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FIXED POINTS THEOREMS FOR n-VALUED MULTIFUNCTIONS

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Abstract. We first show that if Y is a nonempty AR space and $F\colon Y \to Y$ is a compact n-valued multifunction, then F has at least n fixed point. We also prove that if C is a nonempty closed convex subset of a topological vector space E and $F\colon C \to C$ is a continuous Φ-condensing n-valued multifunction, then F has at least n fixed points.

1. Introduction and preliminaries

Let X and Y be two Hausdorff topological spaces.

A multifunction $F\colon X \to Y$ is a map from X into the set 2^Y of nonempty subsets of Y. The range of F is $F(X) = \bigcup\limits_{x\in X} F(x)$.

The multifunction $F\colon X \to Y$ is said to be upper semi-continuous (usc) if for each open subset V of Y with $F(x) \subset V$ there exists an open subset U of X with $x \in U$ and $F(U) \subset V$.

The multifunction $F\colon X \to Y$ is called lower semi-continuous (lsc) if for every $x \in X$ and open subset V of Y with $F(x) \cap V \neq \emptyset$ there exists an open subset U of X with $x \in U$ and $F(x') \cap V \neq \emptyset$ for all $x' \in U$.

A multifunction $F\colon X \to Y$ is continuous if it is both upper semi-continuous and lower semi-continuous.

A multifunction $F\colon X \to Y$ is compact if it is continuous and the closure of its range $\overline{F(X)}$ is a compact subset of Y.

A point x of X is said to be a fixed point of a multifunction $F\colon X \to X$ if $x \in F(x)$. We denote by $Fix(F)$ the set of all fixed points of F.

A multifunction $F\colon X \to Y$ is said to be n-valued if for all $x \in X$, the subset $F(x)$ of Y consists of n points.

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A multifunction $F\colon X \to X$ is said to be an n-function if there exist n continuous maps $f_i\colon X \to X$, where $i = 1, \dots, n$, such that $F(x) = \{f_1(x), \dots, f_n(x)\}$ and $f_i(x) \neq f_j(x)$ for all $x \in X$ and $i, j = 1, \dots, n$ with $i \neq j$.

In this work, we shall use the following result due to H. Schirmer [11].

Lemma 1.1. [11]. *Let X and Y be two compact Hausdorff topological spaces. If X is path and simply connected and $F\colon X \to Y$ is a continuous n-valued multifunction, then F is an n-function.*

In [1], Borsuk first introduced the notion of AR spaces (for the general theory see [1, 2]).

Definition 1.2. [1, 2]. A space Y is called an absolute retract space whenever

(i) Y is metrizable and

(ii) for any metrizable space X and closed subset A of X each continuous map $f\colon A \to Y$ is extendable over X. The class of absolute retracts is denoted by AR.

By Dugundji's extension Theorem [4], we know that every nonempty convex subset of a Banach space is an AR space. In [1], it is shown that every union of two AR spaces, which their intersection is an AR space is also an AR space. Recently, in [9], Park established the following result.

Theorem 1.3. [9]. *Every nonempty compact convex subset of a metrizable topological vector space is an AR space.*

In infinite dimension topology the Hilbert cube I^∞ is an important tool. It is defined by

$$I^\infty = \left\{ (x_1, x_2, x_3, \dots) : x_i \in \mathbb{R} \text{ and for all } i \in \mathbb{N}^*, |x_i| \leq \frac{1}{i} \right\}.$$

In [1], Borsuk proved the following result.

Theorem 1.4. [1]. *Let K be a nonempty compact metric space. Then, there is a closed subset K_1 of the Hilbert cube I^∞ and a homeomorphic map $h\colon K \to K_1$.*

In [11], Schirmer studied the fix-finite approximation property for n-valued multifunction defined on finite polyhedron. Later on, in [12, 13], the first author established some results concerning the fix-finite approximation property for n-valued multifunction defined in normed spaces and metrizable locally convex spaces. In the present work we are interesting to study the existence of fixed point of continuous n-valued multifunctions.

