Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
22 (2006), 265–273
www.emis.de/journals
ISSN 1786-0091

WEIGHTED $(0,1,3)$-INTERPOLATION ON AN ARBITRARY SET OF NODES

PANKAJ MATHUR

In Memory of Prof. A. Sharma.

Abstract. J. Balázs [2] considered the problem of modified weighted $(0,2)$-interpolation on a general set of nodes by removing the weighted second derivative at one of the end points and prescribing first derivative at that point. In this paper (following the suggestion of Prof. A. Sharma) I have studied the case of $(0,1,3)$-interpolation on a general set of nodes, when the condition of weighted third derivative has been replaced at both the end points by the second derivative at those point.

1. Introduction

Let

$$-\infty < a < x_{n,n} < x_{n-1,n} < \ldots < x_{1,n} < b < +\infty,\ n \in N \tag{1.1}$$

be a given set of points (nodal points) in a finite or infinite interval (a,b) and $w(x) \in C^2(a,b)$ be a weight function. On the suggestion of P. Turán, J. Balázs [1] initiated the study of weighted $(0,2)$-interpolation which means the determination of a polynomial of degree $\leq 2n-1$ satisfying the conditions

$$R_n(x_{i,n}) = \alpha_{i,n}, \quad (wR_n)''(x_{i,n}) = \beta_{i,n} \qquad i = 1\ldots n, \tag{1.2}$$

where $\{\alpha_{i,n}\}_{i=1}^n$and $\{\beta_{i,n}\}_{i=1}^n$ are arbitrary real numbers, $x_{i,n}$ $(i = 1\ldots n)$ are the zeros of the nth ultraspherical polynomial $P^{(\alpha)}(x)$, $\alpha > -1$ and the weight is the function $w(x) = (1-x^2)^{\frac{\alpha+1}{2}}$. He proved that there generally does not exist any polynomial of degree $\leq 2n-1$ satisfying (1.2). However, by taking an additional condition:

$$R_n(0) = \sum_{i=1}^{n} \alpha_{i,n} l_{i,n}^2(0), \tag{1.3}$$

2000 *Mathematics Subject Classification.* 41A05.
Key words and phrases. Weight Function, interpolation.

where 0 is not a nodal point, he showed that there exists a uniquely determined polynomial of degree $\leq 2n$ (n is even), and proved a convergence theorem. If n is odd, the uniqueness is not true. L. Szili [7] investigated the above problem of J. Balázs on an infinite interval, taking the nodes as the zeros of nth Hermite polynomial $H_n(x)$ and the weight function $w(x) = \exp(-x^2/2)$. His results were further sharpened by I. Joó [4]. In another paper, S. Datta and P. Mathur [3] have further improved the results of [7] by replacing the artificial looking condition (1.3) by an interpolatory condition $R_n(0) = \alpha_0$, for n even and obtained that the necessary and sufficient condition for the existence of weighted $(0,2)$-interpolation in the case of n odd, is $R_n'(0) = \beta_0$. They have also proved a convergence theorem in both cases.

K. K. Mathur and R. B. Saxena [5] studied the case of weighted $(0,1,3)$-interpolation on an infinite interval by taking the nodes as the zeros of nth Hermite polynomial $H_n(x)$ and showed that if n is even, there exists a unique polynomial $G_n(x)$ of degree $\leq 3n$ satisfying the conditions:

$$G_n(x_{i,n}) = a_{i,n},\ G_n'(x_{i,n}) = b_{i,n},\ (wG_n)'''(x_{i,n}) = c_{i,n}, \quad i = 1 \ldots n \tag{1.4}$$

and

$$G_n(0) = \sum_{i=1}^{n} \left\{(1+3x_{i,n}^2)a_{i,n} - b_{i,n}x_{i,n}\right\} l_{i,n}^2(0), \tag{1.5}$$

where 0 is not a nodal point. They also obtained the explicit representation of the fundamental polynomials and proved a convergence theorem.

