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NEW APPROACH FOR CLOSURE SPACES BY RELATIONS

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Abstract. Recently, the general topology has become the appropriated frame for every collection related to relations because topology is required not only for mathematics and physics but also for biology, rough set theory, biochemistry, and dynamics. In this paper, we have introduced another concept of the closure operator. In so doing, the idempotent condition, which has never been realized, is achieved. The topologies associated with these closure operators are studied. And we study the subspace, continuous functions and lower separation axioms in this space. Also we study these space in digital topology.

1. Introduction

Relations are used in construction of topological structures in several fields such as, rough set theory [10, 11], digital topology [13, 14], biochemistry [15], biology [16] and dynamics [5]. It should be noted that the generation of topology by relations and the representation of topological concepts via relations will narrow the gap between topologists and who are interested in applications of topology. The concepts of aftersets and foresets are used to define closure operators [12].

In this paper, we present a review of closure spaces and some definitions related with this work (section 2). We define and investigate a new closure operator with respect to relation concepts. In so doing, the idempotent condition, which has never been realized, is achieved. The topologies associated with these closure operators are studied. Minimal neighborhood and accumulation points were defined (section 3). Also we study the closure subspace of such space (section 4). We reformulate continuous function via relational concepts and their properties are studied. Moreover open and closed functions and homeomorphism and their properties are studied (section 5). Lower separation axioms in such spaces are reformulated via relation concepts (section

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6). Also we study these space in digital topology (section 7). A final section groups conclusions.

2. Preliminaries

A number of ideas familiar in the topological setting can be straightforwardly generalized to closure spaces. Note that closure, interior and neighborhoods are equivalent constructions on a set X. It is possible to translate properties of the closure function cl into properties of the neighborhood function, and vice versa. The following are cited from [1, 2, 6, 9].

Definition 2.1. A closure space is a pair (X, cl), where X is any set, and $\mathrm{cl}\colon P(X) \to P(X)$ is a function associating with each subset $A \subseteq X$ a subset $\mathrm{cl}(A) \subseteq X$, called the closure of A, such that

(1) $\mathrm{cl}(\phi) = \phi$,
(2) $A \subseteq \mathrm{cl}(A)$,
(3) $\mathrm{cl}(A \cup B) = \mathrm{cl}(A) \cup \mathrm{cl}(B)$.

Definition 2.2. Let (X, cl) be a closure space, and let $A \subseteq X$. Then,

(1) The interior $\mathrm{int}(A)$ of A is the set $(\mathrm{cl}(A^c))^c$.
(2) A is a neighborhood of an element $x \in X$ if $x \in int(A)$.
(3) A is closed set if $A = \mathrm{cl}(A)$.
(4) A is open set if $A = \mathrm{int}(A)$.

Lemma 2.1. *In a closure space the following are holds.*

(1) *A is an open set if and only if A^c is a closed set.*
(2) *If $A \subseteq B$ then $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$.*

Definition 2.3. A closure space (X, cl) is topological space iff $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$ for all $A \subseteq X$.

Definition 2.4. A closure space (X, cl) is called Alexandroff topological space iff one of the following conditions holds:

(1) Each point in X has a minimal neighborhood.
(2) For each $A \subseteq X$, $\mathrm{cl}(A) = \cup_{x \in A} \mathrm{cl}(\{x\})$.

Definition 2.5. A topological space (X, τ) is T_0 if, for any two distinct points $x, y \in X$, either x contained in an open set which does not contain y, or y is contained in an open set which does not contain x.

Definition 2.6. A topological space (X, τ) is $T_{1/2}$ if, every singleton $\{x\}$ is open or closed.

Definition 2.7. A topological space (X, τ) is T_1 if, for every two distinct points $x, y \in X$, each is contained in an open set not containing the other.

Definition 2.8. A topological space is R_0 if, for every two distinct points x and y of the space, either $\mathrm{cl}(x) = \mathrm{cl}(y)$ or $\mathrm{cl}(x) \cap \mathrm{cl}(y) = \phi$.

Definition 2.9. A topological space (X, τ) is called T_2-space if and only if for any two distinct points $x, y \in X$ there exist two disjoint open sets U, V such that $x \in U$ and $y \in V$.

Definition 2.10. A topological space (X, τ) is called $T_{5/2}$-space if and only if for any two distinct points $x, y \in X$ there exist two open sets U, V such that $x \in U$ and $y \in V$ and $\mathrm{cl}(U) \cap \mathrm{cl}(V) = \phi$.

Definition 2.11. A function f of a topological space X_1 into a topological space X_2 is said to be continuous at a point $x \in X_1$ if, given any neighborhood V of $f(x)$ in X_2, there is a neighborhood U of x inX_1 such that $f(U) \subseteq V$.

3. New approach for closure spaces

A relation R from a universe X to a universe X (a relation on X) is a subset of $X \times X$, i.e., $R \subseteq X \times X$. The formula $(x, y) \in R$ is abbreviated as xRy and means that x is in relation R with y.

Definition 3.1 ([3])**.** If R is a relation on X, then the aftersets of $x \in X$ is $xR = \{y : xRy\}$ and the foresets of $x \in X$ is $Rx = \{y : yRx\}$.

Definition 3.2. Let R be any binary relation on X, a set $\langle p\rangle R$ is the intersection of all aftersets containing p, i.e.,

$$\langle p\rangle R = \begin{cases} \cap_{p\in xR}(xR) & \text{if } \exists x : p \in xR, \\ \phi & \text{otherwise.} \end{cases}$$

Also $R\langle p\rangle$ is the intersection of all foresets containing p, i.e.,

$$R\langle p\rangle = \begin{cases} \cap_{p\in Rx}(Rx) & \text{if } \exists x : p \in Rx, \\ \phi & \text{otherwise.} \end{cases}$$

Definition 3.3. Let X be any set and $R \subseteq X \times X$ be any binary relation on X. The relation R gives rise to a closure operation cl_R on X as follows:

$$\mathrm{cl}_R(A) = A \cup \{x \in X : \langle x\rangle R \cap A \neq \phi\}$$

Lemma 3.1. *For any binary relation $R \subseteq X \times X$ on X, (X, cl_R) is a closure space.*

Proof. (1) $\mathrm{cl}_R(\phi) = \phi \cup \{x \in X : \langle x\rangle R \cap \phi \neq \phi\} = \phi$.
(2) $\mathrm{cl}_R(A) = A \cup \{x \in X : \langle x\rangle R \cap A \neq \phi\} \supseteq A$, i.e., $A \subseteq \mathrm{cl}_R(A)$.
(3)

$$\begin{aligned} \mathrm{cl}_R(A \cup B) &= (A \cup B) \cup \{x \in X : \langle x\rangle R \cap (A \cup B) \neq \phi\} \\ &= (A \cup B) \cup \{x \in X : (\langle x\rangle R \cap A) \cup (\langle x\rangle R \cap B) \neq \phi\} \\ &= (A \cup B) \cup \{x \in X : \langle x\rangle R \cap A \neq \phi\} \cup \{x \in X : \langle x\rangle R \cap B \neq \phi\} \\ &= (A \cup \{x \in X : \langle x\rangle R \cap A \neq \phi\}) \cup (B \cup \{x \in X : \langle x\rangle R \cap B \neq \phi\}) \end{aligned}$$

i.e., $\mathrm{cl}_R(A \cup B) = \mathrm{cl}_R(A) \cup \mathrm{cl}_R(B)$. □

Lemma 3.2. *For any binary relation R on X if $x \in \langle y \rangle R$, then*

$$\langle x \rangle R \subseteq \langle y \rangle R.$$

Proof. Let $z \in \langle x \rangle R = \cap_{x \in wR}(wR)$. Then z is contained in any wR which contains x, and since also x is contained in any uR which contains y, then z is contained in any uR which contains y, i.e., $z \in \langle y \rangle R$. Then

$$\langle x \rangle R \subseteq \langle y \rangle R.$$

□

Lemma 3.3. *For any binary relation R on X, (X, cl_R) is idempotent, i.e., $\mathrm{cl}_R(\mathrm{cl}_R(A)) = \mathrm{cl}_R(A)$.*

Proof. We want to show that $\mathrm{cl}_R(\mathrm{cl}_R(A)) \subseteq \mathrm{cl}_R(A)$. Suppose $y \in \mathrm{cl}_R(\mathrm{cl}_R(A))$. Then since

$$\mathrm{cl}_R(\mathrm{cl}_R(A)) = \mathrm{cl}_R(A) \cup x \in X : \langle x \rangle R \cap \mathrm{cl}_R(A) \neq \phi,$$

we have either

$$y \in \mathrm{cl}_R(A) \tag{3.1}$$

or

$$y \in x \in X : \langle x \rangle R \cap \mathrm{cl}_R(A) \neq \phi.$$

In the latter case we have $\langle y \rangle R \cap \mathrm{cl}_R(A) \neq \phi$, i.e.,

$$\langle y \rangle R \cap (A \cup x \in X : \langle x \rangle R \cap A \neq \phi) \neq \phi,$$

and hence $(\langle y \rangle R \cap A) \cup (\langle y \rangle R \cap \{x \in X : \langle x \rangle R \cap A \neq \phi\}) \neq \phi$. It follows that either $\langle y \rangle R \cap A \neq \phi$ or

$$\langle y \rangle R \cap \{x \in X : \langle x \rangle R \cap A \neq \phi\} \neq \phi.$$

