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POISSON APPROXIMATION FOR SUMS OF DEPENDENT BERNOULLI RANDOM VARIABLES

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Abstract. In this paper, we use the Stein-Chen method to determine a non-uniform bound for approximating the distribution of sums of dependent Bernoulli random variables by Poisson distribution. We give two formulas of non-uniform bounds and their applications.

1. Introduction and Main Results

Let Γ denote an arbitrary finite index set and let $|\Gamma|$ denote the number of all elements in Γ. for each $\alpha \in \Gamma$, let X_α be a Bernoulli random variable with the success probability $P(X_\alpha = 1) = 1 - P(X_\alpha = 0) = p_\alpha$, and let $W = \sum_{\alpha\in\Gamma} X_\alpha$ and $\lambda = \sum_{\alpha\in\Gamma} p_\alpha$. If $\Gamma = \{1, \ldots, n\}$ and X_α's are independent, then W has the distribution sometimes called *Poisson binomial*, and in case where all p_α are identical and equal to p, W has the binomial distribution with parameter n and p. In the case of rare or exceptional events, i.e., the probabilities p_α's are small, it is well-known that the distribution of W can be approximately Poisson with parameter λ. In past many years, many mathematicians tried to investigate and propose a good bound for this approximation, see Barbour, Holst and Janson [4], (p. 2–5).

In 1972, Stein introduced a powerful and general method for obtaining an explicit bound for the error in the normal approximation for dependent random variables [8]. This method was adapted and applied to the Poisson approximation by Chen [5], it is usually referred to as the Stein-Chen or Chen-Stein method. There are many authors used this method to give a bound for this approximation. For examples, in case that $X_1, \ldots, X_n$ are independent and $\lambda = \sum_{\alpha=1}^{n} p_\alpha$, Stein [9] gave an explicit uniform bound

$$\left| P(W \in A) - \sum_{k\in A} \frac{\lambda^k e^{-\lambda}}{k!} \right| \leq \lambda^{-1}(1 - e^{-\lambda}) \sum_{\alpha=1}^{n} p_\alpha^2 \tag{1.1}$$

for the difference of the distribution of W and the Poisson distribution, and Neammanee [7] gave a non-uniform bound

$$\left| P(W = w_0) - \frac{\lambda^{w_0} e^{-\lambda}}{w_0!} \right| \leq \min\{\frac{1}{w_0}, \lambda^{-1}\} \sum_{\alpha=1}^{n} p_\alpha^2 \tag{1.2}$$

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for the difference of the point probability of W and the Poisson probability, where $A \subseteq \mathbb{N} \cup \{0\}$ and $w_0 \in \{1, \ldots, n-1\}$.

In case of dependent summands, we first suppose that, for each $\alpha \in \Gamma$, the set $B_\alpha \subsetneq \Gamma$ with $\alpha \in B_\alpha$ is chosen as a neighborhood of α consisting of the set of indices β such that X_α and X_β are dependent. Let

$$b_1 = \sum_{\alpha \in \Gamma} \sum_{\beta \in B_\alpha} p_\alpha p_\beta, \tag{1.3}$$

$$b_2 = \sum_{\alpha \in \Gamma} \sum_{\beta \in B_\alpha \setminus \{\alpha\}} E[X_\alpha X_\beta], \tag{1.4}$$

and

$$b_3 = \sum_{\alpha \in \Gamma} E|E[X_\alpha | \{X_\beta : \beta \notin B_\alpha\}] - p_\alpha|. \tag{1.5}$$

Barbour, Holst and Janson [4] gave a uniform bound in the form of

$$\left| P(W \in A) - \sum_{k \in A} \frac{\lambda^k e^{-\lambda}}{k!} \right| \leq \lambda^{-1}(1 - e^{-\lambda})(b_1 + b_2) + \min\{1, \lambda^{-1/2}\} b_3 \tag{1.6}$$

and Janson (1994) used the coupling method to determine a uniform bound in the form of

$$\left| P(W \in A) - \sum_{k \in A} \frac{\lambda^k e^{-\lambda}}{k!} \right| \leq \lambda^{-1}(1 - e^{-\lambda}) \sum_{\alpha \in \Gamma} p_\alpha E|W - W_\alpha^*|, \tag{1.7}$$

where W_α^* is a random variable constructed on the same probability space as W and which has the same distribution as $W - X_\alpha$ conditional on $X_\alpha = 1$.

Observe that the bounds in (1.6) and (1.7) are uniform. In case of non-uniform bounds, Teerapabolarn and Neammanee [10] gave a pointwise bound in terms of

$$\left| P(W = w_0) - \frac{\lambda^{w_0} e^{-\lambda}}{w_0!} \right| \leq \min\{\frac{1}{w_0}, \lambda^{-1}\}[\min\{\lambda, b_1\} + \min\{\lambda, b_2\} + b_3] \tag{1.8}$$

and

$$\left| P(W = w_0) - \frac{\lambda^{w_0} e^{-\lambda}}{w_0!} \right| \leq \min\{\frac{1}{w_0}, \lambda^{-1}\} \sum_{\alpha \in \Gamma} p_\alpha E|W - W_\alpha^*|, \tag{1.9}$$

where $w_0 \in \{1, 2, \ldots, |\Gamma|\}$.

