $1 < p < \infty$, similar to the proof of Theorem 6 in [4], for each $n \in \mathrm{Z}$, $x \in G$ and τ with $0 < \tau < 1$, we have

$$\begin{aligned}
\|f * \varphi_n\|_{\dot{K}_q^{\alpha,p}} &\leq c\| \sum_{j=-\infty}^{n-1} (\sum_{l=0}^{\infty} (|\lambda_{l,j}||a_{l,j} * \varphi_n(x)|^\tau)^p)^{1/p} m_j^{[-(s-\alpha)+\frac{1}{q}](1-\tau)} \|_{\dot{K}_q^{\alpha,p}} \\
&\leq c \sum_{j=-\infty}^{n-1} m_j^{[-(s-\alpha)+\frac{1}{q}](1-\tau)} \|(\sum_{l=0}^{\infty} (|\lambda_{l,j}||a_{l,j} * \varphi_n(x)|^\tau)^p)^{1/p}\|_{\dot{K}_q^{\alpha,p}} \\
&\leq c \sum_{j=-\infty}^{n-1} m_j^{[-(s-\alpha)+\frac{1}{q}](1-\tau)} (\sum_{k=j}^{\infty} m_k^{-\alpha p} m_j^{[-(s-\alpha)+\frac{1}{q}]\tau p} m_k^{-\frac{p}{q}})^{\frac{1}{p}} (\sum_{l=0}^{\infty} |\lambda_{l,j}|^p)^{\frac{1}{p}} \\
&\leq c \sum_{j=-\infty}^{n-1} m_j^{-s} (\sum_{l=0}^{\infty} |\lambda_{l,j}|^p)^{\frac{1}{p}}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\|f\|_{\dot{K}_q^{\alpha,p} \dot{B}_\beta^s} &\leq c(\sum_{n=-\infty}^{\infty} m_n^{s\beta} \sum_{j=-\infty}^{n-1} m_j^{-s\beta} (\sum_{l=0}^{\infty} |\lambda_{l,j}|^p)^{\frac{\beta}{p}})^{\frac{1}{\beta}} \\
&\leq c(\sum_{j=-\infty}^{\infty} m_j^{-s\beta} (\sum_{l=0}^{\infty} |\lambda_{l,j}|^p)^{\frac{\beta}{p}}) \sum_{n=j+1}^{\infty} m_n^{s\beta})^{\frac{1}{\beta}} \\
&\leq c\|\lambda\|_{p,\beta}.
\end{aligned}$$

(iii) If $p = \infty$, the proof is simpler. Here we omit it. □

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