In the former case we have

$$y \in \mathrm{cl}_R(A), \tag{3.2}$$

and in the latter, there is a z such that $z \in \langle y \rangle R$ and

$$z \in \{x \in X : \langle x \rangle R \cap A \neq \phi\},$$

i.e., $\langle z \rangle R \cap A \neq \phi$; in this case, since $z \in \langle y \rangle R$, we have

$$\langle z \rangle R \subseteq \langle y \rangle R$$

(by lemma 3.2), and hence $\langle y \rangle R \cap A \neq \phi$, so

$$y \in \mathrm{cl}_R(A). \tag{3.3}$$

From (3.1), (3.2) and (3.3), therefore, we have $y \in \mathrm{cl}_R(A)$, so $\mathrm{cl}_R(\mathrm{cl}_R(A)) \subseteq \mathrm{cl}_R(A)$. Since (X, cl_R) is a closure space, the reverse inclusion also holds, so $\mathrm{cl}_R(\mathrm{cl}_R(A)) = \mathrm{cl}_R(A)$. □

From lemma 3.3 we can prove the next theorem.

Theorem 1. *Every closure space (X, cl_R) is topological space.*

We can introduce the interior operation from the closure operation as follows: Since $\mathrm{int}_R(A) = (\mathrm{cl}_R(A^c))^c$ then,

$$\begin{aligned}\mathrm{int}_R(A) &= (A^c \cup \{x \in X : \langle x\rangle R \cap A^c \neq \phi\})^c \\ &= A \cap (\{x \in X : \langle x\rangle R \cap A^c \neq \phi\})^c \\ &= A \cap \{x \in X : \langle x\rangle R \subseteq A\} \\ &= \{x \in A : \langle x\rangle R \subseteq A\}.\end{aligned}$$

Definition 3.4. A point $x \in A$ is an interior point of a subset A of X if $\langle x\rangle R \supseteq A$. i.e., $\mathrm{int}_R(A) = \{x \in A : \langle x\rangle R \subseteq A\}$.

Lemma 3.4. *Let R be any binary relation on a nonempty set X, then*

$$\{x\} \cup \langle x\rangle R$$

is a minimal neighborhood of x for all $x \in X$, i.e., $N_R(X) = \{x\} \cup \langle x\rangle R$.

Proof. We want to show that $\{x\} \cup \langle x\rangle R$ is a minimal neighborhood of x for all $x \in X$. There are two cases, the first is if $\langle x\rangle R = \phi$, then $\{x\} = \{x \in \{x\} : \langle x\rangle R \subseteq \{x\}\} = \mathrm{int}_R(\{x\})$, i.e., $\{x\}$ is the smallest open set containing x and so

$$N_R(x) = \{x\}. \tag{3.4}$$

The second is if $\langle x\rangle R \neq \phi$, then from lemma 3.2 we have $\langle x\rangle R = \{y \in \langle x\rangle R : \langle y\rangle R \subseteq \langle x\rangle R\} = \mathrm{int}_R(\langle x\rangle R)$, i.e., $\langle x\rangle R$ is the smallest open set containing x and so

$$N_R(x) = \langle x\rangle R. \tag{3.5}$$

From (3.4) and (3.5) we have,$N_R(X) = \{x\} \cup \langle x\rangle R$. □

From the last lemma we can write the minimal neighborhood of a point x in a closure space (X, cl_R) as follows:

$$N_R(x) = \begin{cases} \langle x\rangle R & \text{if } \langle x\rangle R \neq \phi, \\ \{x\} & \text{if } \langle x\rangle R = \phi. \end{cases}$$

Lemma 3.5. *Let R be any binary relation on X and for each a subset A of a closure space (X, cl_R), then $\mathrm{cl}_R(A) = \cup_{x\in A}(\mathrm{cl}_R(\{x\}))$.*

Proof. Since $\mathrm{cl}_R(A) = A \cup \{y \in X : \langle y\rangle R \cap A \neq \phi\}$, then

$$\begin{aligned}\mathrm{cl}_R(A) &= A \cup \{y \in X : \langle y\rangle R \cap A \neq \phi\} \\ &= \cup_{x\in A}(\{x\} \cup \{y \in X : \langle y\rangle R \cap (\cup_{x\in A}(\{x\})) \neq \phi\}) \\ &= (\cup_{x\in A}(\{x\})) \cup \ (\cup_{x\in A}\{y \in X : \langle y\rangle R \cap \{x\} \neq \phi\}) \\ &= \cup_{x\in A}((\{x\}) \cup \{y \in X : \langle y\rangle R \cap \{x\} \neq \phi\}) \\ &= \cup_{x\in A}(\mathrm{cl}_R(\{x\})).\end{aligned}$$

□

By lemma 3.4 and 3.5 we can prove the following theorem.

Theorem 2. *Let R be any binary relation then a closure space (X, cl_R) is an Alexandroff topological space.*

Lemma 3.6. *For any binary relation R on X we have, $y \in \langle x \rangle R$ if and only if $x \in \mathrm{cl}_R(\{y\})$.*

Proof. Let $y \in \langle x \rangle R$, then $\langle x \rangle R \cap \{y\} \neq \phi$, and hence $x \in \mathrm{cl}_R(\{y\})$. Conversely, if $x \in \mathrm{cl}_R(\{y\})$, then $\langle x \rangle R \cap \{y\} \neq \phi$, and so $y \in \langle x \rangle R$. □

Lemma 3.7. *In a closure space (X, cl_R) if $\langle x \rangle R = \phi$, then $\{x\}$ is closed.*

Proof. Let $\langle x \rangle R = \phi$, then for all $y \in X$ $(x \notin \langle y \rangle R)$. Thus $\langle y \rangle R \cap \{x\} = \phi$ for all $y \in X$, hence $\mathrm{cl}_R(\{x\}) = \{x\}$. And so $\{x\}$ is closed. □

Lemma 3.8. *In a closure space (X, cl_R) the open sets are precisely the unions $\cup_{x \in A}(N_R(x))$ for all $A \subseteq X$.*

Proof. Let A be any open set in (X, cl_R), then

$$A = \mathrm{int}_R(A) = \{x \in A : \langle x \rangle R \subseteq A\}.$$

Hence A is a neighborhood of each of its elements, so for each $x \in A$, $N_R(x) \subseteq A$ then $\cup_{x \in A}(N_R(x)) \subseteq A$. But since $x \in N_R(x)$ for all x, we have $A \subseteq \cup_{x \in A}(N_R(x))$. And so A is the union of the minimal neighborhoods of its elements. Conversely, consider any subset $A \subseteq X$. We want to show that $\cup_{x \in A}(N_R(x))$ is an open set. We want to show that $N_R(x)$ is open. First if $\langle x \rangle R \neq \phi$, then for any point $y \in N_R(x) = \langle x \rangle R$ we have

$$\langle y \rangle R \subseteq \langle x \rangle R \text{ and } y \in \mathrm{int}_R(\langle x \rangle R) = \mathrm{int}_R(N_R(x)),$$

thus $N_R(x)$ is open. Second if $\langle x \rangle R = \phi$ then

$$N_R(x) = \{x\} = \{x \in \{x\} : \langle x \rangle R \subseteq \{x\}\} = \mathrm{int}_R(\{x\}),$$

i.e., $N_R(x) = \mathrm{int}_R(N_R(x))$. Then $N_R(x)$ is an open set. □

Definition 3.5. Let R be any binary relation on X then a point $x \in X$ is called an accumulation point of A iff $(\langle x \rangle R - \{x\}) \cap A \neq \phi$. The set of all accumulation points of A is denoted by A', i.e.,

$$A' = \{x \in X : (\langle x \rangle R - \{x\}) \cap A \neq \phi\}.$$

Lemma 3.9. *Let R be any binary relation on X then $\mathrm{cl}_R(A) = A \cup A^{'}$.*

Proof. Suppose $y \in \mathrm{cl}_R(A)$. Then since

$$\mathrm{cl}_R(A) = A \cup \{x \in X : \langle x \rangle R \cap A \neq \phi\},$$

we have either $y \in A$, i.e.,

$$y \in A \cup A^{'} \tag{3.6}$$

or $y \in \{x \in X : \langle x\rangle R \cap A \neq \phi\}$. In the latter case we have $\langle y\rangle R \cap A \neq \phi$. Either $y \in A$, then

$$y \in A \cup A^{'} \tag{3.7}$$

or $y \notin A$, hence $(\langle y\rangle R - \{y\}) \cap A \neq \phi$, and so $y \in A^{'}$, i.e.,

$$y \in A \cup A^{'} \tag{3.8}$$

From (3.6), (3.7), and (3.8), therefore, we have $\mathrm{cl}_R(A) \subseteq A \cup A^{'}$.