In this paper, we give another formulas of non-uniform bounds of (1.6) and (1.7) where $A = \{0, 1, \ldots, w_0\}$. The followings are our main results.

Theorem 1.1. *For $w_0 \in \{0, 1, \ldots, |\Gamma|\}$,*

$$\begin{aligned} \left| P(W \leq w_0) - \sum_{k=0}^{w_0} \frac{\lambda^k e^{-\lambda}}{k!} \right| &\leq \lambda^{-1}(1 - e^{-\lambda}) \min\left\{1, \frac{e^\lambda}{w_0 + 1}\right\}(b_1 + b_2) \\ &\quad + \min\left\{1, \lambda^{-1/2}, \max\{1, \lambda^{-1}\}\frac{(e^\lambda - 1)}{w_0 + 1}\right\} b_3. \end{aligned} \tag{1.10}$$

For each $\alpha \in \Gamma$, if X_α is independent of the collection $\{X_\beta : \beta \notin B_\alpha\}$, then we have

$$\left| P(W \leq w_0) - \sum_{k=0}^{w_0} \frac{\lambda^k e^{-\lambda}}{k!} \right| \leq \lambda^{-1}(1 - e^{-\lambda}) \min\left\{1, \frac{e^\lambda}{w_0 + 1}\right\}(b_1 + b_2). \tag{1.11}$$

Theorem 1.2. *For* $w_0 \in \{0, 1, \ldots, |\Gamma|\}$,

$$\begin{aligned} &\left| P(W \le w_0) - \sum_{k=0}^{w_0} \frac{\lambda^k e^{-\lambda}}{k!} \right| \\ &\le \lambda^{-1}(1 - e^{-\lambda}) \min\left\{1, \frac{e^\lambda}{w_0+1}\right\} \sum_{\alpha\in\Gamma} p_\alpha E|W - W_\alpha^*|. \end{aligned} \tag{1.12}$$

If $\Gamma = \{1, \ldots, n\}$ and X_α's are all independent, a non-uniform bound of Poisson approximation to Poisson binomial distribution can be obtained by setting W_α^* in Theorem 1.2 to be $W - X_\alpha$. So, we have $E|W - W_\alpha^*| = p_\alpha$ and then the following holds.

Theorem 1.3. *Let* $X_1, X_2, \ldots, X_n$ *be independent Bernoulli random variables. Then, for* $w_0 \in \{0, 1, \ldots, n\}$,

$$\left| P(W \le w_0) - \sum_{k=0}^{w_0} \frac{\lambda^k e^{-\lambda}}{k!} \right| \le \lambda^{-1}(1 - e^{-\lambda}) \min\left\{1, \frac{e^\lambda}{w_0+1}\right\} \sum_{\alpha=1}^{n} p_\alpha^2. \tag{1.13}$$

We see that, for $A = \{0, \ldots, w_0\}$ and $\frac{e^\lambda}{w_0+1} < 1$, the bounds in (1.10), (1.12) and (1.13) are better than the bounds in (1.6), (1.7) and (1.1) respectively.

In many applications of the Poisson approximation, we know that this approximation can be good when λ is small, and from above theorems, we observe that $\frac{e^\lambda}{w_0+1} < 1$ when $\lambda < log(w_0 + 1)$. So, the following corollaries hold.

Corollary 1.1. *Let* $w_0 \in \{0, 1, \ldots, |\Gamma|\}$ *and* $\lambda < log(w_0 + 1)$. *Then*
1.

$$\left| P(W \le w_0) - \sum_{k=0}^{w_0} \frac{\lambda^k e^{-\lambda}}{k!} \right| \le \frac{\lambda^{-1}(e^\lambda - 1)(b_1 + b_2 + b_3)}{w_0+1}, \tag{1.14}$$

and, if X_α *is independent of the collection* $\{X_\beta : \beta \notin B_\alpha\}$,

$$\left| P(W \le w_0) - \sum_{k=0}^{w_0} \frac{\lambda^k e^{-\lambda}}{k!} \right| \le \frac{\lambda^{-1}(e^\lambda - 1)(b_1 + b_2)}{w_0+1}. \tag{1.15}$$

2.

$$\left| P(W \le w_0) - \sum_{k=0}^{w_0} \frac{\lambda^k e^{-\lambda}}{k!} \right| \le \frac{\lambda^{-1}(e^\lambda - 1)}{w_0+1} \sum_{\alpha\in\Gamma} p_\alpha E|W - W_\alpha^*|. \tag{1.16}$$

Corollary 1.2. *For* n *independent Bernoulli summands, let* $w_0 \in \{0, 1, \ldots, n\}$ *and* $\lambda < \log(w_0 + 1)$. *Then*

$$\left| P(W \le w_0) - \sum_{k=0}^{w_0} \frac{\lambda^k e^{-\lambda}}{k!} \right| \le \frac{\lambda^{-1}(e^\lambda - 1)}{w_0+1} \sum_{\alpha=1}^{n} p_\alpha^2. \tag{1.17}$$