Recently, considering the nodes as the zeros of $W_{n-1}(x)$ where

$$W_{n-1}(x) = \prod_{i=1}^{n-1}(x - x_{i,n}) \tag{1.6}$$

and the weight function $w(x) \in C^2(a,b)$ satisfying the conditions:

$$w(x_{i,n}) \neq 0,\ (wW_{n-1})''(x_{i,n}) = 0 \quad i = 1 \ldots n-1 \tag{1.7}$$

J. Balázs [2] studied the problem of modified weighted $(0,2)$-interpolation. He showed that, for any n (even or odd), there exists a unique polynomial $S_n(x)$ of degree $\leq 2n-1$ satisfying the conditions:

$$\begin{aligned} S_n(x_{i,n}) &= \lambda_{i,n}, \quad i = 1 \ldots n \\ S_n'(x_{n,n}) &= \mu_{n,n} \\ (wS_n)''(x_{i,n}) &= \nu_{i,n}, \quad i = 1 \ldots n-1 \end{aligned}$$

where $\{\lambda_{i,n}\}_{i=1}^{n}$, $\mu_{n,n}$ and $\{\nu_{i,n}\}_{i=1}^{n-1}$ are arbitrarily given real numbers.

A. Sharma suggested the study of the above problem when the condition of weighted second derivative was replaced at both end points of (1.1) by the first derivatives at those points. This motivated us to consider the problem of determining the modified weighted $(0,1,3)$-interpolation. Precisely, we shall study the following:

Problem. Let $x_{i,n}$ be the zeros of

$$W_{n-2}(x) = \prod_{i=2}^{n-1}(x - x_{i,n}) \tag{1.8}$$

and suppose that the weight function $w(x) \in C^3(a,b)$ satisfies the conditions:

$$w(x_{i,n}) \neq 0,\ (w^2W_{n-2}^2)'''(x_{i,n}) = 0, \quad i = 2 \dots n-1 \tag{1.9}$$

and

$$\int_{x_1}^{x_n} (t - x_{1,n})^2(t - x_{n,n})^2 W_{n-2}(t)dt \neq 0. \tag{1.10}$$

Does there exist a unique polynomial $T_n(x)$ of degree $\leq 3n-1$ satisfying the conditions:

$$\begin{aligned} T_n(x_{i,n}) = y_{i,n},\ T_n'(x_{i,n}) = y_{i,n}',\ i = 1 \dots n; \\ T_n''(x_{1,n}) = y_{1,n}'',\ T_n''(x_{n,n}) = y_{n,n}''; (w^2T_n)'''(x_{i,n}) = y_{i,n}''',\ i = 2 \dots n-1; \end{aligned} \tag{1.11}$$

where $\{y_{i,n}\}_{i=1}^n$, $\{y_{i,n}'\}_{i=1}^n$, $\{y_{i,n}'''\}_{i=2}^{n-1}$, $y_{1,n}''$, $y_{n,n}''$ are given arbitrarily real numbers?

In this paper we answer this problem in affirmative. For the sake of convenience we shall use i in the place of i,n in the subscript from now on.

2. New results

Theorem 1. *If the nodes are the zeros of the polynomial $W_{n-2}(x)$ and $w(x) \in C^3(a,b)$ is a weight function satisfying the conditions (1.9) and (1.10), then there exists a unique polynomial $T_n(x)$ of degree $\leq 3n-1$ satisfying the conditions (1.11), which can be explicitly represented as:*

$$T_n(x) = \sum_{i=1}^{n} y_i A_i(x) + \sum_{i=1}^{n} y_i' B_i(x) + \sum_{i=2}^{n-1} y_i''' C_i(x) + y_1'' D_1(x) + y_n'' D_n(x) \tag{2.1}$$

where the fundamental polynomials $A_i(x)$, $B_i(x)$, $C_i(x)$ and $D_i(x)$ are given in the theorems below.

Theorem 2. *Let $W_{n-2}(x)$ be given by (1.8) and let the weight function $w(x) \in C^3(a,b)$ satisfy the conditions (1.9) and (1.10), then the polynomial $A_k(x)$, for*

$k = 2 \ldots n-1$, has the form

$$A_k(x) = \frac{1}{(x_k - x_1)^3(x_k - x_n)^3}\left[(x - x_1)^3(x - x_n)^3 l_k^3(x) + W_{n-2}^2(x)\right.$$
$$\times\left\{\int_{x_1}^{x}(t - x_1)^3(t - x_n)^3\frac{\{l_k'(x_k) - (l_k'(x_k)^2 - l_k''(x_k))(t - x_k)\}\, l_k(t) - l_k'(t)}{(t - x_k)^2 W_{n-2}'(x_k)^2}dt\right.$$
$$+ a_k\int_{x_1}^{x} l_k(t)dt +$$
$$\int_{x_1}^{x}\left[b_k + c_k(t - x_n) + d_k(t - x_1)^2(t - x_n) + e_k(t - x_1)(t - x_n)^2\right.$$
$$\left.\left. +\ g_k(t - x_1)^2(t - x_n)^2\right] W_{n-2}(t)dt\right\}] + h_k B_k(x)$$