Conversely, assume that $y \in A \cup A^{'}$. We have either $y \in A$, i.e.,

$$y \in \mathrm{cl}_R(A) \tag{3.9}$$

or $y \in A^{'}$. In the latter case, if $y \in A$, then

$$y \in \mathrm{cl}_R(A) \tag{3.10}$$

and if $y \notin A$, then $(\langle y\rangle R - \{y\}) \cap A \neq \phi$, thus $\langle y\rangle R \cap A \neq \phi$, hence

$$y \in \mathrm{cl}_R(A). \tag{3.11}$$

From (3.9), (3.10) and (3.11) we have $A \cup A^{'} \subseteq \mathrm{cl}_R(A)$. And so $\mathrm{cl}_R(A) = A \cup A^{'}$. □

4. Closure subspace

In the next definition we will introduce the definition of the closure subspace via relation concepts.

Definition 4.1. Let $A \subseteq X$ and $R_A \subseteq R$, then (A, cl_{R_A}) is called a closure subspace of a closure space (X, cl_R) if $\langle x\rangle R_A = \langle x\rangle R \cap A$ for all $x \in A$.

Lemma 4.1. *Let (A, cl_{R_A}) be a closure subspace of a closure space (X, cl_R), then $\langle x\rangle R_A = \langle x\rangle R \cap A$ for all $x \in A$ if and only if $\mathrm{cl}_{R_A}(B) = \mathrm{cl}_R(B) \cap A$ for all $B \subseteq A$.*

Proof. Assume that $\langle x\rangle R_A = \langle x\rangle R \cap A$ for all $x \in A$. We want to show that $\mathrm{cl}_{R_A}(B) = \mathrm{cl}_R(B) \cap A$ for all $B \subseteq A$. Then

$$\begin{aligned}
\mathrm{cl}_{R_A}(B) &= B \cup \{x \in A : \langle x\rangle R_A \cap B \neq \phi\} \\
&= B \cup \{x \in A : \langle x\rangle R \cap A \cap B \neq \phi\} \\
&= B \cup (\{x \in X : \langle x\rangle R \cap B \neq \phi\} \cap A) \\
&= (B \cup \{x \in X : \langle x\rangle R \cap B \neq \phi\}) \cap (B \cup A) \\
&= \mathrm{cl}_R(B) \cap A.
\end{aligned}$$

Conversely, suppose that $\mathrm{cl}_{R_A}(B) = \mathrm{cl}_R(B) \cap A$ for all $B \subseteq A$. Then

$$\begin{aligned}
B \cup \{x \in A : \langle x\rangle R_A \cap B \neq \phi\} &= (B \cup (\{x \in X : \langle x\rangle R \cap B \neq \phi\}) \cap A \\
&= (A \cap B) \cup (A \cap \{x \in X : \langle x\rangle R \cap B \neq \phi\}) \\
&= B \cup \{x \in A : (\langle x\rangle R \cap A) \cap B \neq \phi\}.
\end{aligned}$$

Thus we have $\langle x\rangle R_A = \langle x\rangle R \cap A$. □

From the previous lemma we can prove the following theorem.

Theorem 3. *Let* (X, cl_R) *be a closure space and* $A \subseteq X$*, then* (A, cl_{R_A}) *is a closure subspace if and only if* $\mathrm{cl}_{R_A}(B) = \mathrm{cl}_R(B) \cap A$ *for all* $B \subseteq A$.

Proof. Assume that $\mathrm{cl}_{R_A}(B) = \mathrm{cl}_R(B) \cap A$. We want to show that (A, cl_{R_A}) is a closure space.
(1) $\mathrm{cl}_{R_A}(\phi) = \mathrm{cl}_R(\phi) \cap A = \phi \cap A = \phi$
(2) $\mathrm{cl}_{R_A}(B) = \mathrm{cl}_R(B) \cap A \supseteq B$ (since $B \subseteq A$ and $B \subseteq \mathrm{cl}_R(B)$).
(3)

$$\begin{aligned}
\mathrm{cl}_{R_A}(B_1 \cup B_2) &= \mathrm{cl}_R(B_1 \cup B_2) \cap A \\
&= (\mathrm{cl}_R(B_1) \cup \mathrm{cl}_R(B_2)) \cap A \\
&= (\mathrm{cl}_R(B_1) \cap A) \cup (\mathrm{cl}_R(B_2) \cap A) \\
&= \mathrm{cl}_{R_A}(B_1) \cup \mathrm{cl}_{R_A}(B_2).
\end{aligned}$$

Conversely, immediately derived from lemma 4.1. □

Also we can show that the closure subspace (A, cl_{R_A}) is a topological space by the following lemma.

Lemma 4.2. *A closure subspace* (A, cl_{R_A}) *of a closure space* (X, cl_R) *is a topological space.*

Proof. We want only show that the closure operator cl_{R_A} is idempotent. Then

$$\begin{aligned}
\mathrm{cl}_{R_A}(\mathrm{cl}_{R_A}(B)) &= \mathrm{cl}_{R_A}(\mathrm{cl}_R(B) \cap A) \\
&= \mathrm{cl}_R(\mathrm{cl}_R(B) \cap A) \cap A \\
&\subseteq \mathrm{cl}_R(\mathrm{cl}_R(B)) \cap \mathrm{cl}_R(A) \cap A \\
&\subseteq \mathrm{cl}_R(B) \cap A \\
&\subseteq \mathrm{cl}_{R_A}(B).
\end{aligned}$$

Thus we have $\mathrm{cl}_{R_A}(\mathrm{cl}_{R_A}(B)) = \mathrm{cl}_{R_A}(B)$. □

5. Continuous functions

The concept of continuous function is basic to much of mathematics. More general kinds of continuous functions arise as one goes further in mathematics.

Definition 5.1. Let (X_1, cl_{R_1}) and(X_2, cl_{R_2}) be two closure spaces. The function $f\colon X_1 \to X_2$ is continuous at $x \in X_1$ if and only if

$$f(\langle x \rangle R_1) \subseteq \langle f(x) \rangle R_2.$$

Proposition 5.1. *Let* f *be a function of a closure space* (X_1, cl_{R_1}) *into a closure space*(X_2, cl_{R_2}). *If* f *is continuous at* $x \in X_1$ *and* $x \in \mathrm{cl}_{R_1}(A)$*, then* $f(x) \in \mathrm{cl}_{R_2}(f(A))$.

Proof. Suppose $x \in \mathrm{cl}_{R_1}(A)$. Then since $\mathrm{cl}_{R_1}(A) = A \cup \{x \in X_1 : \langle x \rangle R_1 \cap A \neq \phi\}$, we have either $x \in A$, i.e.,

$$f(x) \in f(A) \tag{5.1}$$

or $x \in \{x \in X_1 : \langle x \rangle R_1 \cap A \neq \phi\}$. In the latter case we have $\langle x \rangle R_1 \cap A \neq \phi$, hence, $f(\langle x \rangle R_1) \cap f(A) \neq \phi$. Since f is continuous at x, i.e., $f(\langle x \rangle R_1) \subseteq \langle f(x) \rangle R_2$, then

$$\langle f(x) \rangle R_2 \cap f(A). \neq \phi \tag{5.2}$$

From (5.1) and (5.2) we have $f(x) \in \mathrm{cl}_{R_2}(f(A))$. □

Definition 5.2. A function from a closure space (X_1, cl_{R_1}) into a closure space (X_2, cl_{R_2}) is said to be continuous on X_1 if it is continuous at each point of X_1.

Theorem 4. *Let f be a function of a closure space (X_1, cl_{R_1}) into a closure space (X_2, cl_{R_2}), then the following conditions are equivalent:*

(i) *f is continuous,*
(ii) *For every subset A of X_1, $f(\mathrm{cl}_{R_1}(A)) \subseteq \mathrm{cl}_{R_2}(f(A))$,*
(iii) *The inverse image of every closed subset of X_2 is a closed subset of X_1,*
(iv) *The inverse image of every open subset of X_2 is an open subset of X_1.*

Proof. (i)→(ii) Since $\mathrm{cl}_{R_1}(A) = A \cup \{x \in X_1 : \langle x \rangle R_1 \cap A \neq \phi\}$, then

$$\begin{aligned} f(\mathrm{cl}_{R_1}(A)) &= f(A \cup \{x \in X_1 : \langle x \rangle R_1 \cap A \neq \phi\}) \\ &\subseteq f(A) \cup \{f(x) \in X_2 : \langle x \rangle R_1 \cap A \neq \phi\}. \end{aligned}$$

Since f is continuous, i.e., $f(x) \in \mathrm{cl}_{R_2}(f(A))$, hence

$$f(\mathrm{cl}_{R_1}(A)) \subseteq f(A) \cup \{f(x) \in X_2 : \langle f(x) \rangle R_2 \cap f(A) \neq \phi\} = \mathrm{cl}_{R_2}(f(A)).$$

(ii)→(iii) Let $A \subseteq X_2$ be a closed subset of X_2, we want to show that $f^{-1}(A)$ is a closed subset of X_1. Let $x \in \mathrm{cl}_{R_1}(f^{-1}(A))$ then

$$f(x) \in f(\mathrm{cl}_{R_1}(f^{-1}(A))) \subseteq \mathrm{cl}_{R_2}(f(f^{-1}(A))) \subseteq \mathrm{cl}_{R_2}(A) = A,$$

hence $x \in f^{-1}(A)$ and so $\mathrm{cl}_{R_1}(f^{-1}(A)) = f^{-1}(A)$, i.e., $f^{-1}(A)$is a closed subset of X_1.