In [5, Theorem 10.8, p.94], one can find the proof of the generalized Schauder fixed point theorem.

Theorem 1.5. [5]. *Let Y be a nonempty AR space. Then, every compact map $f\colon Y \to Y$ has a fixed point.*

In this note, we first prove that if Y is an absolute retract and $F\colon Y \to Y$ is a compact n-valued multifunction, then F has at least n fixed points (see Theorem 2.1). That is a generalization of the generalized Schauder fixed point theorem [5]. By using the properties of AR spaces [1, 2], we shall show that if C_i, for $i = 1, \dots, m$, is a finite family of nonempty convex compact subsets of a metrizable topological vector space such that $\cap_{i=1}^{i=m} C_i \neq \emptyset$, then every continuous n-valued multifunction $F\colon \cup_{i=1}^{i=m} C_i \to \cup_{i=1}^{i=m} C_i$ has at least n fixed points (see Theorem 2.2).

The notion of measure of noncompactness was first introduced by Kuratowski in [6]. In Banach spaces he defined the set-measure of noncompactness, α, as follows:

$\alpha(A) = +\infty$, if A is unbounded. and if A is bounded, then

$$\alpha(A) = \inf\{d > 0 : A \text{ can be covered with finite number of sets of diameter less than } d\}.$$

Analogously, Gokhberg, Goldenstein and Markus (see Lloyd [7], Ch. 6) introduced the ball measure of noncompactness β. The notion of measure of noncompactness in the following definition is a generalization of the measure of noncompactness α and β defined in terms of a family of seminorms or a norm.

Definition 1.6. Let E be a topological vector space and L be a lattice with a least element, which is denoted by 0. A function $\Phi\colon E \to L$ is called a measure of noncompactness on E provided that the following conditions hold for any $X, Y \in 2^E$:

(1) $\Phi(X) = 0$ if and only if $\overline{X}$ is compact,
(2) $\Phi(\overline{co}X) = \Phi(X)$, where $\overline{co}$ denotes the convex closure of X,
(3) $\Phi(X \cup Y) = \max\{\Phi(X), \Phi(Y)\}$.

Definition 1.7. For $X \subset E$, a multifunction $F\colon X \to E$ is said to be Φ-condensing provided that if $A \subset X$ and $\Phi(A) \leq \Phi(F(A))$, then A is relatively compact; that is, $\Phi(A) = 0$.

Note that every multifunction defined on a compact set is Φ-condensing.

In 2001, Cauty [3] obtained the affirmative solution of the Schauder conjecture as follows:

Theorem 1.8. [3]. *Let E be a Hausdorff topological vector space, C a nonempty convex subset of E, and f a continuous map from C into C. If $f(C)$ is contained in a compact subset of C, then f has a fixed point.*

By using the last result, we prove that if C is a nonempty closed convex subset of a Hausdorff topological vector space E and $F\colon C \to C$ is a continuous Φ-condensing n-valued multifunction, then F has at least n fixed points (see Theorem 2.5).

2. The Results

In this section, we shall establish some fixed point results for n-valued multifunctions. First, we shall show the following.

Theorem 2.1. *Let Y be a nonempty AR space. Then, every compact n-valued multifunction $F\colon Y \to Y$ has at least n fixed points.*

Proof. Let Y be a nonempty AR space and $F\colon Y \to Y$ be a compact n-valued multifunction. Let $K = \overline{F(Y)}$. Since K is a compact metric space, then by Theorem 1.4, there exists a closed subset K_1 of I^∞ and a homeomorphism $h\colon K \to K_1$. Let $i\colon K \to Y$ and $j\colon K_1 \to I^\infty$ be the inclusion maps. Then, the map $i \circ h^{-1}\colon K_1 \to Y$ is continuous. From this and as K_1 is a closed subset of I^∞ and Y is an AR space, then there exists a continuous map $g\colon I^\infty \to Y$ which extends the map $i \circ h^{-1}$. Now, set $G = j \circ h \circ F\colon Y \to I^\infty$.

