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SYMMETRIC UNITS AND GROUP IDENTITIES IN GROUP ALGEBRAS. I

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Dedicated to Professor L.G. Kovács on his 70th birthday

Abstract. We describe those group algebras over fields of characteristic different from 2 whose units symmetric with respect to the classical involution, satisfy some group identity.

1. Introduction

Let $U(A)$ be the group of units of an algebra A with involution $*$ over the field F and let $S_*(A) = \{u \in U(A) \mid u = u^*\}$ be the set of symmetric units of A.

Algebras with involution have been actively investigated. In these algebras there are many symmetric elements, for example: $x + x^*$ and xx^* for any $x \in A$. This raises natural questions about the properties of the symmetric elements and symmetric units. In [10] Ch. Lanski began to study the properties of the symmetric units in prime algebras with involution, in particular when the symmetric units commute. Using the results and methods of [4], in [5] we classified the cases when the symmetric units commute in modular group algebras of p-groups. The solution of this question for integral group rings and for some modular group rings of arbitrary groups was obtained in [6, 3].

Several results on the group of units $U(R)$ show that if $U(R)$ satisfies a certain group theoretical condition (for example, it is nilpotent or solvable), then R's properties are restricted and a polynomial identity on R holds. This suggests that there may be some general underlying relationship between group identities and polynomial identities. In this topic Brian Hartley made the following:

Conjecture 1. *Let FG be a group algebra of a torsion group G over the field F. If $U(FG)$ satisfies a group identity, then FG satisfies a polynomial identity.*

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The theory of PI-algebras has been established for a long time. On the contrary, the study of algebras with units satisfying a group identity has emerged only recently [11, 12]. Our goal here is to show that with a few extra assumptions, these algebras are actually PI-algebras. In fact, these classes of algebras are quite special, because if the group of units is too small in a algebra, a group identity condition can not limit the structure of the whole algebra. In view of Hartley's conjecture, as a natural generalization the works [5, 6, 3, 10] it is a natural question when does the symmetric units satisfy a group identity in group algebra. Note that the structure theorem of the algebras with involution whose symmetric elements satisfy a polynomial identity was obtained earlier by S.A. Amitsur in [1]. A. Giambruno, S.K. Sehgal and A. Valenti in [8] obtained the following result for group algebras of torsion groups:

Theorem 1. *Let FG be a group algebra of a torsion group G over an infinite field F of characteristic $p > 2$ and assume that the involution $*$ on FG is canonical. The symmetric units $S_*(FG)$ satisfy a group identity if and only if G has a normal subgroup A of finite index, the commutator subgroup A' is a finite p-group and one of the following conditions holds:*
(i) G has no quaternion subgroup of order 8 and G' has of bounded exponent p^k for some k.
(ii) G has of bounded exponent $4p^s$ for some $s \geq 0$, the p-Sylow subgroup of G is normal and G/P is a Hamiltonian 2-subgroup.

In the present paper we extend the result of A. Giambruno, S.K. Sehgal and A. Valenti. For non-torsion groups G we describe the group algebras FG over the field F of characteristic different from 2 whose symmetric units

$$S_*(FG) = \{u = \textstyle\sum_{g\in G} \alpha_g g \in U(FG) \quad | \quad u = u^* = \sum_{g\in G} \alpha_g g^{-1}\}$$

satisfy a group identity. The present result was announced at the International Workshop Polynomial Identities in Algebras, 2002, Memorial University of Newfoundland.

2. Main results

In the sequel of this paper $\mathfrak{d}(\omega)$ denotes a positive integer, which depends on the group identity ω and it is defined in the next section. Our results are the following:

Theorem 2. *Let G be a non-torsion nonabelian group and $char(F) = p \neq 2$ and assume that the symmetric units of FG satisfy some group identity $\omega = 1$. Assume that $|F| > \mathfrak{d}(\omega)$, where $\mathfrak{d}(\omega)$ is an integer which depends only on the word ω. Let P be a p-Sylow subgroup of G and let $t(G)$ be the torsion part of G.*