(iii)→ (iv) Let $A \subseteq X_2$ be an open subset of X_2. We want to show that $f^{-1}(A)$ is an open subset of X_1. Since A is open in X_2, then A^c is closed in X_2 and so $f^{-1}(A^c)$ is closed in X_1, hence $(f^{-1}(A^c))^c$is open in X_1. Since for any function f we have $f^{-1}(A) \cap f^{-1}(A^c) = \phi$ and $f^{-1}(A) \cup f^{-1}(A^c) = X_1$, thus $f^{-1}(A) = (f^{-1}(A^c))^c$, i.e., $f^{-1}(A)$ is open in X_1.

(iv)→ (i) We want to show that f is continuous at any point $x \in X_1$, i.e.,

$$f(\langle x \rangle R_1) \subseteq \langle f(x) \rangle R_2.$$

Let $y \notin \langle f(x) \rangle R_2$ for all $f(x) \in A$ and A be an open subset of X_2 which contains $f(x)$, then $y \notin A$. Since $f^{-1}(A)$ is an open subset containing x, i.e., $\langle x \rangle R_1 \subseteq f^{-1}(A)$, then $f(\langle x \rangle R_1) \subseteq A$, hence $y \notin f(\langle x \rangle R_1)$ i.e., $f(\langle x \rangle R_1) \subseteq \langle f(x) \rangle R_2$, thus f is continuous at $x \in X_1$. □

Proposition 5.2. *Let $f : (X_1, \mathrm{cl}_{R_1}) \to (X_1, \mathrm{cl}_{R_2})$ be a one-to-one correspondence function, then f^{-1} is a continuous function at $f(x)$ for all $x \in X^1$ if and only if $\langle f(x)\rangle R_2 \subseteq f(\langle x\rangle R_1)$.*

Proof. Assume that f^{-1} is a continuous function at $f(x)$, then

$$f^{-1}(\langle f(x)\rangle R_2) \subseteq \langle f^{-1}(f(x))\rangle R_1,$$

hence

$$f(f^{-1}(\langle f(x)\rangle R_2)) \subseteq f(\langle f^{-1}(f(x))\rangle R_1),$$

thus $\langle f(x)\rangle R_2 \subseteq f(\langle x\rangle R_1)$ since f is one-to-one correspondence.

Conversely, assume that $\langle f(x)\rangle R_2 \subseteq f(\langle x\rangle R_1)$. Since f is one-to-one correspondence, then $\langle f(x)\rangle R_2 \subseteq f(\langle f^{-1}(f(x))\rangle R_1)$ and

$$f(f^{-1}(\langle f(x)\rangle R_2)) \subseteq f(\langle f^{-1}(f(x))\rangle R_1),$$

hence

$$f^{-1}(\langle f(x)\rangle R_2) \subseteq \langle f^{-1}(f(x))\rangle R_1,$$

then f^{-1} is a continuous function at $f(x)$ for all $x \in X_1$. □

Proposition 5.3. *Let $f\colon (X_1, \mathrm{cl}_{R_1}) \to (X_2, \mathrm{cl}_{R_2})$ be a function, then f is continuous if and only if $f^{-1}(\mathrm{int}_{R_2}(B)) = \mathrm{int}_{R_1}(f^{-1}(B))$ for all $B \subseteq X_2$.*

Proof. Since $\mathrm{int}_{R_2}(B) = \{y \in B : \langle y\rangle R_2 \subseteq B\}$, then $f^{-1}(\mathrm{int}_{R_2}(B)) = \{f^{-1}(y) : \langle y\rangle R_2 \subseteq B\}$. Since $\{ff^{-1}(y)\} \subseteq \{y\}$, then $\langle ff^{-1}(y)\rangle R_2 \subseteq \langle y\rangle R_2$, hence

$$\langle f(f^{-1}(y))\rangle R_2 \subseteq B,$$

then $f^{-1}(\langle f(f^{-1}(y))\rangle R_2) \subseteq f^{-1}(B)$, thus $f^{-1}(f(\langle f^{-1}(y)\rangle R_1)) \subseteq f^{-1}(B)$ (since f is continuous), thus $\langle f^{-1}(y)\rangle R_1 \subseteq f^{-1}(B)$, then

$$f^{-1}(\mathrm{int}_{R_2}(B)) = \{f^{-1}(y) : \langle f^{-1}(y)\rangle R_1 \subseteq f^{-1}(B)\} = \mathrm{int}_{R_1}(f^{-1}(B)),$$

for all $B \subseteq X_2$.

Conversely, assume that $f^{-1}(\mathrm{int}_{R_2}(B)) = \mathrm{int}_{R_1}(f^{-1}(B))$, for all $B \subseteq X_2$. We want to show that f is continuous. Let $B \subseteq X_2$ be an open subset of X_2, i.e., $B = \mathrm{int}_{R_2}(B)$, then $f^{-1}(B) = f^{-1}(\mathrm{int}_{R_2}(B)) = \mathrm{int}_{R_1}(f^{-1}(B))$, i.e., $f^{-1}(B)$ is an open subset of X_1. Thus f is continuous on X_1. □

Lemma 5.1. *Let f be an identity function from a closure space (X_1, cl_{R_1}) into a closure space (X_2, cl_{R_2}), then f is continuous if and only if $R_1 \subseteq R_2$.*

Proof. Suppose f is a continuous function, then $f(\langle x\rangle R_1) \subseteq \langle f(x)\rangle R_2$. Since f is an identity function, then $f(\langle x\rangle R_1) = \langle x\rangle R_1$ also $f(x) = x$, hence

$$\langle x\rangle R_1 \subseteq \langle x\rangle R_2,$$

i.e., $\cap_{x\in yR_1}(yR_1) \subseteq \cap_{x\in yR_2}(yR_2)$, thus $yR_1 \subseteq yR_2$ for all $y \in X_1$, then $R_1 \subseteq R_2$.

Conversely, assume that $R_1 \subseteq R_2$, then $yR_1 \subseteq yR_2$ for all $y \in X_1$ and so $\cap_{x\in yR_1}(yR_1) \subseteq \cap_{x\in yR_2}(yR_2)$, i.e.,$\langle x\rangle R_1 \subseteq \langle x\rangle R_2$. Since f is an identity function, then $f(\langle x\rangle R_1) = \langle x\rangle R_1$ and $f(x) = x$, hence $f(\langle x\rangle R_1) \subseteq \langle f(x)\rangle R_2$, and so f is a continuous function on X_1. □

Lemma 5.2. *Let $f\colon (X_1, \mathrm{cl}_{R_1}) \to (X_2, \mathrm{cl}_{R_2})$ be a continuous function and $A = f(X_1)$, then $f : (X_1, \mathrm{cl}_{R_1}) \to (A, \mathrm{cl}_{R^*})$ is also continuous function.*

Proof. We want to show that $f(\langle x\rangle R_1) \subseteq \langle f(x)\rangle R^*$. Since

$$\langle f(x)\rangle R^* = (\langle f(x)\rangle R_2) \cap A$$

and $\langle x\rangle R_1 \subseteq X_1$, then $f(\langle x\rangle R_1) \subseteq f(X) = A$ also $f(\langle x\rangle R_1) \subseteq \langle f(x)\rangle R_2$ (since f is continuous), then $f(\langle x\rangle R_1) \subseteq (\langle f(x)\rangle R_2) \cap A$, i.e.,

$$f(\langle x\rangle R_1) \subseteq \langle f(x)\rangle R^*.$$

□

Proposition 5.4. *Let $A \subseteq X_1$ be a subset of X_1 and $f\colon X_1 \to X_2$ be a continuous function, then $f/A : A \to X_2$ is a continuous function.*

Proof. We want to show that $f/A(\langle x\rangle R_A) \subseteq \langle f/A(x)\rangle R_2$ for all $x \in A$. Since $\langle x\rangle R_A = \langle x\rangle R_1 \cap A$, then

$$\begin{aligned} f/A(\langle x\rangle R_A) &= f(\langle x\rangle R_A) \\ &= f(\langle X\rangle R_1 \cap A) \\ &\subseteq f(\langle x\rangle R_1) \\ &\subseteq \langle f(x)\rangle R_2 \qquad \text{since } f \text{ is continuous} \\ &\subseteq \langle f/A(x)\rangle R_2. \end{aligned}$$

Hence f/A is a continuous function on A. □

Definition 5.3. A function $f : (X_1, \mathrm{cl}_{R_1}) \to (X_2, \mathrm{cl}_{R_2})$ is called open (closed) if the image of an open (closed) subset of X_1 is an open (closed) subset of X_2.

