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SECOND ORDER PARALLEL TENSORS ON α – SASAKIAN MANIFOLD

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ABSTRACT. Levy had proved that a second order symmetric parallel nonsingular tensor on a space of constant curvature is a constant multiple of the metric tensor. Sharma [12] has proved that a second order parallel tensor in a Kaehler space of constant holomorphic sectional curvature is a linear combination with constant coefficients of the Kaehlarian metric and the fundamental 2 – form. In this paper we show that a second order symmetric parallel tensor on an $\alpha - K$ contact $(\alpha \in R_o)$ manifold is a constant multiple of the associated metric tensor and we also prove that there is no nonzero skew symmetric second order parallel tensor on an α – Sasakian manifold.

1. INTRODUCTION

In 1923, Eisenhart [10] showed that a Riemannian manifold admitting a second order symmetric parallel tensor other than a constant multiple of the metric tensor is reducible. In 1926, Levy [11] had obtained the necessary and sufficient conditions for the existence of such tensors, Recently Sharma [12] has generalized Levy's result by showing that a second order parallel (not necessarily symmetric and non singular) tensor on an n – dimensional $(n \succ 2)$ space of constant curvature is a constant multiple of the metric tensor. Sharma has also proved in [12] that on a Sasakian manifold there is no nonzero parallel 2 – form. In this paper we have considered an almost contact metric manifold and have proved the following two theorems.

Theorem 1.1. *On an $\alpha - K$ contact $(\alpha \in R_o)$ manifold a second order symmetric parallel tensor is a constant multiple of the associated positive definite Riemannian metric tensor.*

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Now the question arises whether there is a skew symmetric second order parallel tensor on a $\alpha - k$ contact manifold. We do not have an answer to it. However we do have an answer if the manifold is $\alpha-$ Sasakian where $\alpha \in R_0$.

Theorem 1.2. *On an $\alpha-$ Sasakian manifold there is no nonzero parallel 2 – forms.*

2. Preliminaries

A C^∞ manifold M of dimension $2n+1$ is called a contact manifold if it carries a global 1 – form A such that $A \wedge (dA)^n \neq 0$. On a contact manifold there exists a unique vector field T called the characteristic vector field such that

$$A\left(T\right)=1,\ \left(dA\right)\left(T,X\right)=0 \tag{2.1}$$

for any vector field X on M. By polarization we obtain a Riemannian metric g called an associated metric and a $(1,1)$ tensor field ϕ on M such that

$$\begin{gathered}\phi^2=-I+A\otimes T\\ \left(dA\right)\left(X,Y\right)=g\left(X,\phi Y\right)\\ A\left(X\right)=g\left(X,T\right)\end{gathered} \tag{2.2}$$

for the arbitrary vector fields X and Y on M. If in addition to (2.1) and (2.2), M^n admits a positive definite Riemannian metric g such that

$$\begin{gathered}g\left(\phi X,\phi Y\right)=g\left(X,Y\right)-A\left(X\right)A\left(Y\right)\\ \phi\left(T\right)=0,\ A\left(\phi\left(X\right)\right)=0,\forall\ X,Y\in\mathfrak{X}\left(M\right)\\ \text{and rank}\left(\phi\right)=2n\text{ everywhere on }M.\end{gathered} \tag{2.3}$$

Such a manifold satisfying (2.1), (2.2), and (2.3) is called an almost contact metric manifold. The structure endowed in M is called (ϕ, A, T, g) – structure.

For a (ϕ, A, T, g) – structure, the skew symmetric bilinear form

$$\Phi\left(X,Y\right)=g\left(X,\phi Y\right) \tag{2.4}$$

is called the fundamental 2 – form of the almost contact metric structure.

3. Some Definitions and Theorems

Definition 3.1. An almost contact metric structure is said to be an α – K contact structure if the vector field T is killing with respect to g.

In proving Theorems 1.1 and 1.2, we need the following theorems.

Theorem 3.1. *On an α – K contact structure the following holds.*

$$\nabla_X T=-\alpha\phi x\ \textit{for all}\ X\in\mathfrak{X}\left(M\right) \tag{3.1}$$

where ∇ is the Riemannian connection of g.

Theorem 3.2. *An almost contact metric structure – (ϕ, A, T, g) is α – Sasakian iff*

$$(\nabla_x \phi) Y = \alpha\{g(X,Y)T - A(Y)X\} \tag{3.2}$$

where ∇ denotes the Riemannian connection of g.

Proof. The proofs of the above theorems follows in a similar fashion as in the Theorem 6.3 by Blair [3]. □

Definition 3.2 ([2])**.** An almost $\alpha -$ Sasakian manifold M is an almost contact metric manifold such that $\phi(X,Y) = \frac{1}{\alpha} d\eta(X,Y), \alpha \in R_0$ and M is a α – Sasakian manifold if the structure is normal.