(I) *If $p > 2$ then P and $t(G)$ are normal subgroups of G such that:*

(a) *$B = t(G)/P$ is an abelian p'-subgroup and its subgroups are normal in G;*
(b) *if B is noncentral in G/P then the algebraic closure L of the prime subfield F_p in F is finite and for all $g \in G/P$ and for any $a \in B$ there exists an $r \in \mathbb{N}$ such that $a^g = a^{p^r}$ and $|L : F_p|$ is a divisor of r;*
(c) *the p-Sylow subgroup P is a finite group;*
(d) *the p-Sylow subgroup P is infinite and G has a subgroup A of finite index, such that A' is a finite p-group and the commutator subgroup H' of $H = AP$ is a bounded p-group. Moreover, if P is unbounded, then G' is a bounded p-group;*

(II) *If $char(F) = 0$ then $t(G)$ is a subgroup, every subgroup of $t(G)$ is normal in G and one of the following conditions holds:*
(a) *$t(G)$ is abelian and each idempotent of $Ft(G)$ is central in FG;*
(b) *$t(G)$ is a Hamiltonian 2-group, and each symmetric idempotent of $Ft(G)$ is central in FG.*

3. Notation, preliminary results and the proof

Let FG be the group algebra of G over F. We introduce the following notation:

- $(g, h) = g^{-1}g^h = g^{-1}h^{-1}gh$ for all $g, h \in G$;
- $|g|$ and $C_G(g)$ are the order and the centralizer of $g \in G$, respectively;
- G', $Syl_p(G)$ are the commutator subgroup and the Sylow p-subgroup of G;
- $t(G)$ is the set of elements of finite order in G;
- $\Delta(G) = \{g \in G \mid \quad [G : C_G(g)] < \infty\}$ is the FC-radical of G;
- $\Delta^p(G) = \langle g \in \Delta(G) \mid \quad g$ has order of a power of p$\rangle$;
- $T_l(G/H)$ is a left transversal of the subgroup H in G;
- $\mathfrak{N}(FG)$ is the sum of all nilpotent ideals of the group algebra FG;
- $A(FG)$ is the augmentation ideal of the group algebra FG.

Let A be an algebra over a field F, let F_0 be the ring of integers of the field F, and suppose that $U(A)$ satisfies a group identity $\omega = 1$. Then, as it was proved in Lemma 3.1 of [11], there exists a polynomial $f(x)$ over F_0 of degree $\mathfrak{d}(\omega)$ which is determined by the word ω. In several papers (see for example [8]) the authors assumed that the field F is infinite so they could apply the "Vandermonde determinant argument". We shall use some lemmas from [8], which are easy to prove using the method of the paper [11] even without the assumption that the field F is infinite.

In our proof we will use the following facts:

Lemma 1. *([1]) Let A be an algebra with involution over F of $char(F) \neq 2$, such that the set of symmetric units of A satisfy a group identity $\omega = 1$. If I is a stable nil ideal of A then the symmetric units of A/I satisfy a group identity.*

Lemma 2. *(see* [8]*) Let A be an algebra over the field F of characteristic $p \neq 2$, such that the set of symmetric units of A satisfy a group identity $\omega = 1$ and $|F| > \mathfrak{d}(\omega)$, where $\mathfrak{d}(\omega)$ is an integer which depends only on the word ω. Then:*

(i) *if A is semiprime, then $asa = 0$ for every nilpotent element $s \in S_*(A)$ and square-zero $a \in S_*(A)$;*
(ii) *if $a \in A$ is square-zero, then $(aa^*)^m = 0$, for some $m \in \mathbb{N}$;*
(iii) *if A is semiprime and $u, v \in A$ such that $uv = 0$, then $usv = 0$ for any square-zero symmetric element s;*
(iv) *if the subring L of A is nil, then L satisfy a polynomial identity;*
(v) *each symmetric idempotent is central;*
(vi) *if A is artinian, then A is isomorphic to a direct sum of division algebras and 2×2 matrices algebras over a field with symplectic involution. Each nilpotent element of A has index at most 2;*
(vii) *if $A = FG$ is the group algebra of the group $G = Q_8 \times \langle c\rangle$, where Q_8 is the quaternion group of order 8, then the order of the cyclic subgroup $\langle c\rangle$ is finite.*

Lemma 3. *(see* [8]*) Let A be a normal abelian subgroup of G of finite index such that $G = A \cdot H$, where H is a finite group. Let $char(F) = p$ and assume that the set of symmetric units of FG satisfy a group identity $\omega = 1$. If $|F| > \mathfrak{d}(\omega)$, where $\mathfrak{d}(\omega)$ is an integer which depends only on the word ω, then G' has bounded exponent p^m, where m depends only on $\mathfrak{d}$.*