where, $l_k(x)$ are the fundamental polynomials of Lagrange interpolation based on the zeros of $W_{n-2}(x)$ and

$$a_k = -\frac{1}{6w^2(x_k)W_{n-2}'(x_k)^2}\left[2\left\{w^2(x)(x - x_1)^3(x - x_n)^3 l_k^3(x)\right\}'''(x_k)\right.$$
$$\left. - 3w^2(x_k)(x_k - x_1)^3(x_k - x_n)^3\left\{2l_k'(x_k)^3 - 3l_k'(x_k)l_k''(x_k) + l_k'''(x_k)\right\}\right],$$
$$b_k = -\frac{a_k l_k(x_n)}{W_{n-2}(x_n)},$$
$$c_k = \frac{a_k}{x_1 - x_n}\left[\frac{l_k(x_n)}{W_{n-2}(x_n)} - \frac{l_k(x_1)}{W_{n-2}(x_1)}\right],$$
$$d_k = -\frac{a_k}{(x_1 - x_n)^2}\left[\frac{l_k'(x_n)}{W_{n-2}(x_n)} - \frac{l_k(x_n)W_{n-2}'(x_n)}{W_{n-2}^2(x_n)} + \frac{c_k}{a_k}\right],$$
$$e_k = -\frac{a_k}{(x_1 - x_n)^2}\left[\frac{l_k'(x_1)}{W_{n-2}(x_1)} - \frac{l_k(x_1)W_{n-2}'(x_1)}{W_{n-2}^2(x_1)} + \frac{c_k}{a_k}\right],$$
$$h_k = -3\left[\frac{2x_k - (x_1 + x_n)}{(x_k - x_1)(x_k - x_n)} + l_k'(x_k)\right]$$

and

$$g_k = -\left[\left\{\int_{x_1}^{x_n}(t - x_1)^3(t - x_n)^3\frac{\{l_k'(x_k) - (l_k'(x_k)^2 - l_k''(x_k))(t - x_k)\}\, l_k(t)}{(t - x_k)^2 W_{n-2}'(x_k)^2}dt\right.\right.$$
$$+ a_k\int_{x_1}^{x_n} l_k(t)dt + \int_{x_1}^{x_n}\left[b_k + c_k(t - x_n) + d_k(t - x_1)^2(t - x_n)\right.$$
$$\left.\left.\left. +\ e_k(t - x_1)(t - x_n)^2\right] W_{n-2}(t)dt\right\}\right]$$
$$\times\left[\int_{x_1}^{x_n}(t - x_1)^2(t - x_n)^2 W_{n-2}(t)dt\right]^{-1}.$$

Also

$$A_1(x) = \frac{W_{n-2}^2(x)}{W_{n-2}^2(x_1)} \left[1 + \int_{x_1}^{x} \left\{ c_1(t-x_n) + d_1(t-x_1)^2(t-x_n) \right. \right.$$
$$\left. \left. + e_1(t-x_1)(t-x_n)^2 + \; g_1(t-x_1)^2(t-x_n)^2 \right\} W_{n-2}(t)dt \right]$$

where

$$c_1 = -\frac{2W'_{n-2}(x_1)}{(x_1-x_n)W_{n-2}^2(x_1)},$$
$$d_1 = -\frac{c_1}{(x_1-x_n)^2},$$
$$e_1 = \frac{2W'_{n-2}(x_1)}{(x_1-x_n)^2 W_{n-2}^2(x_1)} \left[\frac{4W'_{n-2}(x_1)}{W_{n-2}(x_1)} + \frac{1}{x_1-x_n} - \frac{W''_{n-2}(x_1)}{W'_{n-2}(x_1)} \right]$$

and

$$g_1 = -\left[1 + \int_{x_1}^{x_n} \left\{ c_1(t-x_n) + d_1(t-x_1)^2(t-x_n) \right. \right.$$
$$\left. \left. + e_1(t-x_1)(t-x_n)^2 \right\} W_{n-2}(x)dt \right]$$
$$\times \left[\int_{x_1}^{x_n} (t-x_1)^2(t-x_n)^2 W_{n-2}(t)dt \right]^{-1}.$$

The expression for $A_n(x)$ can be given by interchanging x_1 and x_n in $\mathrm{A}_1(x)$.