2. Proof of Main Results

The Stein's method for Poisson case started by Stein's equation for Poisson distribution which is defined by

$$\lambda f(w+1) + wf(w) = h(w) - \mathcal{P}_\lambda(h), \tag{2.1}$$

where $\mathcal{P}_\lambda(h) = e^{-\lambda}\sum_{l=0}^{\infty} h(l)\frac{\lambda^l}{l!}$ and f and h are real valued bounded functions defined on $\mathbb{N}\cup\{0\}$. For $A \subseteq \mathbb{N}\cup\{0\}$, let $h_A : \mathbb{N}\cup\{0\} \to \mathbb{R}$ be defined by

$$h_A(w) = \begin{cases} 1 & \text{if } w \in A, \\ 0 & \text{if } w \notin A. \end{cases} \tag{2.2}$$

From Barbour, Holst and Janson [4] p. 7, we know that the solution $U_\lambda h_A$ of (2.1) is of the form

(2.3)

$$U_\lambda h_A(w) = \begin{cases} (w-1)!\lambda^{-w}e^{\lambda}[\mathcal{P}_\lambda(h_{A\cap C_{w-1}}) - \mathcal{P}_\lambda(h_A)\mathcal{P}_\lambda(h_{C_{w-1}})] & \text{if } w \geq 1, \\ 0 & \text{if } w = 0, \end{cases}$$

and

$$0 < U_\lambda h_{C_{w_0}}(w) \leq \min\{1, \lambda^{-1/2}\}. \tag{2.4}$$

Hence, by (2.3), we have

$$U_\lambda h_{C_{w_0}}(w) = \begin{cases} (w-1)!\lambda^{-w}e^{\lambda}[\mathcal{P}_\lambda(h_{C_{w_0}})\mathcal{P}_\lambda(1-h_{C_{w-1}})] & \text{if } w_0 < w, \\ (w-1)!\lambda^{-w}e^{\lambda}[\mathcal{P}_\lambda(h_{C_{w-1}})\mathcal{P}_\lambda(1-h_{C_{w_0}})] & \text{if } w_0 \geq w, \\ 0 & \text{if } w = 0. \end{cases} \tag{2.5}$$

In the proof of main results, we also need the following lemmas.

Lemma 2.1. *Let $w_0 \in \mathbb{N}\cup\{0\}$. Then the followings hold.*
1. For $w \geq 1$,

$$0 < U_\lambda h_{C_{w_0}}(w) \leq \min\left\{1, \lambda^{-1/2}, \max\{1, \lambda^{-1}\}\frac{(e^\lambda - 1)}{w_0+1}\right\}. \tag{2.6}$$

2. For any $s, t \in \mathbb{N}$,

$$|V_\lambda h_{C_{w_0}}(t,s)| \leq \sup_{w\geq 1} |V_\lambda h_{C_{w_0}}(w+1,w)||t-s|,$$

where $V_\lambda h_{C_{w_0}}(t,s) = U_\lambda h_{C_{w_0}}(t) - U_\lambda h_{C_{w_0}}(s)$.

3. For $w \geq 1$,

$$|V_\lambda h_{C_{w_0}}(w+1,w)| \leq \lambda^{-1}(1-e^{-\lambda})\min\left\{1, \frac{e^\lambda}{w_0+1}\right\}. \tag{2.7}$$

Proof. 1. Form (2.4), it suffices to show that

$$0 < U_\lambda h_{C_{w_0}}(w) \leq \max\{1, \lambda^{-1}\}\frac{(e^\lambda - 1)}{w_0+1}.$$

For $w > w_0$, we see that

$$\begin{aligned} 0 < U_\lambda h_{C_{w_0}}(w) &\leq (w-1)!\lambda^{-w}e^{\lambda}\mathcal{P}_\lambda(1-h_{C_{w-1}}) \\ &\leq (w-1)!\sum_{k=w}^{\infty}\frac{\lambda^{k-w}}{k!} \\ &= (w-1)!\left\{\frac{1}{w!} + \frac{\lambda}{(w+1)!} + \frac{\lambda^2}{(w+2)!} + \cdots\right\} \\ &= \frac{(w-1)!}{w!}\left\{1 + \frac{\lambda}{w+1} + \frac{\lambda^2}{(w+1)(w+2)} + \cdots\right\} \\ &\leq \frac{1}{w_0+1}\left\{1 + \frac{\lambda}{2!} + \frac{\lambda^2}{3!} + \cdots\right\} \end{aligned}$$

$$= \frac{\lambda^{-1}}{w_0+1}\left\{\lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots\right\}$$
$$= \frac{\lambda^{-1}(e^\lambda - 1)}{w_0+1}$$

and, for $w \le w_0$, we have

$$\begin{aligned}
0 < U_\lambda h_{C_{w_0}}(w) &\le (w-1)!\lambda^{-w}e^\lambda \mathcal{P}_\lambda(1-h_{C_{w_0}}) \\
&= (w-1)!\sum_{k=w_0+1}^{\infty}\frac{\lambda^{k-w}}{k!} \\
&\le (w-1)!\left\{\frac{\lambda^{(w_0+1)-w}}{(w_0+1)!} + \frac{\lambda^{(w_0+2)-w}}{(w_0+2)!} + \cdots\right\} \\
&= \frac{(w-1)!\lambda^{(w_0+1)-w}}{(w_0+1)!} + \frac{(w-1)!\lambda^{(w_0+2)-w}}{(w_0+2)!} + \cdots \\
&= \frac{\lambda^{(w_0+1)-w}}{(w_0+1)\binom{w_0}{w-1}[(w_0+1)-w]!} \\
&\qquad + \frac{\lambda^{(w_0+2)-w}}{(w_0+2)\binom{w_0+1}{w-1}[(w_0+2)-w]!} + \cdots \\
&\le \frac{1}{w_0+1}\left\{\lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots\right\} \\
&= \frac{e^\lambda - 1}{w_0+1}.
\end{aligned}$$

Hence, (2.6) holds.