Now we are ready to prove the following

Lemma 4. *Let $char(F) = p > 2$ and let the set of symmetric units of FG satisfy a group identity $\omega = 1$. Assume that $|F| > \mathfrak{d}(\omega)$, where $\mathfrak{d}(\omega)$ is an integer which depends only on the word ω. Then the p-Sylow subgroup P of $\Delta(G)$ is normal in G and the set of symmetric units of $F[G/P]$ satisfy a group identity.*

Proof. Let H be a finite subgroup of $\Delta(G)$ and let $J = J(F_pH)$ be the radical of the finite group algebra F_pH over the prime subfield F_p. According to Lemma *2(vi)*, the factor algebra F_pH/J is isomorphic to a direct sum of fields and 2×2 matrices algebras over a finite field with symplectic involution and a nilpotent element $\bar{u} = u + J \in F_pH/J$ has index at most 2. Moreover, from this decomposition follows that $\bar{u}\bar{u}^*$ is central. By Lemma 2*(ii)* the element $\bar{u}\bar{u}^*$ is nilpotent and central in the semiprime algebra F_pH/J. Therefore $\bar{u}\bar{u}^* = 0$ and $uu^* \in J(FH)$.

Let $h \in H$ with $|h| = p^t$. Then $u = h - 1$ is nilpotent and

$$uu^* = (h - 1)(h^{-1} - 1) \in J(FH).$$

It follows that $huu^* = -(h - 1)^2 \in J(FH)$. Using Passman's result (see Lemma 5 in [8], p.453) we obtain that $h - 1 \in J(FH)$ for all $h \in H$ and $H \cap (1 + J)$ is a normal p-subgroup of H, which coincides with the p-Sylow subgroup of H. Thus the p-Sylow subgroup P of $\Delta(G)$ is normal in G, so the proof is complete. $\square$

Lemma 5. *Let FG be a semiprime group algebra over the field F with $char(F) > 2$ such that the set of symmetric units of FG satisfy a group identity $\omega = 1$. Suppose that $|F| > \mathfrak{d}(\omega)$, where $\mathfrak{d}(\omega)$ is an integer which depends only on the word ω. Then one of the following conditions holds:*

(i) *$t(G)$ is abelian and each idempotent of $Ft(G)$ is central in FG.*
(ii) *$t(G)$ is a Hamiltonian 2-group and each symmetric idempotent of $Ft(G)$ is central in FG.*

Proof. (i) Let $a \in t(G)$, such that $(|a|, p) = 1$. Then, by Lemma 2*(v)*, the symmetric idempotent $e = \frac{1}{n}(1 + a + \cdots + a^{|a|-1})$ is central in FG, so $\langle a \rangle$ is normal in G. Now let $p > 2$ and let $a \in t(G)$ be of order p. If $N_G(\langle a \rangle) = G$ then $\overline{\langle a \rangle}$ is a central nilpotent element of the semiprime algebra FG, a contradiction.

Let us prove that each torsion element belongs to $N_G(\langle a \rangle)$. Pick $h \notin N_G(\langle a \rangle)$ such that $|h| = p^t$. The elements $(h-1)(h^{-1}-1)$ and $\overline{\langle a \rangle}$ are symmetric and $(2 - h - h^{-1})^{p^t} = (\overline{\langle a \rangle})^2 = 0$. By Lemma 2*(i)* we get $\overline{\langle a \rangle}(2 - h - h^{-1})\overline{\langle a \rangle} = 0$ and

$$\overline{\langle a \rangle} h \overline{\langle a \rangle} + \overline{\langle a \rangle} h^{-1} \overline{\langle a \rangle} = 0. \tag{1}$$

An element of $Supp(\overline{\langle a \rangle} h \overline{\langle a \rangle})$ can be written as $a^i h a^j$, where $0 \le i, j \le p-1$. If all the elements in $Supp(\overline{\langle a \rangle} h \overline{\langle a \rangle})$ and in $Supp(\overline{\langle a \rangle} h^{-1} \overline{\langle a \rangle})$ are distinct, then on the left-hand side of (1) each element appears at most two times, but this leads to a contradiction if $char(F) \neq 2$. Therefore, in the subset $Supp(\overline{\langle a \rangle} h \overline{\langle a \rangle})$ not all elements are different, whence there exist i, j, k, l such that $a^i h a^j = a^k h a^l$ and either $i \neq k$ or $j \neq l$. If, for example, $i > k$, then $h^{-1} a^{i-k} h = a^{l-j}$ and $h \in N_G(\langle a \rangle)$.