Theorem 5. *Let $f\colon X_1 \to X_2$ be a function from a closure space (X_1, cl_{R_1}) into a closure space (X_2, cl_{R_2}), then the following conditions are equivalent:*

(i) *f is open.*
(ii) *$f(\mathrm{int}_{R_1}(A)) \subseteq \mathrm{int}_{R_2}(f(A))$ for all $A \subseteq X_1$.*
(iii) *if N is a neighborhood of x then there is a neighborhood W of $f(x)$ such that $W \subseteq f(N)$.*

Proof. (i)→ (ii) Since $\mathrm{int}_{R_1}(A) \subseteq A$ for all $A \subseteq X_1$, then $f(\mathrm{int}_{R_1}(A)) \subseteq f(A)$ and so $\mathrm{int}_{R_2}(f(\mathrm{int}_{R_1}(A))) \subseteq \mathrm{int}_{R_2}(f(A))$, then $f(\mathrm{int}_{R_1}(A)) \subseteq \mathrm{int}_{R_2}(f(A))$, (since f is open).

(ii)→ (iii) Let N be a neighborhood of x, then $x \in \mathrm{int}_{R_1}(N)$, i.e.,

$$x \in \langle x\rangle R_1 \subseteq \mathrm{int}_{R_1}(N) \subseteq N,$$

hence $f(x) \in f(\langle x\rangle R_1) \subseteq f(N)$ since by (ii) $f(\langle x\rangle R_1) \subseteq \mathrm{int}_{R_2}(f(\langle x\rangle R_1))$, i.e., $f(x) \in \mathrm{int}_{R_2}(f(\langle x\rangle R_1)) \subseteq f(N)$. We can take $W = \mathrm{int}_{R_2}(f(\langle x\rangle R_1))$, then $W \subseteq f(N)$.

(iii)→(i) Let A be an open subset of X_1, then $\mathrm{int}_{R_1}(A) = A$ and A is a neighborhood of all points lies in A, i.e., $x \in \langle x\rangle R_1 \subseteq A$, for all $x \in A$. Thus there is a neighborhood W of $f(x)$ such that $W \subseteq f(A)$ and so

$$f(x) \in \langle f(x)\rangle R_2 \subseteq W \subseteq f(A),$$

then $\langle f(x)\rangle R_2 \subseteq f(A)$ for all $f(x) \in f(A)$, hence $\mathrm{int}_{R_2}(f(A)) = f(A)$, thus $f(A)$ is an open subset of X_2. □

Lemma 5.3. *If $f\colon (X_1, \mathrm{cl}_{R_1}) \to (X_2, \mathrm{cl}_{R_2})$ is one-to-one correspondence function, then f is open if and only if f is closed.*

Proof. Let f be an open function from X_1 onto X_2. We want to show that if A is closed in (X_1, cl_{R_1}) then $f(A)$ is closed in (X_2, cl_{R_2}). Since A is closed, then $X_1 - A$ is open, hence $f(X_1 - A)$ is open but

$$\begin{aligned} f(X_1 - A) &= f(X_1) - f(A) \qquad \text{since } f \text{ is one-to-one} \\ &= X_2 - f(A) \qquad \text{since } f \text{ is onto,} \end{aligned}$$

then $f(A)$ is a closed subset of X_2.

Conversely, similarly. □

We introduce the following example to show that the condition one-to-one correspondence is necessary.

Example 5.1. Let $X_1 = \{a, b, c, d\}$ and $X_2 = \{1, 2, 3, 4\}$ be two universal sets and

$$R_1 = \{(a, a), (a, b), (b, c), (c, d), (d, a)\}$$

and

$$R_2 = \{(1, 1), (1, 4), (2, 3), (3, 1), (3, 4), (4, 2), (4, 1)\}$$

be two any binary relations on X_1 and X_2 respectively, and

$$f\colon (X_1, \mathrm{cl}_{R_1}) \to (X_2, \mathrm{cl}_{R_2})$$

defined as $f(a) = f(c) = 1, f(b) = 2$ and $f(d) = 3$. Note that the function f is not one-to-one correspondence. Also this function is an open function but not a closed function because there is a closed subset $\{c\}$ of X_1 but $f(\{c\}) = \{1\}$ is not a closed subset of X_2.

Proposition 5.5. *Let $f\colon X_1 \to X_2$ be a function from a closure space (X_1, cl_{R_1}) into a closure space (X_2, cl_{R_2}), then f is closed if and only if*

$$\mathrm{cl}_{R_2}(f(A)) \subseteq f(\mathrm{cl}_{R_1}(A))$$

for all $A \subseteq X_1$.

Proof. Suppose f is a closed function. Since $A \subseteq \mathrm{cl}_{R_1}(A)$, then $f(A) \subseteq f(\mathrm{cl}_{R_1}(A))$ also $\mathrm{cl}_{R_2}(f(A)) \subseteq \mathrm{cl}_{R_2}(f(\mathrm{cl}_{R_1}(A)))$ but since f is a closed function we have $\mathrm{cl}_{R_2}(f(\mathrm{cl}_{R_1}(A))) = f(\mathrm{cl}_{R_1}(A))$, then $\mathrm{cl}_{R_2}(f(A)) \subseteq f(\mathrm{cl}_{R_1}(A))$. Conversely, assume that A is a closed subset of X_1, then $A = \mathrm{cl}_{R_1}(A)$. Since $\mathrm{cl}_{R_2}(f(A)) \subseteq f(\mathrm{cl}_{R_1}(A))$ but $f(A) = f(\mathrm{cl}_{R_1}(A))$, then we have $\mathrm{cl}_{R_2}(f(A)) = f(A)$. Hence f is a closed function. □

Definition 5.4. A function $f\colon (X_1, \mathrm{cl}_{R_1}) \to (X_2, \mathrm{cl}_{R_2})$ is said to be a homeomorphism if f is one-to-one correspondence, continuous and open.

A homeomorphism $f\colon (X_1, \mathrm{cl}_{R_1}) \to (X_2, \mathrm{cl}_{R_2})$ gives us a bijective correspondence not only between X_1 andX_2 but also between the collections of open sets of X_1 and X_2. As a result, any property of X_1 that is entirely expressed in terms of the closure space (X_1, cl_{R_1}) (that is, in terms of the open sets of X_1) yields, via the correspondence f, the correspondence property for the closure space (X_2, cl_{R_2}). Such a property of X_1 is called a topological property of X_1.

Theorem 6. *Let f be a one-to-one correspondence function from a closure space (X_1, cl_{R_1}) into a closure space (X_2, cl_{R_2}) , then the following conditions are equivalent.*

(i) *f is a homeomorphism,*
(ii) *f and f^{-1} are continuous,*
(iii) *f is continuous and closed,*
(iv) *$f(\mathrm{cl}_{R_1}(A)) = \mathrm{cl}_{R_2}(f(A))$, for all $A \subseteq X_1$.*

Proof. (i)$\to$ (ii) We want to show that only f^{-1} is continuous. Since f is open, i.e., if $A \subseteq X_1$ is open then $f(A) \subseteq X_2$ is also open. Suppose $A \subseteq X_1$ is open, hence $(f^{-1})^{-1}(A) = f(A)$ is also open and so f^{-1} is continuous.

(ii)$\to$ (iii) We want to show that f is closed. Assume that $A \subseteq X_1$ is closed, then $X_1 - A$ is open and so $(f^{-1})^{-1}(X_1 - A)$ is open (since f^{-1} is continuous) but $(f^{-1})^{-1}(X_1 - A) = f(X_1 - A) = f(X_1) - f(A) = X_2 - f(A)$ (since f is one-to-one correspondence), thus $f(A)$ is a closed subset of X_2.

(iii)$\to$ (iv) The proof immediately derived from theorem 5.1 and proposition 5.5.

(iv)$\to$ (i) We want to show that f is continuous and open. Since $f(\mathrm{cl}_{R_1}(A)) \subseteq \mathrm{cl}_{R_2}(f(A))$, then f is continuous. Let A be an open subset of X_1, then $X_1 - A$ is a closed subset of X_1 and so

$$f(\mathrm{cl}_{R_1}(X_1 - A)) = f(X_1 - A) = f(X_1) - f(A) = X_2 - f(A)$$

and

$$\mathrm{cl}_{R_2}(f(X_1 - A)) = \mathrm{cl}_{R_2}(f(X_1) - f(A)) = \mathrm{cl}_{R_2}(X_2 - f(A)),$$

then $\mathrm{cl}_{R_2}(X_2 - f(A)) = X_2 - f(A)$, hence $X_2 - f(A)$ is closed in X_2 and so $f(A)$ is an open subset in X_2. ☐

Proposition 5.6. *Let $f\colon (X_1, \mathrm{cl}_{R_1}) \to (X_2, \mathrm{cl}_{R_2})$ be a one-to-one correspondence function, then f is a homeomorphism if and only if $f(\langle x\rangle R_1) = \langle f(x)\rangle R_2$ for all $x \in X_1$.*

Proof. The proof is immediately derived from definition 5.1 and proposition 5.2. ☐

6. Lower Separation Axioms

In this section, we give a new definitions of some concepts of the lower separation axioms via relational concepts and study some of their properties.