2. Assume that $t > s$. Then

$$\begin{aligned}
|V_\lambda h_{C_{w_0}}(t,s)| &= \left|\sum_{w=s}^{t-1} V_\lambda h_{C_{w_0}}(w+1,w)\right| \\
&\le \sum_{w=s}^{t-1}|V_\lambda h_{C_{w_0}}(w+1,w)| \\
&\le \sup_{w\ge 1}|V_\lambda h_{C_{w_0}}(w+1,w)||t-s|.
\end{aligned}$$

3. From Barbour, Holst and Janson [4] p.7, we have $|V_\lambda h_{C_{w_0}}(w+1,w)| \le \lambda^{-1}(1-e^{-\lambda})$. Next we shall show that

$$|V_\lambda h_{C_{w_0}}(w+1,w)| \le \frac{\lambda^{-1}(e^\lambda - 1)}{w_0+1}.$$

From (2.5) we see that

$$V_\lambda h_{C_{w_0}}(w+1,w) =$$
$$\begin{cases} (w-1)!\lambda^{-(w+1)}e^\lambda\mathcal{P}_\lambda(h_{C_{w_0}})[w\mathcal{P}_\lambda(1-h_{C_w}) - \lambda\mathcal{P}_\lambda(1-h_{C_{w-1}})] & \text{if } w \ge w_0+1, \\ (w-1)!\lambda^{-(w+1)}e^\lambda\mathcal{P}_\lambda(1-h_{C_{w_0}})[w\mathcal{P}_\lambda(h_{C_w}) - \lambda\mathcal{P}_\lambda(h_{C_{w-1}})] & \text{if } w \le w_0. \end{cases}$$

We divide the proof of 3 into two cases as follows:

Case 1. $w \ge w_0 + 1$. Since

$$\begin{aligned}
w\mathcal{P}_\lambda(1-h_{C_w}) - \lambda\mathcal{P}_\lambda(1-h_{C_{w-1}}) &= e^{-\lambda}\left\{w\sum_{k=w+1}^{\infty}\frac{\lambda^k}{k!} - \sum_{k=w}^{\infty}\frac{\lambda^{k+1}}{k!}\right\} \\
&= e^{-\lambda}\sum_{k=w+1}^{\infty}(w-k)\frac{\lambda^k}{k!}
\end{aligned}$$

$$
\begin{aligned}
0 &< -V_\lambda h_{C_{w_0}}(w+1,w) \\
&\leq (w-1)! \sum_{k=w+1}^{\infty} (k-w) \frac{\lambda^{k-(w+1)}}{k!} \\
&\leq (w-1)! \left\{ \frac{1}{(w+1)!} + \frac{2\lambda}{(w+2)!} + \cdots \right\} \\
&= \frac{(w-1)!}{w!} \left\{ \frac{1}{w+1} + \frac{2\lambda}{(w+1)(w+2)} + \cdots \right\} \\
&\leq \frac{\lambda^{-1}}{w_0+1} \left\{ \frac{\lambda}{2} + \frac{2\lambda^2}{3!} + \cdots \right\} \\
&\leq \frac{\lambda^{-1}(e^\lambda - 1)}{w_0+1}.
\end{aligned}
$$

Case 2. $w \leq w_0$.

$$
\begin{aligned}
0 < V_\lambda h_{C_{w_0}}(w+1,w) &\leq e^\lambda w! \lambda^{-(w+1)} \mathcal{P}_\lambda(1 - h_{C_{w_0}}) \\
&\leq w! \sum_{k=w_0+1}^{\infty} \frac{\lambda^{k-(w+1)}}{k!} \\
&= \sum_{k=w_0+1}^{\infty} \frac{w!\lambda^{k-(w+1)}}{k(k-1)\cdots(k-w)[k-(w+1)]!} \\
&\leq \frac{1}{w_0+1} \sum_{k=w_0+1}^{\infty} \frac{\lambda^{k-(w+1)}}{\binom{k-1}{w}[k-(w+1)]!} \\
&\leq \frac{1}{w_0+1} \left\{ 1 + \frac{\lambda}{2!} + \frac{\lambda^2}{3!} + \cdots \right\} \\
&= \frac{\lambda^{-1}(e^\lambda - 1)}{w_0+1}.
\end{aligned}
$$

Hence, form case 1 to case 2, we have (2.7). □

Lemma 2.2. *Let* $Z_\alpha = \sum_{\beta \in B_\alpha \setminus \{\alpha\}} X_\beta, Y_\alpha = W - X_\alpha - Z_\alpha = \sum_{\beta \notin B_\alpha} X_\beta$ *and* $f = U_\lambda h_{C_{w_0}}$. *Then, for* $w_0 \in \{0, 1, \ldots, |\Gamma|\}$,