Now, let $h \notin N_G(\langle a \rangle)$ be a p'-element. As we have seen before, $\langle h \rangle$ is normal in G, so $\langle a, h \rangle$ is a finite subgroup. By Lemma 4 the p-Sylow subgroup P of $\langle a, h \rangle$ is normal in $\langle a, h \rangle$ and $(a, h) \in P \cap \langle h \rangle = \langle 1 \rangle$, a contradiction.

Therefore, each element of finite order belongs to $N_G(\langle a \rangle)$. Moreover, the elements of order p in G form an elementary abelian normal p-subgroup E of G.

Finally, if $h \notin N_G(\langle a \rangle)$, then h has infinite order and h acts on E. The subgroups $\langle a^h \rangle$ and $\langle a \rangle$ are different and we can choose a subgroup $\langle b \rangle \subset E$, which differs from $\langle a \rangle$. Clearly, $\overline{\langle a \rangle}(h + h^{-1})\overline{\langle a \rangle}$ and $\overline{\langle b \rangle}$ are square-zero symmetric elements and according to Lemma 2*(i)*,

$$\overline{\langle b \rangle \langle a \rangle}(h + h^{-1})\overline{\langle a \rangle \langle b \rangle} = 0. \tag{2}$$

Since hE and $h^{-1}E$ are different cosets, from (2) follows that

$$\overline{\langle b \rangle \langle a \rangle} h \overline{\langle a \rangle \langle b \rangle} = 0. \tag{3}$$

The subgroup $H=\langle a,b\rangle \subset E$ has order p^2 and by (3) we have $\overline{H}h_1\overline{H}h_2=0$ for all h_1,h_2. Since elements of finite order belong to $N_G(H)$, we get $(\overline{H}FG)^2=0$, which is impossible by the semiprimeness of FG. Thus G have no p-elements and all finite cyclic subgroups of G are normal in G. Applying Lemmas 6 and 7 from [8] and the fact that G have no p-elements ($p\neq 2$), we obtain that $t(G)$ is either an abelian group or a Hamiltonian 2-group.

Let $t(G)$ be an abelian group and let $e\in Ft(G)$ be a noncentral idempotent in FG. Set $H=\langle Supp(e)\rangle$. Since every subgroup of $t(G)$ is normal in G, the subgroup H is also normal in G and FH has a primitive idempotent f, which does not commute with some $g\in G$ of infinite order. Then $g^{-1}fg\neq f$ is also a primitive idempotent of FH and $(g^{-1}fg)f=0$, i.e. $(fg)^2=(gf)^2=0$.

Let $g^{-1}fg=\overline{f}\neq f^*$. By Lemma 2(v) we have $f\neq f^*$, so $g^{-1}f+f^*g$ is a square-zero symmetric element and by Lemma 2*(iii)*, we get that

$$fg(g^{-1}f+f^*g)fg=0.$$

It follows that $f+g(\overline{f}f^*)gf=f=0$, a contradiction. Therefore, $g^{-1}fg=f^*$, so $(f^*)^*=(g^{-1}fg)^*=g^{-1}f^*g=f$. Furthermore, $g^{-2}fg^2=g^{-1}f^*g=f$ and $f^*g^2=g^2f^*$. Since $f^*g^2=g^2f^*$, $(gf^*)^2=0$ and $gf+f^*g^{-1}$ is square-zero symmetric element, by Lemma 2*(iii)* we obtain that

$$gf^*(gf+f^*g^{-1})gf^*=gf^*g^2(g^{-1}fg)f^*+gf^*=gf^*g^2f^*+gf^*=0.$$

Thus $(g^2+1)f^*=0$, which is impossible, since g^2H and H are different cosets. $\square$

Lemma 6. *Let F be a field of characteristic p, and suppose that G contains a normal locally finite p-subgroup P such that the centralizer of each element of P in every finitely generated subgroup of G is of finite index. Then $\mathfrak{I}(P)$ is a locally nilpotent ideal.*