Lemma 6.1. *Let R be any binary relation and for every two distinct points x and y in a closure space (X, cl_R), then $x \notin \mathrm{cl}_R(y)$ or $y \notin \mathrm{cl}_R(x)$ if and only if either $x \notin \langle y \rangle R$ or $y \notin \langle x \rangle R$.*

Proof. Suppose $x \notin \mathrm{cl}_R(y)$ or $y \notin \mathrm{cl}_R(x)$ for every $x, y \in X$. In the former case we have $\langle x \rangle R \cap \{y\} = \phi$, i.e., $y \notin \langle x \rangle R$. And in the latter, then $\langle y \rangle R \cap \{x\} = \phi$, i.e., $x \notin \langle y \rangle R$. Conversely, Assume that $x \notin \langle y \rangle R$ or $y \notin \langle x \rangle R$ for every $x, y \in X$. In the former case we have $\langle y \rangle R \cap \{x\} = \phi$, i.e., $y \notin \mathrm{cl}_R(x)$. And in the latter we get $\langle x \rangle R \cap \{y\} = \phi$, i.e., And let $x \in \mathrm{cl}_R(y)$ and $y \in \mathrm{cl}_R(x)$, then $y \in \langle x \rangle R$ $x \notin \mathrm{cl}_R(y)$. □

Definition 6.1. Let R be any binary relation, then a closure space (X, cl_R) is called T_0-space if and only if for every two distinct points $x, y \in X$ either $x \notin \langle y \rangle R$ or $y \notin \langle x \rangle R$.

Proposition 6.1. *Let $f : (X_1, \mathrm{cl}_{R_1}) \to (X_2, \mathrm{cl}_{R_2})$ be one-to-one correspondence, f^{-1} be continuous on X_2 and (X_1, cl_{R_1}) be a T_0-space then (X_2, cl_{R_2}) is also T_0-space.*

Proof. Assume that $x, y \in X_2$ are two distinct points, then $f^{-1}(x)$ and $f^{-1}(y)$ are two distinct points in X_1, since f is one-to-one correspondence. But (X_1, cl_{R_1}) is a T_0-space, then $f^{-1}(x) \notin \langle f^{-1}(y) \rangle R_1$ or $f^{-1}(y) \notin \langle f^{-1}(x) \rangle R_1$ and hence $f(f^{-1}(x)) \notin f(\langle f^{-1}(y) \rangle R_1)$ or $f(f^{-1}(y)) \notin f(\langle f^{-1}(x) \rangle R_1)$, so

$$x \notin f(\langle f^{-1}(y) \rangle R_1) \text{ or } y \notin f(\langle f^{-1}(x) \rangle R_1),$$

but f^{-1} is continuous on X_2, then $x \notin \langle f(f^{-1}(y)) \rangle R_2$ or $y \notin \langle f(f^{-1}(x)) \rangle R_2$ and hence $x \notin \langle y \rangle R_2$ or $y \notin \langle x \rangle R_2$, so (X_2, cl_{R_2}) is a T_0-space. □

Corollary 6.1. *If $f : (X_1, \mathrm{cl}_{R_1}) \to (X_2, \mathrm{cl}_{R_2})$ is homeomorphism then the property of a T_0-space is a topological property.*

Lemma 6.2. *Let R be any binary relation, then in a closure space (X, cl_R) every singleton $\{x\}$ is open or closed if and only if $x \notin \langle y \rangle R$ or $\langle x \rangle R = \{x\}$ or ϕ for all $x, y \in X$.*

Proof. Assume that $\{x\} \subset X$ is a closed or an open subset of X (i.e., $\mathrm{cl}_R(x) = \{x\}$ or $\mathrm{int}_R(x) = \{x\}$). In the former case we have $\{x\} = \{x\} \cup \{y : \langle y \rangle R \cap \{x\} \neq \phi\}$, i.e., $\langle y \rangle R \cap \{x\} = \phi$, so $x \notin \langle y \rangle R$. In the latter case we get $\{x\} = \{x : \langle x \rangle R \subseteq \{x\}\}$, then $\langle x \rangle R = \{x\}$ orϕ. Conversely, if $x \notin \langle y \rangle R$, then $\langle y \rangle R \cap \{x\} = \phi$, hence $y \notin \mathrm{cl}_R(x)$ for all $y \in X$, i.e., $\mathrm{cl}_R(x) = \{x\}$. Also if $\langle x \rangle R = x$ orϕ, then $\langle x \rangle R \subseteq \{x\}$, and hence $\{x \in \{x\} : \langle x \rangle R \subseteq \{x\}\} = \{x\}$, i.e., $\mathrm{int}_R(x) = \{x\}$. So every singleton $\{x\}$ is open or closed. □

Definition 6.2. Let R be any binary relation, then a closure space (X, cl_R) is called $T_{1/2}$-space if and only if for every two distinct points $x, y \in X$ either $x \notin \langle y \rangle R$ or $\langle x \rangle R = \{x\}$ or ϕ.

Proposition 6.2. *Let $f\colon (X_1, \mathrm{cl}_{R_1}) \to (X_2, \mathrm{cl}_{R_2})$ be one-to-one correspondence, f^{-1} be continuous on X_2 and (X_1, cl_{R_1}) be a $T_{1/2}$-space then (X_2, cl_{R_2}) is also $T_{1/2}$-space.*

Proof. Assume that $x, y \in X_2$ are two distinct points, then $f^{-1}(x)$ and $f^{-1}(y)$ are two distinct points in X_1, since f is one-to-one correspondence. But (X_1, cl_{R_1}) is a $T_{1/2}$-space, then $f^{-1}(x) \notin \langle f^{-1}(y)\rangle R_1$ or $\langle f^{-1}(x)\rangle R_1 = \{f^{-1}(x)\}$ or ϕ and hence $f(f^{-1}(x)) \notin f(\langle f^{-1}(y)\rangle R_1)$ or $f(\langle f^{-1}(x)\rangle R_1) = f\{f^{-1}(x)\}$ or $f(\phi\}$, so $x \notin f(\langle f^{-1}(y)\rangle R_1)$ or $f(\langle f^{-1}(x)\rangle R_1 = \{x\})$ or ϕ but f^{-1} is continuous on X_2, then $x \notin \langle f(f^{-1}(y))\rangle R_2$ or $\langle f(f^{-1}(x))\rangle R_2 \subseteq \{x\}$ or ϕ and hence $x \notin \langle y\rangle R_2$ or $\langle x\rangle R_2 = \{x\}$ or ϕ, so (X_2, cl_{R_2}) is a $T_{1/2}$-space. □

Corollary 6.2. *If $f : (X_1, \mathrm{cl}_{R_1}) \to (X_2, \mathrm{cl}_{R_2})$ is homeomorphism then the property of a $T_{1/2}$-space is a topological property.*

Lemma 6.3. *Let R be any binary relation and for every two distinct points x and y in a closure space (X, cl_R), then either $\mathrm{cl}_R(x) = \mathrm{cl}_R(y)$ or $\mathrm{cl}_R(x) \cap \mathrm{cl}_R(y) = \phi$ if and only if if $x \in \langle y\rangle R$ then $y \in \langle x\rangle R$.*

Proof. Let x and y be two distinct points in a closure space (X, cl_R) and $\mathrm{cl}_R(x) = \mathrm{cl}_R(y)$ or $\mathrm{cl}_R(x) \cap \mathrm{cl}_R(y) = \phi$. In the former case we have $y \in \mathrm{cl}_R(x)$ and $x \in \mathrm{cl}_R(y)$, i.e., $x \in \langle y\rangle R$ and $y \in \langle x\rangle R$. In the latter case we get $x \notin \mathrm{cl}_R(y)$ and $y \notin \mathrm{cl}_R(x)$, i.e., $x \notin \langle y\rangle R$ and $y \notin \langle x\rangle R$. So if $x \in \langle y\rangle R$ then $y \in \langle x\rangle R$. Conversely, if $x \in \langle y\rangle R$ then $y \in \langle x\rangle R$ holds, then either

$$(x \in \langle y\rangle R \text{ and } y \in \langle x\rangle R)$$

or

$$(x \notin \langle y\rangle R \text{ and } y \notin \langle x\rangle R)$$

are holds. In the former case we have $y \in \mathrm{cl}_R(x)$ and $x \in \mathrm{cl}_R(y)$ for all $x, y \in X$, then

$$\mathrm{cl}_R(x) = \mathrm{cl}_R(y). \tag{6.1}$$

In the latter case we get $y \notin \mathrm{cl}_R(x)$ and $x \notin \mathrm{cl}_R(y)$ for all $x, y \in X$, then

$$\mathrm{cl}_R(x) \cap \mathrm{cl}_R(y) = \phi. \tag{6.2}$$

From (6.1) and (6.2) the proof is complete. □

Definition 6.3. Let R be any binary relation, then a closure space (X, cl_R) is called R_0-space if and only if for every two distinct points $x, y \in X$ if $x \in \langle y\rangle R$ then $y \in \langle x\rangle R$.