Claim 1. The multifunction $G\colon Y \to I^\infty$ is an n-valued continuous multifunction. Indeed, if $x \in Y$, then $F(x) = \{y_1, \dots, y_n\}$ and $y_i \neq y_j$ for all $i, j = 1, \dots, n$ with $i \neq j$. So, we have

$$G(x) = j(h(\{y_1, \dots, y_n\})) = j(\{h(y_1), \dots, h(y_n)\}) = \{h(y_1), \dots, h(y_n)\}.$$

As h is a homeomorphism, hence for every $x \in Y$ the set $G(x)$ has exactly n elements. Thus, G is an n-valued continuous multifunction and our claim is proved.

Claim 2. We have: $F = g \circ G$. Indeed, if $x \in Y$, then $F(x) = \{y_1, \dots, y_n\}$ and $y_i \neq y_j$ for all $i, j = 1, \dots, n$ with $i \neq j$. Then, we obtain,

$$g(G(x)) = g(\{h(y_1), \dots, h(y_n)\}) = \{g(h(y_1)), \dots, g(h(y_n))\}.$$

On the other hand, we know that for every $i \in \{1, \dots, n\}$, we have $h(y_i) \in K_1$. From this and as $g/_{K_1} = i \circ h^{-1}$, then for every $i \in \{1, \dots, n\}$, we get

$$g(h(y_i)) = i \circ h^{-1}(h(y_i)) = y_i.$$

Therefore, $F = g \circ G$ and our claim is proved.

Claim 3. The multifunction $H = G \circ g\colon I^\infty \to I^\infty$ has at least n fixed point. Indeed, since G is an n-valued multifunction, then H is an n-valued multifunction. On the other hand G and g are continuous, so H is continuous. Since I^∞ is compact convex set, then by Lemma 1.1 H is an n-function. Hence, there exist n continuous maps $h_i\colon I^\infty \to I^\infty$, where $i = 1, \dots, n$, such that $H(x) = \{h_1(x), \dots, h_n(x)\}$ and $h_i(x) \neq h_j(x)$ for all $x \in I^\infty$ and $i, j = 1, \dots, n$ with $i \neq j$. By using the Schauder fixed point theorem [5], we deduce that we have $Fix(h_i) \neq \emptyset$, for every $i \in \{1, \dots, n\}$. From this and as $Fix(h_i) \cap Fix(h_j) = \emptyset$ for $i, j = 1, \dots, n$ and $i \neq j$ and $Fix(H) = \cup_{i=1}^{i=n} Fix(h_i)$, then H has at least n fixed points.

Claim 4. The multifunction F has at least n fixed point. Indeed, if x is a fixed point of H, then $g(x) \in (g \circ G)(g(x))$. On the other hand, by Claim 2, we know that we have $F = g \circ G$. Then,

$$x \in Fix(H) \Rightarrow x \in H(x) \Rightarrow g(x) \in F(g(x)) \Rightarrow g(x) \in Fix(F).$$

Thus, we have

$$g(Fix(H)) \subseteq Fix(F).$$

Now, let $x_i, x_j \in Fix(H)$ with $i, j = 1, \ldots, n$, $i \neq j$ and $x_i \neq x_j$. Let $F(g(x_i)) = \{z_1^i, \ldots, z_n^i\}$ and $F(g(x_j)) = \{z_1^j, \ldots, z_n^j\}$. As $H = G \circ g$ and $G = j \circ h \circ F$, then we have

$$H(x_i) = \{h(z_1^i), \ldots, h(z_n^i)\} \text{ and } H(x_j) = \{h(z_1^j), \ldots, h(z_n^j)\}.$$