1. $|E[p_\alpha(f(W+1) - f(Y_\alpha+1))]|$
$$\leq \lambda^{-1}(1 - e^{-\lambda}) \min\left\{1, \frac{e^\lambda}{w_0+1}\right\} (p_\alpha^2 + p_\alpha E[Z_\alpha]),$$
2. $|E[X_\alpha(f(Y_\alpha + Z_\alpha + 1) - f(Y_\alpha+1))]|$
$$\leq \lambda^{-1}(1 - e^{-\lambda}) \min\left\{1, \frac{e^\lambda}{w_0+1}\right\} E[X_\alpha Z_\alpha], \text{ and}$$
3. $|E[X_\alpha f(Y_\alpha+1) - p_\alpha f(Y_\alpha+1)]|$
$$\leq \min\left\{1, \lambda^{-1/2}, \max\{1, \lambda^{-1}\} \frac{(e^\lambda - 1)}{w_0+1}\right\} E|E[X_\alpha|\{X_\beta : \beta \notin B_\alpha\}] - p_\alpha|.$$

Proof. 1. By lemma 2.1 (2 and 3), we have

$$
\begin{aligned}
&|E[p_\alpha(f(W+1) - f(Y_\alpha+1))]| \\
&\qquad \leq E|p_\alpha|f(Y_\alpha + Z_\alpha + X_\alpha + 1) - f(Y_\alpha+1)|| \\
&\qquad \leq \sup_{w \geq 1} |V_\lambda h_{w_0}(w+1,w)| p_\alpha E[X_\alpha + Z_\alpha]
\end{aligned}
$$

$$\leq \lambda^{-1}(1-e^{-\lambda})\min\left\{1, \frac{e^{\lambda}}{w_0+1}\right\}(p_\alpha^2 + p_\alpha E[Z_\alpha]).$$

2. Use the same argument of 1.
3.

$$\begin{aligned}
&|E[X_\alpha f(Y_\alpha+1) - p_\alpha f(Y_\alpha+1)]| \\
&\quad = |E[f(Y_\alpha+1)E[X_\alpha - p_\alpha|\{X_\beta : \beta \notin B_\alpha\}]]| \\
&\quad \leq E|f(Y_\alpha+1)E[X_\alpha|\{X_\beta : \beta \notin B_\alpha\}] - p_\alpha| \\
&\quad \leq \sup_{w\geq 1}|f(w)|E|E[X_\alpha|\{X_\beta : \beta \notin B_\alpha\}] - p_\alpha| \text{ by lemma 2.1 (1):} \\
&\quad \leq \min\left\{1, \lambda^{-1/2}, \max\{1,\lambda^{-1}\}\frac{(e^{\lambda}-1)}{w_0+1}\right\} E|E[X_\alpha|\{X_\beta : \beta \notin B_\alpha\}] - p_\alpha|.
\end{aligned}$$

□

Proof of Theorem 1.1. Let $Z_\alpha = \sum_{\beta\in B_\alpha\setminus\{\alpha\}} X_\beta, Y_\alpha = W - X_\alpha - Z_\alpha = \sum_{\beta\notin B_\alpha} X_\beta$ and $W_\alpha = W - X_\alpha$. From (2.1), when $h = h_{C_{w_0}}$, we have

$$P(W \leq w_0) - \sum_{k=0}^{w_0}\frac{\lambda^k e^{-\lambda}}{k!} = E[\lambda f(W+1) - Wf(W)], \tag{2.8}$$

where $f = U_\lambda h_{C_{w_0}}$ is defined by (2.5).
By the fact that each X_α takes a value on 0 and 1, we can see that

$$\begin{aligned}
E[Wf(W)] &= \sum_{\alpha\in\Gamma} E[X_\alpha f(W_\alpha+1)] \\
&= \sum_{\alpha\in\Gamma} E[X_\alpha f(Y_\alpha+1)] + \sum_{\alpha\in\Gamma} E[X_\alpha(f(Y_\alpha+Z_\alpha+1) - f(Y_\alpha+1))].
\end{aligned}$$

Hence

$$\begin{aligned}
&E[\lambda f(W+1) - Wf(W)] \\
&\quad = \sum_{\alpha\in\Gamma}\{E[p_\alpha(f(W+1) - f(Y_\alpha+1))] - E[X_\alpha(f(Y_\alpha+Z_\alpha+1) - f(Y_\alpha+1))] \\
&\qquad + E[p_\alpha f(Y_\alpha+1) - X_\alpha f(Y_\alpha+1)]\}.
\end{aligned}$$

From this fact, lemma 2.2 and (2.8), we have

$$\begin{aligned}
\left|P(W\leq w_0) - \sum_{k=0}^{w_0}\frac{\lambda^k e^{-\lambda}}{k!}\right| &= |E[\lambda f(W+1) - Wf(W)]| \\
&\leq \lambda^{-1}(1-e^{-\lambda})\min\left\{1, \frac{e^{\lambda}}{w_0+1}\right\}(b_1+b_2) \\
&\quad + \min\left\{1, \lambda^{-1/2}, \max\{1,\lambda^{-1}\}\frac{(e^{\lambda}-1)}{w_0+1}\right\} b_3.
\end{aligned}$$