Proposition 6.3. *Let $f : (X_1, \mathrm{cl}_{R_1}) \to (X_2, \mathrm{cl}_{R_2})$ be one-to-one correspondence, f^{-1} be continuous on X_2 and (X_1, cl_{R_1}) is a R_0-space then (X_2, cl_{R_2}) is also R_0-space.*

Proof. Assume that $x, y \in X_2$ are two distinct points then $f^{-1}(x)$ and $f^{-1}(y)$ are two distinct points in X_1, since f is one-to-one correspondence. But (X_1, cl_{R_1}) is a R_0-space, i.e., if $f^{-1}(x) \in \langle f^{-1}(y)\rangle R_1$ then $f^{-1}(y) \in \langle f^{-1}(x)\rangle R_1$ and hence if $f(f^{-1}(x)) \in f(\langle f^{-1}(y)\rangle R_1)$ then $f(f^{-1}(y)) \in f(\langle f^{-1}(x)\rangle R_1)$, so if $x \in f(\langle f^{-1}(y)\rangle R_1)$ then $y \in f(\langle f^{-1}(x)\rangle R_1)$ but f^{-1} is continuous on X_2, thus we have if $x \in \langle f(f^{-1}(y))\rangle R_2$ then $y \in \langle f(f^{-1}(x))\rangle R_2$ and hence if $x \in \langle y\rangle R_2$ then $y \in \langle x\rangle R_2$. Then (X_2, cl_{R_2}) is a R_0-space. □

Corollary 6.3. *If $f : (X_1, \mathrm{cl}_{R_1}) \to (X_2, \mathrm{cl}_{R_2})$ is homeomorphism then the property of a R_0-space is a topological property.*

Lemma 6.4. *Let R be any binary relation and for every two distinct points x and y in a closure space (X, cl_R), then $x \notin \mathrm{cl}_R(y)$ and $y \notin \mathrm{cl}_R(x)$ if and only if both $x \notin \langle y\rangle R$ and $y \notin \langle x\rangle R$ are holds.*

Proof. Suppose $x \notin \mathrm{cl}_R(y)$ and $y \notin \mathrm{cl}_R(x)$ for every $x, y \in X$. In the former case we have $\langle x\rangle R \cap \{y\} = \phi$, then $y \notin \langle x\rangle R$. Also in the latter case we get $\langle y\rangle R \cap \{x\} = \phi$, hence $x \notin \langle y\rangle R$. So $x \notin \langle y\rangle R$ and $y \notin \langle x\rangle R$ for every $x, y \in X$. Conversely, assume that $x \notin \langle y\rangle R$ and $y \notin \langle x\rangle R$ for every $x, y \in X$. In the former case we have $\langle y\rangle R \cap \{x\} = \phi$, then $y \notin \mathrm{cl}_R(x)$ and in the latter case we get $\langle x\rangle R \cap \{y\} = \phi$, hence $x \notin \mathrm{cl}_R(y)$. Thus $x \notin \mathrm{cl}_R(y)$ and $y \notin \mathrm{cl}_R(x)$ for every $x, y \in X$. □

Definition 6.4. Let R be any binary relation, then a closure space (X, cl_R) is called T_1-space if and only if for every two distinct points $x, y \in X$ both $x \notin \langle y\rangle R$ and $y \notin \langle x\rangle R$ are holds.

Proposition 6.4. *Let $f : (X_1, \mathrm{cl}_{R_1}) \to (X_2, \mathrm{cl}_{R_2})$ be one-to-one correspondence, f^{-1} be continuous on X_2 and (X_1, cl_{R_1}) be a T_1-space then (X_2, cl_{R_2}) is also T_1-space.*

Proof. Assume that $x, y \in X_2$ are two distinct points then $f^{-1}(x)$ and $f^{-1}(y)$ are two distinct points in X_1, since f is one-to-one correspondence. But (X_1, cl_{R_1}) is a T_1-space, i.e., $f^{-1}(x) \notin \langle f^{-1}(y)\rangle R_1$ and $f^{-1}(y) \notin \langle f^{-1}(x)\rangle R_1$. Then $f(f^{-1}(x)) \notin f(\langle f^{-1}(y)\rangle R_1)$ and $f(f^{-1}(y)) \notin f(\langle f^{-1}(x)\rangle R_1)$ and hence $x \notin f(\langle f^{-1}(y)\rangle R_1)$ and $y \notin f(\langle f^{-1}(x)\rangle R_1)$ but f^{-1} is continuous on X_2, so $x \notin \langle f(f^{-1}(y))\rangle R_2$ and $y \notin \langle f(f^{-1}(x))\rangle R_2$ and hence $x \notin \langle y\rangle R_2$ and $y \notin \langle x\rangle R_2$. Thus (X_2, cl_{R_2}) is a T_1-space. □

Corollary 6.4. *If $f\colon (X_1, \mathrm{cl}_{R_1}) \to (X_2, \mathrm{cl}_{R_2})$ is homeomorphism then the property of a T_1-space is a topological property.*

Corollary 6.5. *For any closure space (X, cl_R) the following are holds,*

(1) $T_1 \Rightarrow T_{1/2} \Rightarrow T_0$.
(2) $T_1 \Rightarrow R_0$.
(3) $T_1 = R_0 + T_0$.

Proof. The proof is immediately derived from lemma 6.1, 6.2, 6.3 and 6.4. □

Lemma 6.5. *Let R be any reflexive relation on X, then in a closure space (X, cl_R) for any two distinct points $x, y \in X$ there exist two disjoint open sets U, V such that $x \in U$ and $y \in V$ if and only if $\langle x\rangle R \cap \langle y\rangle R = \phi$.*

Proof. Suppose $x, y \in X$ are two distinct points and U, V are two disjoint open sets containing x, y respectively, i.e., $x \in U = \{x : \langle x\rangle R \subset U\}$ and $y \in V = \{y : \langle y\rangle R \subset V\}$, then $\langle x\rangle R \cap \langle y\rangle R \subset U \cap V = \phi$ for all $x, y \in X$. Conversely, assume $\langle x\rangle R \cap \langle y\rangle R = \phi$ for all $x, y \in X$. Then $x \in \langle x\rangle R = U$ and $y \in \langle y\rangle R = V$ and hence $U \cap V = \langle x\rangle R \cap \langle y\rangle R = \phi$. □

Definition 6.5. Let R be any binary reflexive relation, then a closure space (X, cl_R) is called T_2-space if and only if for every two distinct points $x, y \in X$ we have $\langle x\rangle R \cap \langle y\rangle R = \phi$.

Proposition 6.5. *Let $f : (X_1, \mathrm{cl}_{R_1}) \to (X_2, \mathrm{cl}_{R_2})$ be one-to-one correspondence, f^{-1} be continuous on X_2 and (X_1, cl_{R_1}) be a T_2-space then (X_2, cl_{R_2}) is also T_2-space.*

Proof. Assume that $x, y \in X_2$ are two distinct points then $f^{-1}(x)$ and $f^{-1}(y)$ are two distinct points in X_1, since f is one-to-one correspondence. But (X_1, cl_{R_1}) is a T_2-space, i.e., $\langle f^{-1}(x)\rangle R_1 \cap \langle f^{-1}(y)\rangle R_1 = \phi$, then

$$f(\langle f^{-1}(x)\rangle R_1 \cap \langle f^{-1}(y)\rangle R_1) = f(\phi)$$

and hence $f(\langle f^{-1}(x)\rangle R_1) \cap f(\langle f^{-1}(y)\rangle R_1) = \phi$ but f^{-1} is continuous on X_2, then $\langle f(f^{-1}(x))\rangle R_2 \cap \langle f(f^{-1}(y))\rangle R_2 = \phi$, thus $\langle x\rangle R_2 \cap \langle y\rangle R_2 = \phi$. So (X_2, cl_{R_2}) is a T_2-space. □

Corollary 6.6. *If $f : (X_1, \mathrm{cl}_{R_1}) \to (X_2, \mathrm{cl}_{R_2})$ is homeomorphism then the property of a T_2-space is a topological property.*

Lemma 6.6. *Let R be any reflexive relation on X, then in a closure space (X, cl_R) for any two distinct points $x, y \in X$ there exist two open sets U, V containing x, y respectively such that $\mathrm{cl}_R(U) \cap \mathrm{cl}_R(V) = \phi$ if and only if $\mathrm{cl}_R(\langle x\rangle R) \cap \mathrm{cl}_R(\langle y\rangle R) = \phi$.*

Proof. Assume that $x, y \in X$ are two distinct points and let U, V are two open sets containing x, y respectively such that $\mathrm{cl}_R(U) \cap \mathrm{cl}_R(V) = \phi$. We want to prove that $\mathrm{cl}_R(\langle x\rangle R) \cap \mathrm{cl}_R(\langle y\rangle R) = \phi$ for all $x, y \in X$. If

$$\mathrm{cl}_R(\langle x\rangle R) \cap \mathrm{cl}_R(\langle y\rangle R) \neq \phi,$$

i.e., there is z such that $z \in \mathrm{cl}_R(\langle x\rangle R)$ and $z \in \mathrm{cl}_R(\langle y\rangle R)$, then $\langle z\rangle R \cap \langle x\rangle R \neq \phi$ and $\langle z\rangle R \cap \langle y\rangle R \neq \phi$. Since $x \in U$ and $y \in V$ then $\langle x\rangle R \subset U$ and $\langle y\rangle R \subset V$, hence $\langle z\rangle R \cap U \neq \phi$ and $\langle z\rangle R \cap V \neq \phi$ so $z \in \mathrm{cl}_R(U)$ and $z \in \mathrm{cl}_R(V)$, i.e., $z \in \mathrm{cl}_R(U) \cap \mathrm{cl}_R(V)$ and so $\mathrm{cl}_R(U) \cap \mathrm{cl}_R(V) \neq \phi$, which that is contradiction then $\mathrm{cl}_R(\langle x\rangle R) \cap \mathrm{cl}_R(\langle y\rangle R) = \phi$ for all $x, y \in X$.

