Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
22 (2006), 73–76
www.emis.de/journals
ISSN 1786-0091

THE LATTICE-ALMOST ISOMETRIC COPIES OF l^1 AND l^∞ IN BANACH LATTICES

JINXI CHEN

Abstract. In this paper it is shown that if a Banach lattice E contains a lattice copy of l^1 (or l^∞), then it contains a lattice-almost isometric copy of l^1 (resp. l^∞). The above result is a lattice version of the well-known results of James and Partington concerning the almost isometric copies of l^1 and l^∞ in Banach spaces.

1. Introduction

Recall that two Banach spaces X, Y are said to be $(1+\varepsilon)$*-isometric* provided that there exists a linear isomorphism $T\colon X \to Y$ with $\|T\|\,\|T^{-1}\| \leq 1+\varepsilon$, equivalently, that there exists a linear isomorphism $T\colon X \to Y$ such that

$$\|x\| \leq \|Tx\| \leq (1+\varepsilon)\|x\|$$

for all $x \in X$. We say that a Banach space X contains an *almost isometric copy* of Y if for any $\varepsilon > 0$ there exists a subspace Z of X such that Z, Y are $(1+\varepsilon)$ -isometric.

Let E, F be two Banach lattices. It is interesting to know whether or not for any $\varepsilon > 0$ there exists a lattice isomorphism T from E onto F such that $\|T\|\,\|T^{-1}\| \leq 1+\varepsilon$. Namely, it is expected that not only E, F are $(1+\varepsilon)$-isometric, but their respective lattice structures are preserved as well. The above problem leads us to the following notions introduced in [9] and used in the remaining part of this paper.

Definition ([9])**.** E, F are called to be $(1+\varepsilon)$*-lattice isometric* if there exists a lattice isomorphism $T\colon E \to F$ such that $\|T\|\,\|T^{-1}\| \leq 1+\varepsilon$. If M is another Banach lattice, then E is said to contain a *lattice-almost isometric copy* of M if for any $\varepsilon > 0$ there exists a Banach sublattice L of E such that L, M are $(1+\varepsilon)$-lattice isometric.

Let us recall that the well-known result of James [5] asserts that a Banach space X contains an almost isometric copy of c_0 (or l^1) whenever it contains a copy of c_0 (resp. l^1). Here and in what follows the term "copy" means "topological copy", and "lattice copy" means "both lattice and topological copy", and "lattice isometric copy" means "both lattice and isometric copy". Note also that Partington [7] proved that if a Banach space X contains a copy of l^∞, then it also contains an almost isometric copy of l^∞.

2000 *Mathematics Subject Classification.* 46B04, 46B25, 46B42.

Key words and phrases. Banach lattice, lattice-almost isometry, copy of l^1 (or l^∞).

Supported in part by the National Natural Science Foundation of China under Grant NO.10571090.

In [2] the present author established that if a Banach lattice E contains a copy of c_0, then it contains a lattice-almost isometric copy of c_0. One can set a natural question: does the Banach lattice E contain a lattice-almost isometric copy of l^1 (or l^∞) whenever it contains a lattice copy of l^1 (resp. l^∞)? In this paper we give a positive answer. In a sense our results are the lattice versions of the preceding results due to James and Partington, respectively.

All spaces in this paper are over the reals, and our notions are standard. For the undefined notions and basic facts concerning Banach lattices we refer the reader to the monographs [1], [6].

2. Main Results

We recall that by a disjoint sequence of a Banach lattice E we mean a sequence $\{x_n\}$ of E satisfying $|x_n| \wedge |x_m| = \theta$ if $n \neq m$. It is known ([1], Theorem 14.21), that a Banach lattice E contains a lattice copy of l^1 iff it contains a complemented copy of l^1. The proof of the following theorem uses the idea of the classical proof of James' original result [5].

Theorem 1. *If a Banach lattice E contains a lattice copy of l^1, it contains a lattice-almost isometric copy of l^1.*

Proof. Since l^1 is lattice embeddable in E, there exists a pairwise disjoint sequence $\{x_n\}$ of E^+ and two positive constants m, M such that

$$m\sum_{i=1}^{\infty}|\,a_i| \leq \left\|\sum_{i=1}^{\infty}a_ix_i\right\| \leq M\sum_{i=1}^{\infty}|\,a_i|$$

for every $(a_1, a_2, \cdots) \in l^1$. Put

$$K_n = \inf\left\{\left\|\sum_{i=n}^{\infty}a_ix_i\right\| : a_i \geq 0,\ \sum_{i=n}^{\infty}a_i = 1\right\}.$$