Since, $x_i, x_j \in Fix(H)$, so there is $k, l \in \{1, \ldots, n\}$ such that

$$x_i = h(z_k^i) \text{ and } x_j = h(z_l^j).$$

From this and as $h(z_k^i), h(z_l^j) \in K_1$ and $g/_{K_1} = i \circ h^{-1}$, then we get

$$g(x_i) = g(h(z_k^i)) = z_k^i = h^{-1}(x_i) \text{ and } g(x_j) = g(h(z_l^j)) = z_l^j = h^{-1}(x_j).$$

As $x_i \neq x_j$ and h is a homeomorphism, hence we get $g(x_i) \neq g(x_j)$ for $i, j = 1, \ldots, n$ and $i \neq j$. By Claim 3, we know that the set $Fix(H)$ has at least n elements, so $g(Fix(H))$ has also at least n elements. On the other hand, we know that $g(Fix(H)) \subseteq Fix(F)$. Therefore, F has at least n fixed points. □

For finite unions of closed convex subsets of a metrizable topological vector space, we obtain the following result.

Theorem 2.2. *Let C_i, for $i = 1, \ldots, m$, be a finite family of nonempty compact convex subsets of a metrizable topological vector space such that $\cap_{i=1}^{i=m} C_i \neq \emptyset$. Then, every continuous n-valued multifunction $F\colon \cup_{i=1}^{i=m} C_i \to \cup_{i=1}^{i=m} C_i$ has at least n fixed points.*

Proof. Let $C = \cup_{i=1}^{i=m} C_i$ and let $F\colon C \to C$ be a continuous n-valued multifunction. By Theorem 1.3, we know that every nonempty convex subset of a metrizable topological vector space is an AR space. In addition, it is shown in [2] that every union of two AR, which their intersection is an AR is also an AR. From this it follows that C is an AR space. By using Theorem 2.1, we deduce that F has at least n fixed points in C. □

Remark 2.3. In Theorem 2.2, the condition $\cap_{i=1}^{i=m} C_i \neq \emptyset$ is essential. Because if it is not the case, then there exists at least a continuous n-valued multifunction $F\colon \cup_{i=1}^{i=m} C_i \to \cup_{i=1}^{i=m} C_i$ which is fixed free. Indeed, let $C_1 = \overline{B((0,1), \frac{1}{2})}$ and $C_2 = \overline{B((0,-1), \frac{1}{2})}$ be two compact convex in the Banach space $\mathbb{R}^2$ and let $f\colon C_1 \cup C_2 \to C_1 \cup C_2$ the continuous map defined by $f(x) = -x$. If $f(x) = x$, then $x = 0$. That is not possible. Therefore the map f is fixed point free.

Next, we shall show the following result.

Theorem 2.4. *Let C be a nonempty closed convex subset of a Hausdroff topological vector space and $F\colon C \to C$ be a continuous Φ-condensing n-multifunction. Then, F has at least n fixed points.*

To prove Theorem 2.4, we recall the following result.

Lemma 2.5. [8]. *Let C be a nonempty closed convex subset of a topological vector space E, and $F: C \to C$ be a Φ-condensing multifunction. Then, there exists a nonempty compact convex subset K of C such that $F(K) \subset K$.*

Combining Theorems 1.3 and 1.8 and Lemma 2.5, we obtain the proof of Theorem 2.4.

Proof of Theorem 2.4. Let C be a nonempty closed convex subset of a Hausdroff topological vector space and $F: C \to C$ be a continuous Φ-condensing n-multifunction. By Lemma 2.5, there exists a nonempty compact convex subset K of C such that $F(K) \subset K$. From this and by using Lemma 1.1 and Theorems 1.3 and 1.8, we conclude that F has at least n fixed points. $\square$

As a consequence of Theorem 2.4, we obtain the following result.

Corollary 2.6. *Let C be a nonempty closed convex subset of a Hausdroff topological vector space and $F: C \to C$ be a compact n-valued multifunction. Then, F has at least n fixed points.*

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