□

Proof of Theorem 1.2. Note that $E[Wf(W)] = \sum_{\alpha\in\Gamma} E[X_\alpha f(W)]$ and for each α,

$$\begin{aligned}
E[X_\alpha f(W)] &= E[E[X_\alpha f(W)|X_\alpha]] \\
&= E[X_\alpha f(W)|X_\alpha = 0]P(X_\alpha = 0) + E[X_\alpha f(W)|X_\alpha = 1]P(X_\alpha = 1) \\
&= E[f(W)|X_\alpha = 1]P(X_\alpha = 1) \\
&= p_\alpha E[f(W_\alpha^* + 1)].
\end{aligned}$$

Thus

$$\begin{aligned} E[\lambda f(W+1) - Wf(W)] &= \sum_{\alpha\in\Gamma} p_\alpha E[f(W+1)] - \sum_{\alpha\in\Gamma} p_\alpha E[f(W_\alpha^*+1)] \\ &= \sum_{\alpha\in\Gamma} p_\alpha E[f(W+1) - f(W_\alpha^*+1)]. \end{aligned}$$

By lemma 2.1 (2 and 3), we have

$$\begin{aligned} |E[\lambda f(W+1) - Wf(W)]| &\le \sum_{\alpha\in\Gamma} p_\alpha E|f(W+1) - f(W_\alpha^*+1)| \\ &\le \lambda^{-1}(1-e^{-\lambda})\min\left\{1, \frac{e^\lambda}{w_0+1}\right\} \sum_{\alpha\in\Gamma} p_\alpha E|W - W_\alpha^*|. \end{aligned}$$

Hence, by (2.8), the proof is completed. □

3. Applications

In this section, we apply Theorem 1.1 and Theorem 1.2 to some problems.

Example 3.1 (The birthday problem). In the usual formulation of the birthday problem, we assume that birthdays of n individuals are independent over the d days in a year. We consider the general birthday problem of a k-way coincidence when birthdays are uniform. Let $\{1,2,\ldots,n\}$ denote a group of n people, and, for fixed $k \ge 2$, let the index set $\Gamma = \{\alpha \subset \{1,2,\ldots,n\} : |\alpha| = k\}$. For example, in the classical case $k=2$ and Γ is the set of all pairs of people among whom a two-way coincidence could occur. Let X_α be the indicator of the event that the people indexed by α share the same birthday with small probability $p_\alpha = P(X_\alpha = 1) = d^{1-k}$. The number of birthday coincidence, that is, the number of groups of k people that share the same birthday is given by $W = \sum_{\alpha\in\Gamma} X_\alpha$. It seems reasonable to approximate W as a Poisson random variable with mean $\lambda = E[W]$. Since all p_α are identical, we have

$$\lambda = |\Gamma| p_\alpha = \binom{n}{k} d^{1-k}.$$

We can bound the error of Poisson approximation to the distribution of W with the bound (1.10) in Theorem 1.1 by taking the set $B_\alpha = \{\beta \in \Gamma : \alpha\cap\beta \ne \varnothing\}$ as the neighborhood dependence for α. We observe that X_α and X_β are independent if $\alpha\cap\beta = \varnothing$, hence $b_3 = 0$. Since $|B_\alpha| = \binom{n}{k} - \binom{n-k}{k}$, we have

$$b_1 = |\Gamma||B_\alpha|p_\alpha^2 = \lambda|B_\alpha|d^{1-k}. \tag{3.1}$$

For a given α, we have $1 \le |\alpha\cap\beta| \le k-1$ for $\beta \in B_\alpha \setminus \{\alpha\}$ and

$$b_2 = \binom{n}{k}\sum_{j=1}^{k-1}\binom{k}{j}\binom{n-k}{k-j} d^{1+j-2k} = \lambda b, \tag{3.2}$$

where $b = \sum_{j=1}^{k-1}\binom{k}{j}\binom{n-k}{k-j} d^{j-k}$. By (1.10), we have

$$\left| P(W \le w_0) - \sum_{k=0}^{w_0} \frac{\lambda^k e^{-\lambda}}{k!} \right| \le (1-e^{-\lambda})\min\left\{1, \frac{e^\lambda}{w_0+1}\right\}\left(|B_\alpha|d^{1-k} + b\right),$$

where $w_0 \in \left\{0, 1, \ldots, \binom{n}{k}\right\}$. This bound is small when λ is small.

For numerical example, if $k=3$, $n=50$ and $d=365$, we have $\lambda = \binom{50}{3}(365)^{-2} = 0.14711953$, $|B_\alpha| = \binom{50}{3} - \binom{47}{3} = 3385$ and $b = 3\binom{47}{2}(365)^{-2} + 3(47)(365)^{-1} =$

0.41064365. So, a non-uniform bound for approximating the distribution of the number of groups of three people that share the same birthday is

$$\left| P(W \le w_0) - \sum_{k=0}^{w_0} \frac{\lambda^k e^{-\lambda}}{k!} \right| \le 0.05965590 \min\left\{1, \frac{1.15849244}{w_0+1}\right\},$$

where $w_0 \in \left\{0, 1, \ldots, \binom{50}{3}\right\}$ and the following table shows some representative Poisson estimate $P(W \le w_0)$ of this choice.

w_0	Estimate	Uniform Error Bound	Non-Uniform Error Bound
0	0.86319079	0.05965590	0.05965590
1	0.99018302	0.05965590	0.03455546
2	0.99952454	0.05965590	0.02303697
3	0.99998264	0.05965590	0.01727773
4	0.99999949	0.05965590	0.01382218
5	0.99999999	0.05965590	0.01151849
6	1.00000000	0.05965590	0.00987299