Proof. Clearly, $\{\ u(h-1)\ \mid\ u\in T_l(G/P),\ 1\neq h\in P\ \}$ is an F-basis for the ideal $\mathfrak{I}(P)$. Let us show that the subalgebra $W=\langle u_1(h_1-1),\ \dots,\ u_s(h_s-1)\rangle_F$ is nilpotent. According to our assumption, the centralizers of $h_1,\dots,h_s$ in the subgroup $H=\langle u_1,\dots,u_s,h_1,\dots,h_s\rangle$ have finite index. Since P is normal, its subgroup $L=\langle h_1^u,h_2^u,\dots,h_s^u\mid u\in H\rangle$ is a finitely generated FC-group and by a Theorem of B.H. Neumann ([1], Theorem 4, p.19) L is a finite p-group. Thus the augmentation ideal $A(FL)$ is nilpotent with index, say, t. Furthermore, $A(FL)=u^{-1}A(FL)u$ for any $u\in H$ and this implies that $(A(FL)\cdot FH)^n=A^n(FL)\cdot FH$ for any $n>0$, so $W^t\subseteq A^t(FL)\cdot FH=0$, because $W\subseteq A(FL)\cdot FH$. Therefore W is a nilpotent subalgebra and $\mathfrak{I}(P)$ is a locally nilpotent ideal.

$\square$

Lemma 7. *Let G be a group with a nontrivial p-Sylow subgroup P and let $char(F)=p>2$. If the set of symmetric units of FG satisfy a group identity $\omega=1$ and $|F|>\mathfrak{d}(\omega)$, where $\mathfrak{d}(\omega)$ is an integer which depends only on the word ω, then P is normal in G and the ideal $\mathfrak{I}(P)$ is nil.*

Proof. Let P be a maximal normal p-subgroup of G such that the ideal $\mathfrak{I}(P)$ is nil. By Lemma 1 the set of symmetric units of $F[G/P]$ satisfy a group identity. If $F[G/P]$ is semiprime, then by (i) of the Theorem the group G/P has no p-elements and P coincides with the p-Sylow subgroup of G. Now, suppose that $F[G/P]$ is not semiprime. According to Theorem 4.2.13 ([13], p.131) the group $\Delta(G/P)$ has a nontrivial p-Sylow subgroup P_1/P, which is normal in G/P by Lemma 4. Clearly, P_1/P is an FC-subgroup of G/P, so by Lemma 6 the ideal $\mathfrak{I}(P_1/P)$ is nil.

Since $\mathfrak{I}(P_1/P) \cong \mathfrak{I}(P_1)/\mathfrak{I}(P)$ and P_1 is normal in G, the ideal $\mathfrak{I}(P_1)$ is nil and $P \subset P_1$, a contradiction. Thus $P = Syl_p(G)$ and the proof is done. □

Lemma 8. *Let R be an algebra with involution $*$ over a field F of characteristic $p > 2$ and assume that $S_*(R)$ satisfies a group identity and $|F| > \mathfrak{d}(\omega)$. If some nil subring L of R is $*$-stable, then L satisfies a non-matrix polynomial identity.*

Proof. Let $A = F\langle X\rangle[[t]]$ be the ring of power series over the polynomial ring $F\langle X\rangle$ with noncommuting indeterminably $X = \{x_1, x_2\}$. By a result of Magnus, the elements $1 + x_1t, 1 + x_2t$ are units in A and $\langle 1 + x_1t, 1 + x_2t\rangle$ is a free group.

Assume that $S_*(R)$ satisfies the group identity w, where w is a reduced word in 2 variables. Then $w(1+x_1t, 1+x_2t) \neq 1$ according to result of Magnus and it is well-known that $(1+x_it)^{-1} = 1-x_it+x_i^2t^2-\cdots$. If we substitute $(1+x_it)^{-1}$ in the expression $w(1 + x_1t, 1 + x_2t) - 1$, then it can be expanded as

$$\sum_{i\geq s} g_i(x_1, x_2)t^i, \tag{4}$$

where $g_i(x_1, x_2) \in F\langle X\rangle$ is a homogeneous polynomial of degree i. Obviously there exists a smallest integer $s \geq 1$ such that $g_s(x_1, x_2) \neq 0$.

Let L be a $*$-stable nil subring and let $S(L)$ be the set of the symmetric elements of L. Take now $r_1, r_2 \in S(L)$ and let $\lambda \in F$. Obviously, r_1, r_2 are nilpotent elements, so each $1 + \lambda r_i$ is a symmetric unit in R and

$$(1 + r_i\lambda)^{-1} = 1 - r_i\lambda + r_i^2\lambda^2 + \cdots + (-1)^{t-1}r_i^{t-1}\lambda^{t-1}$$

for a suitable t. By evaluating w on these elements, (4) gives us a finite sum $\sum_{i\geq s}^{l} g_i(r_1, r_2)\lambda^i = 0$ for some l. Since $|F| > \mathfrak{d}(\omega)$, we can apply the Vandermonde determinant argument to obtain $g_i(r_1, r_2) = 0$ for all i. Therefore $g_s(x_1, x_2)$ is a $*$-polynomial identity on $S(L)$. Finally, by [1] it follows that $S(L)$ satisfies an ordinary polynomial identity.