Conversely, suppose $\mathrm{cl}_R(\langle x\rangle R) \cap \mathrm{cl}_R(\langle y\rangle R) = \phi$ for all $x, y \in X$ and R is reflexive relation. Then $x \in \langle x\rangle R = U$ (open subset) and $y \in \langle y\rangle R = V$ (open subset). Hence $\mathrm{cl}_R(U) \cap \mathrm{cl}_R(V) = \mathrm{cl}_R(\langle x\rangle R) \cap \mathrm{cl}_R(\langle y\rangle R) = \phi$. □

Definition 6.6. Let R be any binary reflexive relation, then a closure space (X, cl_R) is called $T_{5/2}$-space if and only if for every two distinct points $x, y \in X$ then $\mathrm{cl}_R(\langle x\rangle R) \cap \mathrm{cl}_R(\langle y\rangle R) = \phi$.

Proposition 6.6. *Let*

$$f\colon (X_1, \mathrm{cl}_{R_1}) \to (X_2, \mathrm{cl}_{R_2})$$

be homeomorphism and (X_1, cl_{R_1}) *be a* $T_{5/2}$*-space then* (X_2, cl_{R_2}) *is also* $T_{5/2}$*-space.*

Proof. Assume that $x, y \in X_2$ are two distinct points then $f^{-1}(x)$ and $f^{-1}(y)$ are two distinct points in X_1, since f is one-to-one correspondence. But (X_1, cl_{R_1}) is a $T_{5/2}$-space, i.e., $\mathrm{cl}_{R_1}(\langle f^{-1}(x)\rangle R_1) \cap \mathrm{cl}_{R_1}(\langle f^{-1}(y)\rangle R_1) = \phi$, then

$$f(\mathrm{cl}_{R_1}(\langle f^{-1}(x)\rangle R_1) \cap \mathrm{cl}_{R_1}(\langle f^{-1}(y)\rangle R_1)) = f(\phi)$$

and hence $f(\mathrm{cl}_{R_1}(\langle f^{-1}(x)\rangle R_1)) \cap f(\mathrm{cl}_{R_1}(\langle f^{-1}(y)\rangle R_1)) = \phi$ but f is homeomorphism, then $\mathrm{cl}_{R_2}(f(\langle f^{-1}(x)\rangle R_1)) \cap \mathrm{cl}_{R_2}(f(\langle f^{-1}(y)\rangle R_1)) = \phi$ also f is homeomorphism, thus $\mathrm{cl}_{R_2}(\langle f(f^{-1}(x))\rangle R_2) \cap \mathrm{cl}_{R_2}(\langle f(f^{-1}(y))\rangle R_2) = \phi$, so

$$\mathrm{cl}_{R_2}(\langle x\rangle R_2) \cap \mathrm{cl}_{R_2}(\langle y\rangle R_2) = \phi.$$

Thus we have (X_2, cl_{R_2}) is a $T_{5/2}$-space. □

Corollary 6.7. *If* $f : (X_1, \mathrm{cl}_{R_1}) \to (X_2, \mathrm{cl}_{R_2})$ *is homeomorphism then the property of a* $T_{5/2}$*-space is a topological property.*

Example 6.1. Let $X = \{a, b, c, d\}$ and R be any binary relation on X,

$$R = \{(a, a), (a, b), (b, d), (c, d), (d, d)\},$$

then

$$\langle a\rangle R = \{a, b\},\ \langle b\rangle R = \{a, b\},\ \langle c\rangle R = \phi,\ \langle d\rangle R = \{d\}.$$

Note that the closure space (X, cl_R) is not $T_{1/2}$ because $a \in \langle b\rangle R$, also is not T_0 because $a \in \langle b\rangle R$ and $b \in \langle a\rangle R$. But the closure space (X, cl_R) is R_0-space. And the corresponding topology of this relation is

$$\tau = \{\phi, X, \{c\}, \{d\}, \{c, d\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}.$$

Note that in this topology every τ -open set is τ - closed set, i.e., this topology is a quasi-discrete topology.

7. Digital line

In this section we define a transitive relation to generate the digital line and so called Khalimesky line, which has many application in computer science.

Let $X = Z$ and R be a transitive relation on Z,

$$R = \{(2n, 2n), (2n, 2n + 1), (2n, 2n - 1) : n \in Z\},$$

then

$$\langle 2n+1\rangle R = \{2n+1\},\ \langle 2n-1\rangle R = \{2n-1\} \text{ and } \langle 2n\rangle R = \{2n-1, 2n, 2n+1\}.$$

Note that the closure space (Z, cl_R) is T_0 and $T_{1/2}$-space but not T_1 because $2n+1 \in \langle 2n \rangle R$ and $2n \notin \langle 2n+1 \rangle R$, also not R_0 because $2n+1 \in \langle 2n \rangle R$ but $2n \notin \langle 2n+1 \rangle R$.

In this space $\mathrm{cl}_R(2n) = \{2n\}$ and $\mathrm{cl}_R(2n+1) = \{2n, 2n+1, 2n+2\}$ also $N_R(2n) = \{2n-1, 2n, 2n+1\}$ and $N_R(2n+1) = \{2n+1\}$. Note that this space is called the digital line (Z, K) and so-called Khalimsky line [7, 8] generated by the above relation. Also we can write this relation in an equivalent form as follows: $xRy \Leftrightarrow x \rightarrow y$.

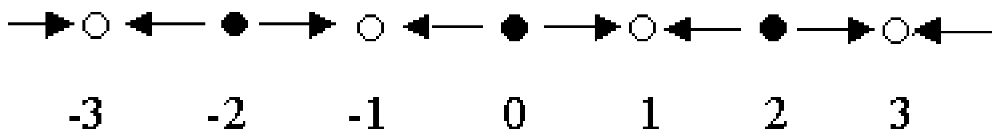
Example 6.1 and this application show that the two separation axioms T_0 and $T_{1/2}$ are independent with R_o.

In Z the 2-neighbors of x is $N_2(x) = \{x-1, x+1\}$, then the following conditions are holds on Z by the above relation.
(1) If two points $x, y \in Z$ are 2-neighbors, then either $x \rightarrow y$ or $y \rightarrow x$.
(2) If two points $x, y \notin Z$ are not 2-neighbors, then both$x \nrightarrow y$ and $y \nrightarrow x$.

Lemma 7.1. *If $x_1, x_2, x_3 \in Z$ such that $x_2 \in N_2(x_1)$ and $x_3 \in N_2(x_2)$, then we have either $x_1 \rightarrow x_2 \leftarrow x_3$ or $x_1 \leftarrow x_2 \rightarrow x_3$.*

Proof. By condition (1) we have either $x_1 \rightarrow x_2$ or $x_2 \rightarrow x_1$. Let $x_1 \rightarrow x_2$. Also by condition (1) either $x_2 \rightarrow x_3$ or $x_3 \rightarrow x_2$. If $x_2 \rightarrow x_3$ then by transitivity $x_1 \rightarrow x_3$ which contradiction to condition (2). Hence $x_3 \rightarrow x_2$. □

Then by lemma 7.1 we can draw the graph of this relation as follows:



Note that $\rightarrow \circ \leftarrow$ is equivalent to open set also $\leftarrow \bullet \rightarrow$ is equivalent to the closed set. We can obtain the results in [4] by this method.

8. Conclusion

Topology is a branch of mathematics, whose concepts exist not only in almost all branches of mathematics, but also in many real life applications. Relations are used in construction of topological structures in several fields.

In this paper, we investigate a new concept of a binary relation R ($\langle x \rangle R$) to generate a closure operator. In so doing, the idempotent condition, which has never been realized, is achieved. The topology associated with this closure operator are studied.

So we choose this line aiming to fill the gap between topologists and application. Also to open the door for more topological applications.

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