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ON THE STABILITY AND THE PERIODICITY PROPERTIES OF SOLUTIONS OF A CERTAIN THIRD ORDER NONLINEAR DIFFERENTIAL EQUATION

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Abstract. The object of this paper is to give sufficient conditions for the existence of bounded solutions that are globally exponentially stable, periodic and almost periodic, for a certain third-order non-linear differential equation. A matrix inequality is obtained and proved to satisfy a generalized frequency domain inequality of Yacubovich [10] through the frequency domain technique.

1. Introduction

In a relatively recent note of J. Chen [12], a suitable Lyapunov function of the type "quadratic form only" was used to obtain sufficient conditions for the existence of a solution which is uniformly ultimately bounded, and periodic (or almost periodic) for the third-order non-linear differential equation:

$$x^{'''} + F(r)x^{''} + G(r)x^{'} + H(r)x = e(t), \tag{1.1}$$

where

$$\begin{aligned} F(r) &= 1 - \epsilon[1 + kn(\epsilon)r^2], \\ G(r) &= 1 - \epsilon[1 + km(\epsilon)r^2], \\ H(r) &= 1 - \epsilon kl(\epsilon)r^2. \end{aligned}$$

The functions $F(r)$, $G(r)$, $H(r)$ and $e(t)$ are continuous with $r^2 = x^2 + x^{'2} + x^{''2}$. Moreover $k > 0$ and ϵ are real parameters.

Equations of the form (1.1) with various combinations of nonlinear terms have been of great interest to many mathematicians for decades (see for instance [9] with over 250 references). The reader can find many interesting expositions in [2, 3, 4, 6, 5, 7, 11, 19, 18, 20, 17]. These equations are not only

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of theoretical interest, but also of a great practical importance as they can be applied to model automatic control in T.V. systems realized by means of R-C filters. For instance, it is well known that phase synchronization systems, modeled, by phase-locked loops (PLL systems), can be described by second order differential equations, if they include only single-chain R-C filters. However in several cases, it is useful to employ more complicated filters, leading to differential equations of higher order, inspite of the decreasing stability of the corresponding PLL systems (see, e.g. [9]).

In this paper, we would also like to consider the equation (1.1) and find necessary and sufficient conditions that guarantee the existence of a bounded solution, which is globally exponentially stable and periodic (or almost periodic). Our method of approach shall be the frequency domain technique. The frequency domain technique employed in this work is beneficial in applications and circumvents the limitations experienced in the practical construction of the well-known method of Lyapunov functions. Besides, it has been asserted (see e.g. [13]) that no Lyapunov function or its variants can better the frequency domain inequality criteria. This fact is more profound in the works of Barbălat and Halanay [10] and Yacubovich [21, 22]. For more exposition on the frequency domain method, see [1, 2, 14, 15]. Our approach in this work has an advantage over the Lyapunov second method used in [12], because the best choice of Lyapunov function of the type 'quadratic form plus the integral of the nonlinear term' was not used in [12]. Consequently, the results obtained in [12] cannot better the result obtained in this work.

In an interesting paper, Afuwape [1] derived conditions for the existence of solutions that are bounded, globally exponentially stable and periodic (or almost periodic) for special cases of the equation (1.1), when the nonlinear terms F, G, H and $e(t)$ depend only on one argument. The results obtained in this work improved some of those contained in [1] and [12]. Our work shall depend on the generalized Yacubovich's Theorem [10], which shall be stated without proof.

Generalised Yacubovich's Theorem [10]. *Consider the system:*

$$X' = AX - B\varphi(\sigma) + P(t), \quad \sigma = C^\star X \tag{1.2}$$

where A is an $n \times n$ real matrix, B and C are $n \times m$ real matrices with $C^\star$ as the transpose of C, $\varphi(\sigma) = col\varphi_j(\sigma_j), (j = 1, 2, \ldots, m)$ and $P(t)$ is an n-vector.

Suppose that in the system (1.3), the following assumptions are true:

(i) A is a stable matrix;

(ii) $P(t)$ is bounded for all t in $\mathbb{R}$;

(iii) for some constants $\hat{\mu_j} \geq 0, (j = 1, 2, \ldots, m)$

$$0 \leq \frac{\varphi_j(\sigma_j) - \varphi_j(\hat{\sigma}_j)}{\sigma_j - \hat{\sigma}_j} \leq \hat{\mu}_j, (\sigma_j \neq \hat{\sigma}_j); \tag{1.3}$$

(iv) there exists a diagonal matrix $D > 0$, such that the frequency domain inequality

$$\pi(\omega) = MD + \operatorname{Re} DG(i\omega) > 0 \tag{1.4}$$

holds for all ω in $\mathbb{R}$, where $G(i\omega) = C^{\star}(i\omega I - A)^{-1}B$ is the transfer function and $M = \operatorname{diag}(\frac{1}{\hat{\mu}_j}), (j = 1, 2, \ldots, m)$.

Then, the system (1.2) has a unique solution which is bounded in $\mathbb{R}$, globally exponentially stable and periodic (or almost periodic) whenever $P(t)$ is periodic (or almost periodic).

The paper is organized in the following way. In section 2 we give some preliminary notes which will be needed in the next sections. Next, in Section 3 we state the main result of the work and give part of the proof. In the last section we conclude the proof of the main result.