Suppose that the homogeneous polynomial $g(x_1, x_2)$ vanishes on the matrix algebra $M_2(K)$ over a commutative ring K. Then

$$g(x_1, x_2) = h(x_1, x_2) + g_{11}(x_1, x_2) + g_{12}(x_1, x_2) + g_{21}(x_1, x_2) + g_{22}(x_1, x_2),$$

where $h(x_1, x_2)$ consists of all monomials which contain x_1^2 or x_2^2 while the $g_{ij}(x_1, x_2)$ contain all the remaining monomials beginning with x_i and ending with x_j for $i, j \in \{1, 2\}$. If a and b are two square-zero matrices, then $h(a, b) = 0$, because each term of h has a^2 or b^2 as a factor, so we conclude that $ag_{21}(a,b)b = 0$. Clearly $x_1g_{21}(x_1,x_2)x_2$ is some polynomial $f(x_1x_2)$. Then $f(ab\lambda) = 0$ for each $\lambda \in F$ and, by the Vandermonde determinant argument, we get $(ab)^d = 0$ for some d. Take, for instance, the matrix units $a = e_{12}$ and $b = e_{21}$, then we obtain a contradiction. □

Lemma 9. *Let R be an algebra over a field F of positive characteristic p satisfying a non-matrix polynomial identity. Then R satisfies also a polynomial identity of the form $([x,y]z)^{p^l}$ and $[x,y]^{p^l}$*

Proof. Let $g(x_1, x_2, \dots, x_n)$ be a non-matrix polynomial identity in R. The variety W determined by the polynomial identity $g(x_1, x_2, \dots, x_n)$ contains a relatively free algebra K of rank 3. Of course, K is a finitely generated PI-algebra, and the result of Braun and Razmyslov (Theorem 6.3.39, [14]) states that the radical $J(K)$ of K is nilpotent. Writing $K/J(K)$ as a subdirect sum of primitive rings $\{L_i\}$, we get that every primitive ring L_i satisfies the non-matrix polynomial identity $g(x_1, x_2, \dots, x_n)$, as a homomorphic image of K. By Theorem 2.1.4 of [9], L_i is either isomorphic to the matrix ring $M_m(D)$ over a division ring D, or for any m the matrix ring $M_m(D)$ is an epimorphic image of some subring of L_i.

Thus $M_m(D)$ satisfies a non-matrix polynomial identity g, which is possible only if L_i is a commutative ring. Consequently, $K/J(K)$ is a commutative algebra, so K satisfies a polynomial identity of the form $([x,y]z)^{p^l}$ such that $J(K)^{p^l} = 0$. Since R belongs to the variety W, the algebra R also satisfies a polynomial identity $([x,y]z)^{p^l}$. □

Lemma 10. *Let FG be a non semiprime group algebra over the field F with $char(F) > 2$, such that the set of symmetric units of FG satisfy a group identity $\omega = 1$ and $|F| > \mathfrak{d}(\omega)$, where $\mathfrak{d}(\omega)$ is an integer which depends only on the word ω. If $\mathfrak{N}(FG)$ is not nilpotent then FG is a PI-algebra, where $\mathfrak{N}(FG)$ is the sum of all nilpotent ideals of FG.*

Proof. Clearly the non nilpotent ideal $\mathfrak{N} = \mathfrak{N}(FG)$ is invariant under the involution $*$ and by Lemma 2(*iv*) the ring $\mathfrak{N}$ satisfies a polynomial identity $f(x_1, \dots, x_n)$. Moreover, by Lemma 2.8 of [12] the algebra FG satisfies a non-degenerate multilinear generalized polynomial identity and hence, by Theorem 5.3.15 ([13], p.202), $|G : \Delta(G)| < \infty$ and $\Delta(G)'$ is finite.