Theorem 3.3. *An almost contact metric manifold M is α – Sasakian manifold iff for all $X, Y \in \mathfrak{X}(M)$*

$$R(X,Y)T = \alpha\{A(Y)X - A(X)Y\} \tag{3.3}$$

Proof. The proof of the above theorem follows in view of Lemma 6.1 of Blair [3]

The two conditions of being normal and contact metric may be written as the following:

$$R(T,X)Y = \alpha\{g(X,Y)T - A(Y)X\} \tag{3.4}$$

□

Theorem 3.4. *For an $\alpha - K$ contact manifold we have*

$$R(T,X)T = \alpha\{-X + A(X)T\} \tag{3.5}$$

Proof. In view of (3.4), the proof follows immediately. □

For a detailed study on a contact manifold the reader is referred to [2].

4. Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1. Let h denote a $(0,2)$ – tensor field on an $\alpha - K$ contact manifold M such that $\nabla h = 0$. Then it follows that

$$h(R(W,X)Y,Z) + h(Y,R(W,X)Z) = 0 \tag{4.1}$$

for arbitrary vector fields X, Y, Z, W on M.

We write (4.1) as follows

$$g(R(W,X)Y,Z) + g(Y,R(W,X),Z) = 0.$$

On substituting $W = Y = Z = T$ in (4.1) we get:

$$g(R(T,X)T,T) + g(T,R(T,X),T) = 0. \tag{4.2}$$

In view of Theorem (3.4), the above equation becomes:

$$g(-\alpha X + \alpha A(X)T,T) + g(T,-\alpha X + \alpha A(X)T) = 0. \tag{4.3}$$

In this equation, using (2.2) we get

$$2\alpha g(X,T)\,h(T,T)-\alpha h(X,T)-\alpha h(T,X)=0. \tag{4.4}$$

Differentiating (4.4) covariantly with respect to Y and using Theorem (3.1) we get

$$\begin{aligned}&2\alpha h(T,T)\ \ g(\nabla_Y X,T)-2\alpha^2 h(T,T)\,g(X,\phi Y)\\ &-\alpha g(\nabla_Y X,T)+\alpha^2 g(X,\phi Y)+\alpha^2 g(\phi Y,X)-\alpha g\ \ (T,\nabla_Y X)=0.\end{aligned} \tag{4.5}$$

Replacing Y by ϕY and using equations (2.2), (2.3) and (4.4) we obtain

$$h(X,Y)+h(Y,X)=2h(T,T)\ \ g(X,Y).$$

But h is symmetric, thus on simplifying the above equation we get

$$2h(T,T)\,g(X,Y)=2h(X,Y). \tag{4.6}$$

In view of the fact that $h(T,T)$ is constant by differentiating it along any vector on M^{2n+1} we get

$$h(T,T)\,g(X,Y)=h(X,Y)$$

which completes the proof. □

Proof of Theorem 1.2. Let us consider h to be a parallel 2 – form on an $\alpha-$ Sasakian manifold M^{2n+1} and let H be a $(1,1)$ tensor field metrically equivalent to h since $h(X,Y)=g(HX,Y)$. Now (4.1) can be written as

$$g(R(W,X)\,Y,Z)+g(Y,R(W,X)\,Z)=0. \tag{4.7}$$

Let us put $X=Y=T$ in (4.7) and using the fact that $h(X,Y)=g(HX,Y)$ we get

$$g(HR(W,T)\,T,Z)+g(HT,R(WT)\,Z)=0. \tag{4.8}$$

Applying the skew symmetric property of $R(X,Y)$ and using (3.3) and (3.4) in (4.8) and after simplifying, we obtain

$$\alpha g(HZ,T)\,T+\alpha g(Z,T)\,HT=\alpha HZ. \tag{4.9}$$

Differentiating (4.9) along ϕX we obtain

$$\begin{aligned}&2\alpha A(X)\,A(HZ)\,T-\alpha g(HZ,X)\,T-\alpha g(HZ,T)\,X\\ &=\alpha g(Z,X)\,HT-2\alpha A(X)\,A(Z)\,HT+\alpha A(Z)\,HX.\end{aligned} \tag{4.10}$$

Let $\{e_i\}, i=1,2,\ldots,2n+1$ be an orthonormal basis of the tangent space. In the above equation (4.10), we substitute $X=e_i$ and take the inner product with e_i and eventually summing over i gives us

$$\alpha(2n-1)\,g(HZ,T)=0.$$

Since $\alpha(2n-1)\neq 0$, we have $g(HZ,T)=0$. But $g(HZ,T)=-g(HT,Z)$. Thus, $HT=0$ and hence (4.9) shows that $H=0$, which completes the proof.
□

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