Theorem 3. *The fundamental polynomials $B_k(x)$, for k=2... n-1 has the form:*

$$B_k(x) = \frac{1}{(x_k-x_1)^3(x_k-x_n)^3} \left[(x-x_1)^3(x-x_n)^3(x-x_k)l_k^3(x) \right.$$
$$+ W_{n-2}^2(x) \left\{ p_k \int_{x_1}^{x} l_k(t)dt + \int_{x_1}^{x} (t-x_1)^3(t-x_n)^3 \frac{l'_k(x_k)l_k(t) - l'_k(t)}{(t-x_k)W'_{n-2}(x_k)^2} dt \right.$$
$$+ \int_{x_1}^{x} \left[q_k + r_k(t-x_n) + s_k(t-x_1)^2(t-x_n) + u_k(t-x_1)(t-x_n)^2 \right.$$
$$\left. \left. \left. + \; v_k(t-x_1)^2(t-x_n)^2 \right] W_{n-2}(t)dt \right\} \right],$$

where

$$p_k = -\frac{1}{6w^2(x_k)W'_{n-2}(x_k)^2}\left[\left\{w^2(x-x_k)(x-x_1)^3(x-x_n)^3 l_k^3(x)\right\}'''(x_k)\right.$$
$$\left. + 6w^2(x_k)(x_k-x_1)^3(x_k-x_n)^3\left\{l'_k(x_k)^2 - l''_k(x_k)\right\}\right],$$
$$q_k = -\frac{p_k l_k(x_n)}{W_{n-2}(x_n)},$$
$$r_k = \frac{p_k}{x_1-x_n}\left[\frac{l_k(x_n)}{W_{n-2}(x_n)} - \frac{l_k(x_1)}{W_{n-2}(x_1)}\right],$$
$$s_k = -\frac{p_k}{(x_1-x_n)^2}\left[\frac{l'_k(x_n)}{W_{n-2}(x_n)} - \frac{l_k(x_n)W'_{n-2}(x_n)}{W^2_{n-2}(x_n)} + \frac{r_k}{p_k}\right],$$
$$u_k = -\frac{p_k}{(x_1-x_n)^2}\left[\frac{l'_k(x_1)}{W_{n-2}(x_1)} - \frac{l_k(x_1)W'_{n-2}(x_1)}{W^2_{n-2}(x_1)} + \frac{r_k}{p_k}\right]$$

and

$$v_k = -\left[\left\{\int_{x_1}^{x_n}(t-x_1)^3(t-x_n)^3\frac{l'_k(x_k)l_k(t)-l'_k(t)}{(t-x_k)W'_{n-2}(x_k)^2}dt\right.\right.$$
$$+ p_k\int_{x_1}^{x_n} l_k(t)dt + \int_{x_1}^{x_n}\left[q_k + r_k(t-x_n) + s_k(t-x_1)^2(t-x_n)\right.$$
$$\left.\left.\left. + \ u_k(t-x_1)(t-x_n)^2\right]W_{n-2}(t)dt\right\}\right]\left[\int_{x_1}^{x_n}(t-x_1)^2(t-x_n)^2 W_{n-2}(t)dt\right]^{-1}.$$

Also

$$B_1(x) = W^2_{n-2}(x)\left[\int_{x_1}^{x}\left\{r_1(t-x_n) + s_1(t-x_1)^2(t-x_n) + u_1(t-x_1)(t-x_n)^2\right.\right.$$
$$\left.\left. + \ v_1(t-x_1)^2(t-x_n)^2\right\}W_{n-2}(t)dt\right]$$

where

$$r_1 = \frac{1}{(x_1-x_n)W^3_{n-2}(x_1)},$$
$$s_1 = -\frac{r_1}{(x_1-x_n)^2},$$
$$u_1 = -\frac{r_1}{(x_1-x_n)}\left[\frac{5W'_{n-2}(x_1)}{W_{n-2}(x_1)} + \frac{1}{x_1-x_n}\right]$$

and

$$v_1 = -\left[\int_{x_1}^{x_n}\left\{r_1(t-x_n) + s_1(t-x_1)^2(t-x_n)\right.\right.$$
$$\left.\left. + u_1(t-x_1)(t-x_n)^2\right\}W_{n-2}(x)dt\right]\left[\int_{x_1}^{x_n}(t-x_1)^2(t-x_n)^2 W_{n-2}(t)dt\right]^{-1}.$$

The expression for $B_n(x)$ can be given by interchanging x_1 and x_n in $B_1(x)$.