Observe that $\{K_n\}$ is a nondecreasing sequence all values of which lie between m and M. So it is convergent and $m \leq K = \lim_n K_n \leq M$. Let $0 < \varepsilon < 1$ be fixed, and take $0 < \theta < 1 < \lambda$ such that $\theta/\lambda \geq 1 - \varepsilon$. Choose p_1 so that $K_{p_1} > \theta K$. By the definition of K_{p_1} there exists a sequence $\{a_i^*\}_{i \geq p_1}$ of nonnegative scalars such that $\sum_{i=p_1}^{\infty} a_i^* = 1$ and $\|\sum_{i=p_1}^{\infty} a_i^* x_i\| < \lambda K$. Since

$$\left\|\sum_{i=p_1}^{p}a_i^*x_i\right\| / \sum_{i=p_1}^{p}a_i^* \overset{p\to\infty}{\longrightarrow} \left\|\sum_{i=p_1}^{\infty}a_i^*x_i\right\|,$$

we can select a certain $p_2 > p_1$ and nonnegative scalars $a_i^{(1)}$, $p_1 \leq i \leq p_2 - 1$, satisfying

$$\sum_{i=p_1}^{p_2-1}a_i^{(1)} = 1, \qquad \left\|\sum_{i=p_1}^{p_2-1}a_i^{(1)}x_i\right\| < \lambda K.$$

Now, by induction, using the definitions of K_n and K, we can select a strictly increasing infinite sequence $\{p_n\}$ of integers and finite sequences $(a_i^{(n)})_{i=p_n}^{p_{n+1}-1}$ of nonnegative numbers such that, for the elements $y_n := \sum_{i=p_n}^{p_{n+1}-1} a_i^{(n)} x_i, n = 1, 2, \ldots,$ we have

$$\|y_n\| < \lambda K \quad \textit{and} \quad \sum_{i=p_n}^{p_{n+1}-1}a_i^{(n)} = 1.$$

Let $u_n = (\lambda K)^{-1} y_n$, and it is easy to see that $\{u_n\}$ is a bounded pairwise disjoint sequence of E^+.

For each $\xi = (\xi_1, \xi_2, \cdots) \in l^1$ we have $\| \sum_{n=1}^{\infty} \xi_n u_n \| \leq \sum_{n=1}^{\infty} | \xi_n | = \|\xi\|$. On the other hand, since $\{x_n\}$ and $\{y_n\}$ are disjoint sequences, for any $\xi = (\xi_1, \xi_2, \cdots) \in l^1 \setminus \{0\}$ it follows that

$$\begin{aligned} \left\| \sum_{n=1}^{\infty} \xi_n u_n \right\| &= (\lambda K)^{-1} \left\| \sum_{n=1}^{\infty} \xi_n y_n \right\| \\ &= (\lambda K)^{-1} \|\xi\| \cdot \left\| \sum_{n=1}^{\infty} \frac{\xi_n}{\|\xi\|} y_n \right\| \\ &= (\lambda K)^{-1} \|\xi\| \cdot \left\| \sum_{n=1}^{\infty} \frac{|\xi_n|}{\|\xi\|} y_n \right\| \\ &\geq (\lambda K)^{-1} K_{p_1} \|\xi\| \geq (1-\varepsilon)\|\xi\|. \end{aligned}$$

Now if for each $\xi = (\xi_1, \xi_2, \cdots) \in l^1$ we define $T\xi = \sum_{n=1}^{\infty} \xi_n u_n$, then T is a lattice embedding from l^1 into E such that $(1-\varepsilon)\|\xi\| \leq \|T\xi\| \leq \|\xi\|$. □

Note that if E has order continuous norm, then l^1 is embeddable in E iff l^1 is lattice embeddable in E (see [6], Propositions 2.5.13 and 2.5.15; cf. [8]). Thus we obtain immediately:

Corollary 2. *Let E be a Banach lattice with order continuous norm. If E contains a copy of l^1, then it contains a lattice-almost isometric copy of l^1.*

It is known that Banach lattices containing a lattice copy of l^∞ need not contain a lattice isometric copy of l^∞, not even an isomorphically isometric copy of l^∞ (see [3], pp. 521–522). On the other hand, a Dedekind σ-complete Banach lattice E contains a copy of l^∞ iff E contains a lattice copy of l^∞ (see [1], Theorem 14.9). The theorem below generalizes and strengthens the result due to Hudzik and Mastylo [4] which asserts that if a Dedekind σ-complete Banach lattice with an order semicontinuous norm contains a copy of l^∞, it contains an almost isometric copy of l^∞. Recall that Partington [7] proved that if a Banach space X contains a copy of l^∞, then it also contains an almost isometric copy of l^∞. The Partington's theorem, formulated in terms of disjoint sequences, has also the following lattice version:

Theorem 3. *Every Banach lattice which contains a lattice copy of l^∞ contains a lattice-almost isometric copy of l^∞.*

Proof. Let $\| \, \|$ be a norm on l^∞ equivalent to the usual norm. The Partington's original result asserts that for every $\varepsilon > 0$ there exists a sequence $u_1, u_2, \cdots$ of disjointly supported elements of l^∞ such that for every $(a_n) \in l^\infty$ we have

$$(1-\varepsilon) \sup |a_n| \leq \left\| \sum_{n=1}^{\infty} a_n u_n \right\| \leq (1+\varepsilon) \sup |a_n|,$$

where $\sum_{n=1}^{\infty} a_n u_n$ denotes the pointwise formal sum in l^∞.

Now let $\| \, \|_1$ be an equivalent lattice norm on l^∞, and let u_n denote the nth element in Partington's theorem obtained for this norm. Since the elements $w_n := |u_n|$, $n = 1, 2, \ldots$, are pairwise disjoint, for every $(a_n) \in l^\infty$ we have

$$\begin{aligned} (1-\varepsilon) \sup |a_n| \leq \left\| \sum_{n=1}^{\infty} a_n u_n \right\|_1 &= \left\| \left| \sum_{n=1}^{\infty} a_n u_n \right| \right\|_1 \\ &= \left\| \sum_{n=1}^{\infty} |a_n| \, w_n \right\|_1 = \left\| \sum_{n=1}^{\infty} a_n w_n \right\|_1 \\ &\leq (1+\varepsilon) \sup |a_n|. \end{aligned}$$

Define an operator $T\colon l^\infty \to (l^\infty, \|\,\|_1)$ by $T\alpha = \sum_{n=1}^\infty a_n w_n$ for any $\alpha = (a_n) \in l^\infty$. It is easy to see that T is a $(1+\varepsilon)$-lattice isometry. $\square$

With the preceding result and the above mentioned Theorem 14.9 in [1], the following result should be clear.

Corollary 4. *For a Dedekind σ-complete Banach lattice E the following five conditions are equivalent:*

(1) *E contains a copy of l^∞;*
(2) *E contains a lattice copy of l^∞;*
(3) *E contains a lattice-almost isometric copy of l^∞;*
(4) *E contains an almost isometric copy of l^∞;*
(5) *E does not have order continuous norm.*

Acknowledgement. The author would like to thank Professor G.G. Ding for his helpful remarks. The author is also grateful to the referee for comments which have improved this paper, in particular, for Corollary 4.

References

[1] C. D. Aliprantis and O. Burkinshaw. *Positive operators*, volume 119 of *Pure and Applied Mathematics*. Academic Press Inc., Orlando, FL, 1985.
[2] J. Chen. The almost lattice isometric copies of c_0 in Banach lattices. *Comment. Math. Univ. Carolin.*, 46(3):409–412, 2005.
[3] H. Hudzik. Banach lattices with order isometric copies of l^∞. *Indag. Math. (N.S.)*, 9(4):521–527, 1998.
[4] H. Hudzik and M. Mastyło. Almost isometric copies of l_∞ in some Banach spaces. *Proc. Amer. Math. Soc.*, 119(1):209–215, 1993.
[5] R. C. James. Uniformly non-square Banach spaces. *Ann. of Math. (2)*, 80:542–550, 1964.
[6] P. Meyer-Nieberg. *Banach lattices*. Universitext. Springer-Verlag, Berlin, 1991.
[7] J. R. Partington. Subspaces of certain Banach sequence spaces. *Bull. London Math. Soc.*, 13(2):163–166, 1981.
[8] M. Wójtowicz. The lattice copies of ℓ_1 in Banach lattices. *Comment. Math. Univ. Carolin.*, 42(4):649–653, 2001.
[9] M. Wójtowicz. The lattice copies of $l_\infty(\Gamma)/c_0(\Gamma)$ in a quotient of a Banach lattice. *Indag. Math. (N.S.)*, 16(1):147–155, 2005.

Received August 11, 2005; October 26, 2005 in revised form.

Department of Mathematics,
Nankai University,
Tianjin 300071, P.R. China
E-mail address: `chenjinxi@mail.nankai.edu.cn`