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ON THE CONVERGENCE OF THE THREE-STEP ITERATION PROCESS IN THE CLASS OF QUASI-CONTRACTIVE OPERATORS

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Abstract. We establish a general theorem to approximate fixed points of quasi-contractive operators on a normed space through the three-step iteration process. Our result generalizes and improves upon, among others, the corresponding result of Berinde [1].

1. Introduction and Preliminaries

Throughout this note, $\mathbb{N}$ will denote the set of all positive integers. Let C be a nonempty convex subset of a normed space E and $T\colon C \to C$ be a mapping. Let $\{b_n\}$ and $\{b'_n\}$ be two sequences in $[0,1]$.

The Mann iteration process is defined by the sequence $\{x_n\}_{n=0}^{\infty}$ (see [8])

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = (1-b_n)\,x_n + b_n T x_n,\ n \in \mathbb{N}. \end{cases} \tag{1.1}$$

The sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = (1-b_n)\,x_n + b_n T y_n, \\ y_n = (1-b'_n)\,x_n + b'_n T x_n,\ n \in \mathbb{N} \end{cases} \tag{1.2}$$

is known as the Ishikawa iteration process [3].

The sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = b_n T y_n + (1-b_n)x_n, \\ y_n = b'_n T z_n + (1-b'_n)x_n\,, \\ z_n = b''_n T x_n + (1-b''_n)x_n,\ \ n \in \mathbb{N}, \end{cases} \tag{1.3}$$

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where $\{b_n\}$, $\{b'_n\}$ and $\{b''_n\}$ are appropriate sequences in $[0,1]$ is known as three-step iteration process [9].

The iteration processes (1.1-1.2) can be viewed as the special cases of the iteration process (1.3).

We recall the following definitions in a metric space (X,d). A mapping $T\colon X \to X$ is called an a-contraction if

$$d(Tx,Ty) \leq ad(x,y) \text{ for all } x,y \in X, \tag{1.4}$$

where $a \in (0,1)$.

The map T is called Kannan mapping [4] if there exists $b \in (0,\frac{1}{2})$ such that

$$d(Tx,Ty) \leq b[d(x,Tx) + d(y,Ty)] \text{ for all } x,y \in X. \tag{1.5}$$

A similar definition is due to Chatterjea [2]: there exists a $c \in (0,\frac{1}{2})$ such that

$$d(Tx,Ty) \leq c[d(x,Ty) + d(y,Tx)] \text{ for all } x,y \in X. \tag{1.6}$$

Combining these three definitions, Zamfirescu [17] proved the following important result.

Theorem 1. *Let (X,d) be a complete metric space and $T\colon X \to X$ a mapping for which there exists the real numbers a,b and c satisfying $a \in (0,1)$; $b,c \in (0,\frac{1}{2})$ such that for each pair $x,y \in X$, at least one of the following conditions holds:*

(z_1) $d(Tx,Ty) \leq ad(x,y)$,
(z_2) $d(Tx,Ty) \leq b[d(x,Tx) + d(y,Ty)]$,
(z_3) $d(Tx,Ty) \leq c[d(x,Ty) + d(y,Tx)]$.

Then T has a unique fixed point p and the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = Tx_n, \quad n \in \mathbb{N}$$

converges to p for any arbitrary but fixed $x_1 \in X$.

An operator T satisfying the contractive conditions (z_1), $(z_2)$and (z_3) in the above theorem is called Zamfirescu operator.

In 2004, Berinde [1] introduced a new class of operators on an arbitrary Banach space E satisfying

$$\|Tx - Ty\| \leq \delta \|x - y\| + 2\delta \|Tx - x\| \tag{1.7}$$

for any $x,y \in E$, $0 \leq \delta < 1$. He proved that this class is wider than the class of Zamfiresu operators and used the Ishikawa iteration process (1.2) to approximate fixed points of this class of operators in an arbitrary Banach space given in the form of following theorem:

Theorem 2. *Let C be a nonempty closed convex subset of an arbitrary Banach space E and $T\colon C \to C$ be an operator satisfying (1.7). Let $\{x_n\}_{n=0}^{\infty}$ be defined through the iterative process (1.2) and $x_0 \in C$, where $\{b_n\}$ and $\{b'_n\}$ are sequences of positive numbers in $[0,1]$ with $\{b_n\}$ satisfying $\sum_{n=0}^{\infty} b_n = \infty$. Then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the fixed point of T.*

In this paper, we approximate the fixed points of the class of mappings defined in (1.7) by using the iteration process (1.3). Our result generalizes and improves upon, among others, the corresponding result of Berinde [1].

The following lemma is proved in [7].

Lemma 1. *Let $\{r_n\}, \{s_n\}, \{t_n\}$ and $\{k_n\}$ be sequences of nonnegative numbers satisfying*

$$r_{n+1} \leq (1 - s_n)r_n + s_n t_n + k_n \text{ for all } n \geq 1.$$

If $\sum_{n=1}^{\infty} s_n = \infty$, $\lim_{n\to\infty} t_n = 0$ and $\sum_{n=1}^{\infty} k_n < \infty$ hold, then $\lim_{n\to\infty} r_n = 0$.