Set $P = Syl_p(G)$ and $P_1 = Syl_p(\Delta(G)')$. By Lemma 4, $P \cap \Delta(G)' = P_1 \lhd G$ and P_1 is a finite p-group. Thus $\mathfrak{I}(P_1)$ is a nilpotent ideal and by (i) of the Theorem, the set of symmetric units of $F[\Delta(G)'/P_1]$ satisfy a group identity, so $\Delta(G)'/P_1$ is either an abelian p'-group or a Hamiltonian 2-group.

If $P_1 = \Delta(G)'$, then by Theorem 5.3.9 ([13], p.197) the algebra FG is a PI-algebra. If $P_1 \subsetneq \Delta(G)'$ then we can suppose that G is a group such that $Syl_p(\Delta(G)') = 1$ and $\Delta(G)'$ is either an abelian p'-group or a Hamiltonian 2-group.

Set $P_2 = Syl_p(\Delta(G))$. Clearly, $P_2 = P \cap \Delta(G)$ is normal in $\Delta(G)$. Since $[P : P_2] < \infty$ and P is an infinite group, the group P_2 is infinite, too. If

$a \in P_2$, $b \in \Delta(G)$, then $(a, b) \in P_2 \cap \Delta(G)' = 1$, so $(a, b) = 1$ and P_2 is a central subgroup in $\Delta(G)$.

Let us prove that $F\Delta(G)$ is a PI-algebra. If $\Delta(G)$ is a torsion group, then by [8] the statement is trivial.

Since $\mathfrak{N}(F\Delta(G)) \subseteq \mathfrak{N}(FG)$, the ideal $\mathfrak{N}(F\Delta(G))$ also satisfies the same polynomial identity $f(x_1, \dots, x_n)$. By the standard multilinearization process, we may assume that $f(x_1, \dots, x_n)$ is multilinear.

Assume that P_2 has bounded exponent. Then the maximal elementary abelian p-subgroup E of P_2 is infinite. Let $f(a_1, \dots, a_n) = \sum_i \alpha_i y_i$, where $a_1, \dots, a_n \in F\Delta(G)$, $y_1, \dots, y_n \in T_l(\Delta(G)/E)$ and $\alpha_i \in FE$. Then there exists a finite subgroup B such that $\alpha_i \in FB$ and $E = B \times \prod_j \langle c_j \rangle$. Since $(c_k - 1)a_k \in \mathfrak{N}(F\Delta(G))$ and P_2 is central, we conclude that

$$f((c_1 - 1)a_1, \dots, (c_n - 1)a_n) = (c_1 - 1) \cdots (c_n - 1) f(a_1, \dots, a_n) = 0.$$

It follows that $f(a_1, \dots, a_n) = 0$, because $B \cap \prod_j \langle c_j \rangle = \langle 1 \rangle$.

Now let P_2 be of unbounded exponent and $c \in P_2$. Then $(c - 1)a_k \in \mathfrak{N}(F\Delta(G))$ and also

$$f((c - 1)a_1, \dots, (c - 1)a_n) = (c - 1)^n f(a_1, \dots, a_n) = 0$$

for all $c \in P_2$. Then $f(a_1, \dots, a_n)$ belongs to the annihilator of the augmentation ideal $A(FP_2^{p^t})$, where $n \leq p^t$. Since $P_2^{p^t}$ is infinite, we have

$$Ann_l(A(FP_2^{p^t})) = 0.$$

It follows that $f(a_1, \dots, a_n) = 0$, so $f(x_1, \dots, x_n)$ is a polynomial identity for $F\Delta(G)$. Since $F\Delta(G)$ is a PI-algebra and $[G : \Delta(G)] < \infty$, the algebra FG is PI, too. □

Proof of the theorem. Let FG be a group algebra of a non-torsion group G over a field of positive characteristic p. By Lemma 7 the p-Sylow subgroup P is normal in G and $F[G/P] \cong FG/\mathfrak{I}(P)$, so the symmetric units of semiprime algebra $F[G/P]$ satisfy a group identity. By Lemma 5 $B = t(G/P)$ is a subgroup of G/P and B is either an abelian p'-group or a Hamiltonian 2-group. If B is a Hamiltonian 2-group, then Q_8 is a subgroup of B. Choose an element $c \in G/P$ of infinite order. Since every subgroup of $t(G)/P$ is normal in G/P and $|Aut(Q_8)| < \infty$, there exists a $t \in \mathbb{N}$ such that $c^t \in C_{G/P}(Q_8)$ and $Q_8 \times \langle c^t \rangle \subseteq G/P$. Then Lemma 2(vii) asserts that c has finite order, a contradiction. So B is an abelian p'-group and by Lemma 5 every idempotent of FB is central in $F[G/P]$. Moreover, if B is noncentral, then according to [7] the group B satisfy (i.b) of our Theorem.