TABLE 1. Poisson Estimate of $P(W \le w_0)$ for $k = 3,\ n = 50$ and $d = 365$

Example 3.2 (A random graph problem). Consider the random graph n-cube $\{0, 1\}^n$, it has 2^n vertices, each of degree n, with an edge joining pairs of vertices which differ in exactly one coordinate. Suppose that each of the $n2^{n-1}$ edges is assigned a random direction by tossing a fair coin. Let Γ be the set of all 2^n vertices, and for each $\alpha \in \Gamma$, let X_α be the indicator that vertex α has all of its edges directed inward, with the probability $p_\alpha = P(X_\alpha = 1) = 2^{-n}$. Let $W = \sum_{\alpha \in \Gamma} X_\alpha$ be the number of vertices at which all n edges point inward, and its distribution seems reasonable to approximate by Poisson distribution with mean $\lambda = E[W] = 1$ when n is large.

We can bound the error of Poisson approximation to the distribution of W, follows Arratia, Goldstein and Gordon [1] by taking the set $B_\alpha = \{\beta \in \Gamma : |\alpha - \beta| = 1\}$ as the neighborhood dependence for α, hence $b_2 = b_3 = 0$. Since $|B_\alpha| = n$, we have

$$b_1 = |\Gamma||B_\alpha|p_\alpha^2 = n2^{-n}. \tag{3.3}$$

By (1.10), we have

$$\left| P(W \le w_0) - \sum_{k=0}^{w_0} \frac{\lambda^k e^{-\lambda}}{k!} \right| \le (1 - e^{-1})(n2^{-n}) \min\left\{1, \frac{e}{w_0+1}\right\},$$

where $w_0 \in \{0, 1, \ldots, 2^{n-1}\}$.

Table 2 shows some representative Poisson estimate of $P(W \le w_0)$ of this example.

Example 3.3 (Drawing without replacement). Consider a finite population in which individual member is of one of two types, and code these 1 ("success") and 0 ("failure"). When sampling is done at random with replacement and mixing between selections, a sequence of trials is an i.i.d. sequence in which the common distribution is Bernoulli. When sampling is done at random without replacement, the individual selections are Bernoulli; but they are not independent.

Denote the population size by N and the number type 1 individuals (Ones) by m, the number of type 0 individuals (Zeroes) by $N - m$, and we arrange m Ones and $N - m$ Zeroes at random to form an $N-$vector, so that each of the different

w_0	Estimate	Uniform Error Bound	Non-Uniform Error Bound
0	0.36787944	0.00617305	0.00617305
1	0.73575888	0.00617305	0.00617305
2	0.91969860	0.00617305	0.00559337
3	0.98101184	0.00617305	0.00419502
4	0.99634015	0.00617305	0.00335602
5	0.99940582	0.00617305	0.00279668
6	0.99991676	0.00617305	0.00239716
7	0.99998975	0.00617305	0.00209751
8	0.99999888	0.00617305	0.00186446
9	0.99999989	0.00617305	0.00167801
10	0.99999999	0.00617305	0.00152546
11	1.00000000	0.00617305	0.00139834

TABLE 2. Poisson Estimate of $P(W \le w_0)$ for $n = 10$

outcomes has the same probability $\dfrac{m!(N-m)!}{N!}$. Suppose for each $i \in \{1, \dots, n\}$ that $X_i = 1$ if there is a One at position i and $X_i = 0$ otherwise, and the probability that $P(X_i = 1) = \dfrac{m}{N}$. Let $W = \sum_{i=1}^{n} X_i$ be the total number of Ones at positions $1, \dots, n$. It is well-known result that W has the hypergeometric distribution, which is given by

$$P(W = w_0) = \frac{\binom{m}{w_0}\binom{N-m}{n-w_0}}{\binom{N}{n}}, \quad 0 \le w_0 \le \min\{m, n\}. \tag{3.4}$$

If $\dfrac{m}{N}$ and $\dfrac{n}{N}$ are small then it seems reasonable to approximate this distribution by Poisson distribution with mean $\lambda = E[W]$. For the hypergeometric distribution we have

$$\lambda = \frac{mn}{N} \text{ and } \mathrm{Var}[W] = \frac{N-n}{N-1} \cdot \frac{nm}{N}\left(1 - \frac{m}{N}\right).$$

Form Theorem 1.2, in order to determine $E|W - W_i^*|$, we first construct Bernoulli random variables $Y_1^i, \dots, Y_n^i$ and W_i^*, which introduced by Barbour, Holst and Janson [4], as follows. If $X_i = 1$, then set $Y_j^i = X_j$ for all $j \in \{1, \dots, n\}$. Otherwise, $X_i = 0$, change a randomly chosen One to Zero at position i and then, for $1 \le j \le n$, we set $Y_j^i = 1$ if there is a One at position j and $Y_j^i = 0$ otherwise. Let $W_i^* = \sum_{j=1, j\ne i}^{n} Y_j^i$, then W_i^* has the same distribution as $W - X_i$ conditional on $X_i = 1$. Observe that in case of $X_i = 1$, we have $W_i^* = W - 1$ and in case of $X_i = 0$, we have $W_i^* = W - 1$ if the One at position i is obtained from the first n positions and $W_i^* = W$ otherwise, the One is obtained from the rest $N - n$ positions. So, we have

$$\begin{aligned}
\frac{m}{N}\sum_{i=1}^{n} E|W - W_i^*| &= \lambda E[W+1] - \sum_{i=1}^{n} P(X_i = 1)E[W|X_i = 1] \\
&= \lambda^2 + \lambda - \sum_{i=1}^{n} E[X_i W] \\
&= \lambda - \mathrm{Var}[W] \\
&= \frac{\lambda}{N-1}\left[(n+m-1) - \frac{nm}{N}\right].
\end{aligned} \tag{3.5}$$