Theorem 4. *The polynomials $C_k(x)$ for $k = 2 \dots n-1$, has the form*

$$C_k(x) = W_{n-2}^2(x)\left[\alpha_k \int_{x_1}^{x} (t-x_1)^2(t-x_n)^2 W_{n-2}(t)dt \right. \\ \left. + \beta_k \int_{x_1}^{x} (t-x_1)^2(t-x_n)^2 l_k(t)dt\right],$$

where

$$\alpha_k = -\left[\beta_k \int_{x_1}^{x_n} (t-x_1)^2(t-x_n)^2 l_k(t)dt\right] \\ \times \left[\int_{x_1}^{x_n} (t-x_1)^2(t-x_n)^2 W_{n-2}(t)dt\right]^{-1}$$

and

$$\beta_k = \frac{1}{6w^2(x_k)(x_k-x_1)^2(x_k-x_n)^2 W_{n-2}'(x_k)^2}.$$

Theorem 5. *The polynomials $D_1(x)$ and $D_n(x)$ can be represented as:*

$$D_1(x) = \frac{W_{n-2}^2(x)}{2(x_1-x_n)W_{n-2}^3(x_1)} \\ \times \int_{x_1}^{x} \left[(t-x_1)(t-x_n)^2 + \alpha_1(t-x_1)^2(t-x_n)^2\right] W_{n-2}(t)dt,$$

where

$$\alpha_1 = -\left[\int_{x_1}^{x_n} (t-x_1)(t-x_n)^2 W_{n-2}(t)dt\right] \\ \times \left[\int_{x_1}^{x_n} (t-x_1)^2(t-x_n)^2 W_{n-2}(t)dt\right]^{-1}$$

and

$$D_n(x) = \frac{W_{n-2}^2(x)}{2(x_1-x_n)W_{n-2}^3(x_n)} \\ \times \int_{x_1}^{x} \left[(t-x_1)^2(t-x_n) + \alpha_n(t-x_1)^2(t-x_n)^2\right] W_{n-2}(t)dt,$$

where α_n can be obtained by interchanging x_1 and x_n in α_n.

The proof of Theorems 1, 2, 3 are similar to that of Theorems in [1]. We omit details.

3. Particular cases

Now, we show that if the zeros of the polynomial $W_{n-2}(x)$ are the zeros of any of the classical orthogonal polynomials of degree $\leq n-2$, then there exists a weight function $w(x) \in C^3(a,b)$ satisfying the conditions (1.9) and (1.10). It is known that the zeros of the classical orthogonal polynomials are real and simple.

Case 1. Let $W_{n-2}(x) = P_{n-2}^{(\alpha,\beta)}(x)$ $(\alpha, \beta > -1,\ n = 3, 4, \ldots)$ the $(n-2)$th Jacobi polynomial and the nodal points in (1.1) are

$$-1 = x_n < x_{n-1} < \ldots < x_1 = 1, \quad n \in N,$$

where $P_{n-2}^{(\alpha,\beta)}(x_k) = 0,\ k = 2 \ldots n-2$. Then a weight function satisfying the conditions (1.9) and (1.10) has the form $w(x) = (1-x)^{\frac{1+\alpha}{2}}(1+x)^{\frac{1+\beta}{2}}$. Indeed, by Szegö [[6], 4.24.1], the function $y = w(x)W_{n-2}(x)$ satisfies the differential equation

$$[w(x)W_{n-2}(x)]'' + \left[\frac{(1-\alpha^2)}{4(1-x)^2} + \frac{(1-\beta^2)}{4(1+x)^2} \right.$$
$$\left. + \frac{(n-2)(n+\alpha+\beta-1) + \frac{1}{2}(1+\alpha)(1+\beta)}{(1-x^2)}\right] w(x)W_{n-2}(x) = 0$$

which implies that

$$\left[\{w(x)W_{n-2}(x)\}^2\right]'''(x_k) = 0, \quad k = 2 \ldots n-1.$$

Also $w(x_k) \neq 0,\ k = 2 \ldots n-1$.