Now, let P be infinite. By Corollary 8.1.14 ([13], p.312) the ideal $\mathfrak{N}(FG)$ is non-nilpotent, so by Lemma 10, the algebra FG is a PI-algebra, i.e. G has a subgroup A with finite index such that A' is a finite p-group. According to Lemma 1, it can be assumed that G has an abelian subgroup A of finite index.

We claim that the commutator subgroup of $H = P \cdot A$ is a bounded p-group. Clearly $S_*(FP)$ satisfies a group identity and according to Lemma 3 P' is a bounded p-group. The normal abelian p-subgroup $P' \cap A$ has finite exponent and according to Lemma 6 the ideal $\mathfrak{I}(P' \cap A)$ is locally nilpotent of bounded degree. The subgroup $P' \cap A$ of P' has finite index in P and

$$\mathfrak{I}(P')/\mathfrak{I}(P' \cap A) \cong \mathfrak{I}(P'/(P' \cap A)).$$

Therefore $\mathfrak{I}(P')$ is a locally nilpotent ideal of bounded degree p^t for some t. Clearly $FG/\mathfrak{I}(P') \cong F[G/P']$ and put $P' = \langle 1 \rangle$. Since A has a finite index in $H = P \cdot A$, Lemma 3 ensures that H' is a p-group of bounded exponent and according to Lemma 1, we can put $H' = \langle 1 \rangle$ again.

The p-Sylow subgroup P of G is abelian and by Lemma 8 the ideal $\mathfrak{I}(P)$ satisfies a non-matrix polynomial identity, Moreover, by Lemma 9 the ideal $\mathfrak{I}(P)$ satisfies polynomial identities of the following forms: $[x, y]^{p^l}$ and $([x, y]z)^{p^l}$.

Let $h \in G$ and $a \in P$. Clearly $(a-1)h, h^{-1}(a^{-1}-1) \in \mathfrak{I}(P)$ and

$$[(a-1)h, h^{-1}(a^{-1}-1)]^{p^l} = (a^h)^{p^l} + (a^h)^{-p^l} - a^{p^l} - a^{-p^l} = 0$$

which implies that either $(h, a)^{p^l} = 1$ or $h^{-1}a^{p^l}h = a^{-p^l}$.

Put $z = a^{p^l}$. From $h^{-1}a^{p^l}h = a^{-p^l}$ it follows that $h^{-1}zh = z^{-1}$ and $([z-1, (z^{-1}-1)h])^{p^l} = 0$. Clearly $[z-1, (z^{-1}-1)h] = -z^{-2}(z+1)(z-1)^2h$ so

$$\begin{aligned} 0 &= ([z-1, (z^{-1}-1)h])^{p^l} \\ &= -\big((z+1)(z-1)^2(z^{-1}+1)(z^{-1}-1)^2h^2\big)^{\frac{p^l-1}{2}}\big(z^{-2}(z+1)(z-1)^2h\big) \\ &= -z^{\frac{-3p^l-1}{2}} \cdot (z+1)^{p^l} \cdot (z-1)^{2p^l} \cdot h^{p^l}. \end{aligned}$$

Since $char(K) > 2$, the element $z+1$ is a unit and $(z-1)^{2p^l} = (a-1)^{2p^{2l}} = 0$ and the order of a at most $2p^{2l}$. Therefore $(h, a)^{p^{2l+1}} = 1$ for all $h \in G$, $a \in P$ and $2l+1$ depends on only the group identity. Since (G, P) is a p-group of bounded exponent, we can again make a reduction, so we can assumed that $(G, P) = 1$ and P is central.

Let P be a central subgroup of unbounded exponent and $h_1, h_2 \in G$. Obviously

$$\begin{aligned} ((h_1, h_2)^{p^l} - 1)(a-1)^{p^{3l}} &= ((h_1, h_2) - 1)^{p^l}(a-1)^{p^{3l}} \\ &= ([h_1^{-1}(a-1), h_2^{-1}(a-1)]h_1h_2(a-1))^{p^l} = 0 \end{aligned}$$

for $a \in P$. Since there are infinitely many element of the form $a^{p^{3l}}$ we conclude that $(h_1, h_2)^{p^l} = 1$ and the proof is complete. ☐

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