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CHARACTERIZATION OF FINITE GROUPS BY THEIR COMMUTING GRAPH

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Abstract. The commuting graph of a group G, denoted by $\Gamma(G)$, is a simple graph whose vertices are all non-central elements of G and two distinct vertices x, y are adjacent if $xy = yx$. In [1] it is conjectured that if M is a simple group and G is a group satisfying $\Gamma(G) \cong \Gamma(M)$, then $G \cong M$. In this paper we prove this conjecture for many simple groups.

1. Introduction

We denote by $\pi(n)$ the set of all prime divisors of n and if G is a finite group, then $\pi(G)$ is defined to be $\pi(|G|)$.

In this paper we consider simple graphs which are undirected, with no loops or multiple edges. The following definitions are standard and you can find them for example in [10].

For any graph Γ, we denote the set of vertices of Γ by $V(\Gamma)$. The *degree* $d_\Gamma(v)$ of a vertex v in Γ, is the number of edges incident to v and if the graph is understood, then we denote $d_\Gamma(v)$ simply by $d(v)$. A graph is called *regular* if the degrees of its vertices are the same. Two distinct vertices in Γ are called to be *adjacent*, if they are joined by an edge in Γ. A *path* P is a sequence of distinct vertices $v_0v_1 \ldots v_k$ such that for all i $(0 \leq i \leq k-1)$, v_i and v_{i+1} are adjacent vertices. A graph Γ is a *connected* graph, if there is path between each distinct pair of its vertices; otherwise Γ is *disconnected*. A maximal connected subgraph of a graph Γ is called a *component* of Γ. The complement G' of a simple graph G is a simple graph with the same vertex set as G, two vertices being adjacent in G' if and only if they are not adjacent in G.

We construct the commuting graph, the non-commuting graph and the prime graph of G as follows:

The *commuting graph* of G, denoted by Γ(G), is a graph whose vertex set is $G \setminus Z(G)$, and two distinct vertices x and y are adjacent whenever $xy = yx$ and

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the *non-commuting graph* of a group G, denoted by $\nabla(G)$, is the complement of Γ(G), i.e. the graph with $G \setminus Z(G)$ as its vertex set and two vertices x and y are adjacent, if $xy \neq yx$ (see [1, 32]).

Finally, the *prime graph* of G, denoted by $\Gamma_1(G)$, is a graph whose vertex set is $\pi(G)$, and two distinct vertices p and q are adjacent if and only if G contains an element of order pq (see [30, 34]).

Denote the number of components of the prime graph of a group G with $t(G)$, and let $\pi_1,\ \pi_2, \ldots, \pi_{t(G)}$ be the vertex set of the components of $\Gamma_1(G)$ and $T(G) = \{\pi_i(G) | i = 1, 2, \ldots, t(G)\}$. If $2 \in \pi(G)$, then we always suppose $2 \in \pi_1$. Therefore,

$$\pi(G) = \bigcup_{i=1}^{t(G)} \pi_i.$$

Now, $|G|$ can be expressed as a product of coprime positive integers $m_i, i = 1, 2, \ldots,\ t(G)$ where $\pi(m_i) = \pi_i$. These integers are called the *order components* of G and the set of order components of G is denoted by $OC(G)$:

$$OC(G) = \{m_i | i = 1, 2, \ldots, t(G)\}.$$

If $|G|$ is even, then m_1 is called the *even order component* and $m_2,\ m_3, \ldots,$ $m_{t(G)}$ are called the *odd order components* of G.

In 1996, Chen posed the following question:

Question 1.1. Let M be a finite simple group. If G is a group such that $OC(G) = OC(M)$, do we have $G \cong M$?

Although, the answer to this question is “No” in general, a positive answer has been given for many groups. A simple group M is said to be *characterizable by its order components*, if $M \cong G$ for each group G such that $OC(G) = OC(M)$.