Case 2. Let $W_{n-2}(x) = L_{n-2}^{(\alpha)}(x)$ $(\alpha > -1,\ n = 3, 4, \ldots)$ be the $(n-2)$th Laguerre polynomial and the nodal points in (1.1) are

$$0 = x_n < x_{n-1} < \ldots < x_1 < \infty, \quad n \in N,$$

where $L_{n-2}^{(\alpha)}(x_k) = 0,\ k = 2 \ldots n-2$. Then a weight function satisfying the conditions (1.9) and (1.10) has the form $w(x) = e^{-\frac{x}{2}} x^{\frac{1+\alpha}{2}}$. Indeed, by Szegö [[6],5.1.2], the function $y = w(x)W_{n-2}(x)$ satisfies the differential equation

$$[w(x)W_{n-2}(x)]'' + \left[\frac{(1-\alpha^2)}{4x^2} + \frac{n + \frac{1}{2}(\alpha-3)}{x} - \frac{1}{4}\right] w(x)W_{n-2}(x) = 0,$$

which implies that

$$\left[\{w(x)W_{n-2}(x)\}^2\right]'''(x_k) = 0, \quad k = 2 \ldots n-1.$$

Also $w(x_k) \neq 0,\ k = 2 \ldots n-1$.

Case 3. Let $W_{n-2}(x) = H_{n-2}(x)$ $(n = 3, 4, \ldots)$ be the $(n-2)$th Hermite polynomial and the nodal points in (1.1) are

$$-\infty < x_n < x_{n-1} < \ldots < x_1 < \infty, \quad n \in N,$$

where $H_{n-2}(x_k) = 0$, $k = 2 \dots n-2$. Then a weight function satisfying the conditions (1.9) and (1.10) has the form $w(x) = \exp(-\frac{1}{2}x^2)$. Indeed, by Szegö [[6], 5.5.2], the function $y = w(x)W_{n-2}(x)$ satisfies the differential equation

$$[w(x)W_{n-2}(x)]'' + (2n-3-x^2)w(x)W_{n-2}(x) = 0,$$

which implies that

$$\left[\{w(x)W_{n-2}(x)\}^2\right]'''(x_k) = 0, \quad k = 2 \dots n-1.$$

Also $w(x_k) \neq 0,\ k = 2 \dots n-1$.

Hence in the Cases 1, 2 and 3, a weight function satisfying the conditions (1.9) and (1.10) do exist. Thus, by Theorem 1, a modified weighted (0,1,3)-interpolation polynomial of degree $\leq$ 3n-2, satisfying the conditions (1.11) exists for all the three cases and can be determined uniquely.

Remark 1. The convergence problem of the above problem will be dealt with in the next communication.

Acknowledgement. I am thankful to the Referee for his constructive suggestions.

References

[1] J. Balázs. Súlyozott $(0,2)$ interpoláció ultraszférikus polinomok gyökein. *MTA III. Oszt. Közl.*, 11(3):305–338, 1961.

[2] J. Balázs. Modified weighted (0,2) interpolation. In *Approximation theory. In memory of A. K. Varma*, pages 61–85. Marcel Dekker, 1998.

[3] S. Datta and P. Mathur. On weighted $(0,2)$-interpolation on infinite interval $(-\infty,+\infty)$. *Ann. Univ. Sci. Budap. Rolando Eötvös, Sect. Math.*, 42:45–57, 1999.

[4] I. Joó. On weighted $(0,2)$ interpolation. *Ann. Univ. Sci. Budap. Rolando Eötvös, Sect. Math.*, 38:185–222, 1995.

[5] K. K. Mathur and R. B. Saxena. Weighted (0,1,3) interpolation on the zeros of Hermite polynomials. *Acta Math. Hung.*, 62(1-2):31–47, 1993.

[6] Szegő, Gábor. *Orthogonal polynomials. Revised ed.*, volume 23 of *Colloquium Publ.* American Mathematical Society, 1959.

[7] L. Szili. Weighted (0,2)-interpolation on the roots of Hermite polynomials. *Ann. Univ. Sci. Budap. Rolando Eötvös, Sect. Math.*, 27:153–166, 1985.

Received July 26, 2005.; May 25, 2006 in revised form.

Department of Mathematics and Astronomy,
Lucknow University