Substituting (3.5) into Theorem 1.2, we have

$$\left|P(W \le w_0) - \sum_{k=0}^{w_0} \frac{\lambda^k e^{-\lambda}}{k!}\right| \le (1-e^{-\lambda}) \min\left\{1, \frac{e^\lambda}{w_0+1}\right\} \frac{1}{N-1}\left[(n+m-1) - \frac{nm}{N}\right], \tag{3.6}$$

where $w_0 \in \{0, 1, \ldots, \min\{n, m\}\}$.

w_0	Estimate	Uniform Error Bound	Non-Uniform Error Bound
0	0.36787944	0.03986346	0.03986346
1	0.73575944	0.03986346	0.03986346
2	0.91969860	0.03986346	0.03612004
3	0.98101184	0.03986346	0.02709003
4	0.99634015	0.03986346	0.02167202
5	0.99940582	0.03986346	0.01806002
6	0.99991676	0.03986346	0.01548002
7	0.99998975	0.03986346	0.01354501
8	0.99999887	0.03986346	0.01204001
9	0.99999989	0.03986346	0.01083601
10	0.99999999	0.03986346	0.00985092
11	1.00000000	0.03986346	0.00903001

TABLE 3. Poisson Estimate of $P(W \le w_0)$ for $N = 1,000$, $m = 25$ and $n = 40$

Example 3.4 (The classical occupancy problem). Let m balls be thrown independently of each other into n boxes, with probability $1/n$ falling into the ith box. Let $X_i = 1$ if the ith box is empty and $X_i = 0$ otherwise, then $W = \sum_{i=1}^n X_i$ is the number of empty boxes. The probability that $P(X_i = 1) = (1 - 1/n)^m$ and $\lambda = E[W] = n(1-1/n)^m$. Since $E[X_iX_j] = (1-2/n)^m \neq (1-1/n)^{2m} = E[X_i]E[X_j]$ for $i \neq j$, so X_i's are dependent. It can be approximated the distribution of W by a Poisson distribution with parameter λ if $(1 - 1/n)^m$ is small, or m/n is large.

In order to determine $E|W - W_i^*|$ in Theorem 1.2, we have to construct W_i^* such that W_i^* has the same distribution as $W - X_i$ conditional on $X_i = 1$. Firstly, we construct Bernoulli random variables $Y_1^i, \ldots, Y_n^i$ as follows: if $X_i = 1$, then let $Y_j^i = X_j$ for all $1 \le j \le n$. Otherwise, throw each of the balls which have fallen into the ith box independently into one of the other boxes, in such away that the probability of a ball falling into box j, $j \neq i$, is $1/(n-1)$. Let $Y_j^i = 1$ if box j is empty, $Y_j^i = 0$ otherwise, and let $W_i^* = \sum_{j=1, j\neq i}^n Y_j^i$. Then, evidently, $Y_j^i \le X_j$ for $j \neq i$, and for each i, W_i^* has the same distribution as $W - X_i$ conditional on $X_i = 1$ and $W_i^* \le W$. Hence, we have $E|W - W_i^*| = E[W - W_i^*]$ and

$$\begin{aligned}\sum_{i=1}^n P(X_i = 1)E[W - W_i^*] &= \lambda(\lambda+1) - E[W^2] \\ &= \lambda^2 - n(n-1)(1-2/n)^m \\ &= \lambda\left\{\lambda - (n-1)\left(\frac{n-2}{n-1}\right)^m\right\}.\end{aligned} \tag{3.7}$$

w_0	Estimate	Uniform Error Bound	Non-Uniform Error Bound
0	0.94976779	0.00133687	0.00133687
1	0.99871669	0.00133687	0.00070379
2	0.99997805	0.00133687	0.00046919
3	0.99999972	0.00133687	0.00035189
4	0.99999997	0.00133687	0.00028152
5	0.10000000	0.00133687	0.00023460

TABLE 4. Poisson Estimate of $P(W \leq w_0)$ for $m = 50$ and $n = 10$

Substituting (3.7) into Theorem 1.2, we have

$$
\begin{aligned}
&\left| P(W \leq w_0) - \sum_{k=0}^{w_0} \frac{\lambda^k e^{-\lambda}}{k!} \right| \\
&\qquad \leq (1 - e^{-\lambda}) \left\{ \lambda - (n-1) \left(\frac{n-2}{n-1} \right)^m \right\} \min \left\{ 1, \frac{e^\lambda}{w_0 + 1} \right\},
\end{aligned} \tag{3.8}
$$

where $w_0 \in \{0, 1, \dots, n\}$, and the bound in (3.8) is small when $\dfrac{m}{n}$ is large.

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