Remark 1.2. Suppose M is a finite group. In [1, 32], the authors conjectured that if G is a finite group such that $\nabla(M) \cong \nabla(G)$, then $|M| = |G|$. For every group H, $\nabla(H)$ and $\Gamma(H)$ are complement graphs, therefore for the groups H and K we have $\nabla(H) \cong \nabla(K)$ if and only if $\Gamma(H) \cong \Gamma(K)$. Hence the above conjecture is equivalent to say if G is a finite group such that $\Gamma(M) \cong \Gamma(G)$, then $|M| = |G|$. Although, they proved the statement for the groups S_n, A_n, D_{2n}, all sporadic simple groups and all simple groups of Lie type with disconnected prime graph, recently in [31], the author found some counterexamples to this conjecture. Here we state his counterexamples briefly:

For a prime p and an integer $r > 1$, there exists a non-abelian p-group P of order p^{2r} such that:

(1) $|Z(P)| = p^r$;
(2) $P/Z(P)$ is an elementary abelian p-group;
(3) for every non-central element x of P, $C_P(x) = Z(P)\langle x\rangle$;
(4) the non-commuting graph of P is regular.

According to this statement, there exists a 2-group P of order 2^{10} such that $P/Z(P)$ is an elementary abelian 2-group of order 2^5 and $\nabla(P)$ is a regular graph. Let A be an abelian group such that $|A| = a$ and consider the product $G = P \times A$. Then $\nabla(G)$ is a $(2^5)a(30)$-regular graph, i.e., a graph with $d(v) = (2^5)a(30)$ for every vertex v of $\nabla(G)$ and has $(2^5)a(31)$ vertices. With a similar discussion, there exists a 5-group Q of order 5^6 such that $Q/Z(Q)$ is an elementary abelian 5-group of order 5^3 and $\nabla(Q)$ is a regular graph and if B is any abelian group such that $|B| = b$ and $H = Q \times B$, then $\nabla(H)$ is a $4b(5^3)(30)$-regular graph with $4b(5^3)(31)$ vertices. Now, we must choose A and B so that $a2^3 = b5^3$. If we do that, both non-commuting graphs $\nabla(G)$ and $\nabla(H)$ are $(2^5)a(30)$-regular graphs with the same number of vertices, and in fact they are isomorphic. However, the corresponding groups G and H have different orders.

In [1], the authors put forward another conjecture for the non-commuting graph of a group G. We rewrite this conjecture for the commuting graph of groups as follows:

Conjecture 1.3. Let M be a finite simple group. If G is any finite group such that $\Gamma(M) \cong \Gamma(G)$, then we have $M \cong G$.

In this paper, we will find the relation between the commuting graph and the prime graph of finite groups and then give a positive answer to Conjecture 1.3 for the groups pointed in Remark 1.2, using their characterization by their prime graph. Note that this conjecture is not true if we suppose M is an arbitrary finite group. In particular, the dihedral group and quaternion group of order 8 are not isomorphic while $\Gamma(D_8) \cong \Gamma(Q_8)$.

We will also prove the following theorem that gives a special characterization for all finite non-abelian simple groups:

Theorem 1.4. *Let G and M be two finite simple non-abelian groups. If $\Gamma(G) \cong \Gamma(M)$, then $G \cong M$.*

For a group G, let $N(G) = \{\, n \mid G \text{ has a conjugacy class of size } n\}$.

Lemma 1.5. *Let G_1 and G_2 be finite groups satisfying $|G_1| = |G_2|$ and $N(G_1) = N(G_2)$. Then $t(G_1) = t(G_2)$ and $OC(G_1) = OC(G_2)$.*

This is an immediate consequence of Lemma 1.5 in [9]:

Lemma 1.6. *Let G_1 and G_2 be finite groups satisfying $|G_1| = |G_2|$ and $N(G_1) = N(G_2)$. Then $t(G_1) = t(G_2)$ and $T(G_1) = T(G_2)$.*

The following basic theorem makes a relation between commuting graph of groups and their order components:

Theorem 1.7. *Let G_1 and G_2 be finite groups such that $|G_1| = |G_2|$ and $\Gamma(G_1) \cong \Gamma(G_2)$. Then $N(G_1) = N(G_2)$ and $OC(G_1) = OC(G_2)$.*

Proof. Set $|G_1| = |G_2| = n$. Since $\Gamma(G_1) \cong \Gamma(G_2)$, we have $|G_1 \setminus Z(G_1)| = |G_2 \setminus Z(G_2)|$ and therefore $|Z(G_1)| = |Z(G_2)|$. Also there exists a bijection $\varphi : V(\Gamma(G_1)) \longrightarrow V(\Gamma(G_2))$ such that for all vertices $a, b \in V(\Gamma(G_1))$, a and b are adjacent if and only if $\varphi(a)$ and $\varphi(b)$ are adjacent. Set $|Z(G_1)| = |Z(G_2)| = z$ and suppose $1 \neq k \in N(G_1)$. Thus, there exists an element $x \in G_1$ with the conjugacy class $cl_{G_1}(x)$ in G_1 including x of size k. Therefore, $|C_{G_1}(x)| = n/k$ and thus $d_{\Gamma(G_1)}(x) = n/k - z - 1$. Obviously we have $d_{\Gamma(G_2)}(\varphi(x)) = n/k - z - 1$ and thus $|C_{G_2}(\varphi(x))| = n/k$. Therefore, $|cl_{G_2}(\varphi(x))| = k$ and thus, $k \in N(G_2)$. Hence, we have $N(G_1) \subseteq N(G_2)$ and with a similar reason we have $N(G_2) \subseteq N(G_1)$ and therefore the first statement is proved. The second statement follows immediately from Lemma 1.5. □

2. Characterization of some finite groups by their commuting graph

First, we present the proof of Theorem 1.4:

Proof of Theorem 1.4. Since, $\Gamma(G) \cong \Gamma(M)$, they must have the same number of vertices, so $|G \backslash Z(G)| = |M \backslash Z(M)|$. On the other hand, $|Z(G)| = |Z(M)| = 1$, therefore $|G| = |M|$. Now, it is known that the only pairs of simple groups of the same order are

$$(A_8, PSL(3,4)) \text{ and } (O(2n+1, q) = B_n(q), PSp(2n, q) = C_n(q)),$$

where $n \geq 3$ and q is odd (see [29] and [33]). Thus, if $G \ncong M$, then $G \cong A_8$ and $M \cong PSL(3,4)$ or $G \cong B_n(q)$ and $M \cong C_n(q)$. For the first case, the element $a = (1\ 2)(3\ 4) \in A_8$ is included in a conjugacy class of length 210 and since $|A_8| = 20160$, we have $d_{\Gamma(A_8)}(a) = |C_{A_8}(a)| - 2 = 94$, while $N(PSL(3,4)) = \{1, 315, 1260, 2240, 2880, 4032\}$ and thus $94 \notin \{d_{\Gamma(PSL(3,4))}(x) | x \in PSL(3,4) \setminus \{1\}\}$. Therefore, we have $\Gamma(A_8) \ncong \Gamma(PSL(3,4))$, because they have different degrees. For the second case, it has been proved that $N(B_n(q)) \neq N(C_n(q))$ (see [3]). Therefore, $\Gamma(B_n(q))$ and $\Gamma(C_n(q))$ have different sets of degrees and hence $\Gamma(B_n(q)) \ncong \Gamma(C_n(q))$. Therefore, we must have $G \cong M$. □

Now, we characterize some groups by their commuting graph:

Corollary 2.1. *Let $M = PSL(p,q)$, where p is a prime number and q is a prime power. If G is any group such that $\Gamma(M) \cong \Gamma(G)$, then $M \cong G$.*

Proof. Using Remark 1.2 and since $t(M) \geq 2$, we get $|M| = |G|$. Then by Theorem 1.7, we have $OC(M) = OC(G)$. Finally, by [4, 11, 12, 22, 23] we get the result. □