2. Main Result

Following Berinde [1], we prove a convergence theorem for the iteration process (1.3), under the hypothesis that the operator T has at least a fixed point.

Theorem 3. *Let C be a nonempty closed convex subset of a normed space E. Let $T\colon C \to C$ be an operator satisfying (1.7). Let $\{x_n\}_{n=0}^{\infty}$ be defined by the iterative process (1.3). If $F(T) \neq \phi$ and $\sum_{n=1}^{\infty} b_n = \infty$, then $\{x_n\}_{n=0}^{\infty}$ converges strongly to a fixed point of T.*

Proof. Assume that $F(T) \neq \phi$ and $w \in F(T)$, then using (1.3), we have

$$\begin{aligned} \|x_{n+1} - w\| &= \|b_n T y_n + (1 - b_n)x_n - w\| \\ &= \|(1 - b_n)(x_n - w) + b_n (T y_n - w)\| \\ &\leq (1 - b_n) \|x_n - w\| + b_n \|T y_n - w\| . \end{aligned} \tag{2.1}$$

Now for $x = w$ and $y = y_n$, (1.7) gives

$$\|T y_n - w\| \leq \delta \|y_n - w\| . \tag{2.2}$$

In a similar fashion, we can get

$$\begin{aligned} \|y_n - w\| &= \|b'_n T z_n + (1 - b'_n)x_n - w\| \\ &= \|(1 - b'_n)(x_n - w) + b'_n (T z_n - w)\| \\ &\leq (1 - b'_n) \|x_n - w\| + b'_n \|T z_n - w\| . \end{aligned} \tag{2.2}$$

Using (1.7), for $x = w$ and $y = z_n$, we get

$$\|T z_n - w\| \leq \delta \|z_n - w\| . \tag{2.4}$$

Also

$$\begin{aligned} \|z_n - w\| &= \|b''_n T x_n + (1 - b''_n)x_n - w\| \\ &= \|(1 - b''_n)(x_n - w) + b''_n (T x_n - w)\| \\ &\leq (1 - b''_n) \|x_n - w\| + b''_n \|T x_n - w\| . \end{aligned} \tag{2.3}$$

Again by (1.7), if $x = w$ and $y = x_n$, we get

$$\|T x_n - w\| \leq \delta \|x_n - w\| . \tag{2.6}$$

From (2.1-2.6), we obtain

$$
\begin{aligned}
\|x_{n+1} - w\| &\leq (1 - b_n)\|x_n - w\| + \delta b_n[(1 - b'_n)\|x_n - w\| \\
&\quad + \delta b'_n \|z_n - w\|] \\
&= [1 - b_n[1 - \delta(1 - b'_n)]]\|x_n - w\| + \delta^2 b_n b'_n \|z_n - w\| \\
&\leq [1 - b_n[1 - \delta(1 - b'_n)]]\|x_n - w\| \\
&\quad + \delta^2 b_n b'_n [1 - (1 - \delta) b''_n]\|x_n - w\| \\
&= [1 - (1 - \delta) b_n[1 + \delta b'_n(1 + \delta b''_n)]]\|x_n - w\|.
\end{aligned} \tag{2.4}
$$

It may be noted that for $\delta \; \epsilon \; [0, 1)$ and $\{\eta_n\} \; \epsilon \; [0, 1]$, the following inequality is always true

$$1 \leq 1 + \delta\eta_n \leq 1 + \delta. \tag{2.8}$$

From (2.7) and (2.8), we get

$$\|x_{n+1} - w\| \leq [1 - (1 - \delta) b_n]\|x_n - w\|.$$

By Lemma 1, with $\sum_{n=1}^{\infty} b_n = \infty$, we get that $\lim_{n\to\infty} \|x_n - w\| = 0$. Consequently $x_n \to w \in F(T)$ and this completes the proof. □

Corollary 1. *Let C be a nonempty closed convex subset of a normed space E. Let $T : C \to C$ be an operator satisfying* (1.7). *Let $\{x_n\}_{n=0}^{\infty}$ be defined by the iterative process* (1.2). *If $F(T) \neq \phi$ and $\sum_{n=1}^{\infty} b_n = \infty$, then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique fixed point of T.*

Corollary 2. *Let C be a nonempty closed convex subset of a normed space E. Let $T : C \to C$ be an operator satisfying (1.7). Let $\{x_n\}_{n=0}^{\infty}$ be defined by the iterative process (1.1). If $F(T) \neq \phi$ and $\sum_{n=1}^{\infty} b_n = \infty$,then $\{x_n\}_{n=0}^{\infty}$ converges strongly to a fixed point of T.*

Remarks. 1. The contractive condition (1.4) makes T a continuous function on X while this is not the case with the contractive conditions (1.5–1.6) and (1.7).

2. The Chatterjea's and the Kannan's contractive conditions (1.6) and (1.5) are both included in the class of Zamfirescu operators and so their convergence theorems for the Ishikawa iteration process are obtained in Corollary 1.

3. Theorem 4 of Rhoades [12] in the context of Mann iteration on a uniformly convex Banach space has been extended in Corollary 2.

4. In Corollary 2, Theorem 8 of Rhoades [13] is generalized to the setting of normed spaces.

5. Our result also generalizes Theorem 5 of Osilike [10] and Theorem 2 of Osilike [11].

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