Corollary 2.2. *Let $M = PSU(p,q)$ where p is an odd prime and q is a prime power. If G is any group such that $\Gamma(M) \cong \Gamma(G)$, then $M \cong G$.*

Proof. Similar to the proof of Corollary 2.1 and since $t(M) \geq 2$, we have $OC(M) = OC(G)$. Thus, by [13, 18, 19, 20, 28], we get the result. □

Now, we prove that the groups $B_n(q)$ and $C_n(q)$ where $n = 2^m \geq 2$, are characterized by their commuting graphs, although we know that they cannot be characterized by their order components (see [24]):

Corollary 2.3. *Let M be a simple group of type $B_n(q)$ or $C_n(q)$ where $n = 2^m \geq 4$ or $n = 2$ and $q > 5$. If G is any group such that $\Gamma(M) \cong \Gamma(G)$, then $M \cong G$.*

Proof. Suppose $M \cong B_n(q)$ where $n = 2^m \geq 4$ or $n = 2$ and $q > 5$ and suppose $\Gamma(M) \cong \Gamma(G)$. Similar to the proof of Corollary 2.1 and since $t(M) = 2$, we have $OC(M) = OC(G)$. Thus, by [24], if q is even, then $G \cong M$, and if q is odd, then we have $G \cong M$ or $G \cong C_n(q)$. But, if q is odd and $G \cong C_n(q)$, then $\Gamma(C_n(q)) \cong \Gamma(G) \cong \Gamma(B_n(q))$ and this is a contradiction by Theorem 1.4. Thus, $G \cong M$. The proof for the case $M \cong C_n(q)$ is the same. □

Corollary 2.4. *Let M be a simple group of one of the following types:*

- **(a):** *$E_6(q)$ or $E_8(q)$;*
- **(b):** *$F_4(q)$ where $q > 2$;*
- **(c):** *${}^2D_n(q)$ where $n = 2^m \geq 4$;*
- **(d):** *${}^2D_p(3)$ where $p = 2^n + 1 \geq 5$;*
- **(e):** *${}^2E_6(q)$ where $q > 2$;*
- **(f):** *${}^3D_4(q)$;*
- **(g):** *A Suzuki–Ree group, i.e. a group of type ${}^2B_2(q)$, ${}^2F_4(q)$ or ${}^2G_2(q)$;*
- **(h):** *A sporadic simple group.*

If G is any group such that $\Gamma(M) \cong \Gamma(G)$, then $M \cong G$.

Proof. Similar to the proof of Corollary 2.1 and since $t(M) \geq 2$, we have $OC(M) = OC(G)$. Thus, by [2, 26, 16, 17, 25, 14, 27, 7, 21, 6, 5], we get the result. □

Professor J. G. Thompson has conjectured that:

Conjecture 2.5. If G is a finite group with $Z(G) = 1$ and M a finite non-abelian simple group such that $N(G) = N(M)$, then $M \cong G$.

Lemma 2.6. *Suppose M is a finite non-abelian simple group with $t(M) \geq 2$ which satisfies Thompson's Conjecture. If G is a group such that $\Gamma(M) \cong \Gamma(G)$, then $M \cong G$.*

Proof. Since $\Gamma(M) \cong \Gamma(G)$, we have $|M \setminus Z(M)| = |G \setminus Z(G)|$ and $Z(M) = 1$. On the other hand, $|M| = |G|$ by Remark 1.2. Hence, $Z(G) = 1$ and by Theorem 1.7 we have $N(G) = N(M)$. Therefore, $M \cong G$. □

Therefore we have proved the following corollary:

Corollary 2.7. *Let M be a simple group of type $G_2(q)$ where $q > 2$. If G is any group such that $\Gamma(M) \cong \Gamma(G)$, then $M \cong G$.*

Proof. By [8], M satisfies Thompson's Conjecture. Now, the result follows from Lemma 2.6. □

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