2. Preliminary Notes

The equation (1.1) can be transformed into its equivalent nonlinear system by setting

$$\begin{aligned} x^{'} &= y, \\ y^{'} &= z, \\ z^{'} &= -x - (1-\epsilon)y - (1-\epsilon)z \\ &\quad + k\epsilon[l(\epsilon)x + m(\epsilon)y + n(\epsilon)z](x^2 + y^2 + z^2) + e(t). \end{aligned} \tag{2.1}$$

This system is a periodic system with period ω say, and satisfies the uniqueness condition of the solution with respect to the initial value problem on product space $I \times F$, where $t \in I$, $I = [0, +\infty)$, $(x, y, z) \in F$. F is an arbitrary compact subset of $\mathbb{R}^3$.

The linear part of the system (2.1) is the system

$$\begin{aligned} x^{'} &= y, \\ y^{'} &= z, \\ z^{'} &= -x - (1-\epsilon)y - (1-\epsilon)z, \end{aligned} \tag{2.2}$$

from which we can derive coefficient matrix A as

$$A(\epsilon) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -(1-\epsilon) & -(1-\epsilon) \end{pmatrix}, \tag{2.3}$$

with characteristic polynomial

$$\det(\lambda I - A) = \lambda^3 + (1-\epsilon)\lambda^2 + (1-\epsilon)\lambda + 1. \tag{2.4}$$

The characteristic roots of the equation (2.4) are given as

$$\lambda_1 = -1,$$
$$\lambda_2 = \frac{\epsilon}{2} + \left(1 - \left(\frac{\epsilon}{2}\right)^2\right)^{\frac{1}{2}}, \tag{2.5}$$
$$\lambda_3 = \frac{\epsilon}{2} - \left(1 - \left(\frac{\epsilon}{2}\right)^2\right)^{\frac{1}{2}}.$$

Let us note that for a matrix to be stable, all its eigenvalues should have negative real parts, i.e. $\mathrm{Re}(\lambda_j) < 0$. For our case $j = 1, 2, 3$ and moreover, $A(\epsilon)$ will be stable if $\epsilon \leq -2$. Thus we can have $(i\omega I - A(\epsilon))$ to be

$$(i\omega I - A(\epsilon)) = \begin{pmatrix} i\omega & -1 & 0 \\ 0 & i\omega & -1 \\ 1 & (1-\epsilon) & i\omega + (1-\epsilon) \end{pmatrix} \tag{2.6}$$

and

$$\det(i\omega I - A(\epsilon)) := \Delta = \omega^2(\epsilon - 1) + 1 - i\omega(\omega^2 + \epsilon - 1), \tag{2.7}$$

from which we get

$$|\Delta|^2 = [\omega^2(\epsilon - 1) + 1]^2 + \omega^2[\omega^2 + \epsilon - 1]^2.$$

3. The Main Result

Theorem 3.1. *Suppose that in the equation (1.1), there exist positive parameters μ_1, μ_2 and μ_3 such that for all $\zeta, \hat{\zeta} \in \mathbb{R}$, $\zeta \neq \hat{\zeta}$ we have*

$$0 \leq \frac{H(\zeta) - H(\hat{\zeta})}{\zeta - \hat{\zeta}} \leq \mu_1, \tag{3.1}$$

$$0 \leq \frac{G(\zeta) - G(\hat{\zeta})}{\zeta - \hat{\zeta}} \leq \mu_2, \tag{3.2}$$

$$0 \leq \frac{F(\zeta) - F(\hat{\zeta})}{\zeta - \hat{\zeta}} \leq \mu_3, \tag{3.3}$$

and the inequality

$$1 - \mu_1 - \frac{\tau_1 \mu_1}{4}\left(\frac{\mu_3}{\tau_3} + \frac{\mu_2}{\tau_2}\right) > 6(\epsilon - 1)\mu_1\mu_2\mu_3 \tag{3.4}$$

is satisfied. Then the equation (1.1) has a solution which is bounded in $\mathbb{R}$, globally exponentially stable and periodic or almost periodic according as $e(t)$ is periodic or almost periodic.

The Proof of the Main Result. Let us set $x^{'} = y$ in the equation (1.1) to have the system (2.1). We can rewrite this in the vector form (1.2):

$$X' = AX - B\varphi(\sigma) + P(t), \quad \sigma = C^{\star}X$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -(1-\epsilon) & -(1-\epsilon) \end{pmatrix};\ B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix};$$

(3.5)

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix};\ P(t) = \begin{pmatrix} 0 \\ 0 \\ e(t) \end{pmatrix};\ \varphi(\sigma) = \begin{pmatrix} \hat{H}(x) \\ \hat{G}(y) \\ \hat{F}(y) \end{pmatrix}.$$

The transfer function $G(i\omega) = C^{\star}(i\omega I - A)^{-1}B$ of this system thus becomes

$$G(i\omega) = \frac{1}{\Delta}\begin{pmatrix} 1 & 1 & 1 \\ i\omega & i\omega & i\omega \\ -\omega^2 & -\omega^2 & -\omega^2 \end{pmatrix}. \tag{3.6}$$

On choosing

$$D = \begin{pmatrix} \tau_1 & 0 & 0 \\ 0 & \tau_2 & 0 \\ 0 & 0 & \tau_3 \end{pmatrix} \text{ and } M = \begin{pmatrix} \frac{1}{\mu_1} & 0 & 0 \\ 0 & \frac{1}{\mu_2} & 0 \\ 0 & 0 & \frac{1}{\mu_3} \end{pmatrix},$$

we obtain the frequency domain inequality (1.4) as

$$\pi(\omega) = \begin{pmatrix} \pi_{11} & \pi_{12} & \pi_{13} \\ \pi_{21} & \pi_{22} & \pi_{23} \\ \pi_{31} & \pi_{32} & \pi_{33} \end{pmatrix} > 0; \tag{3.7}$$

where

$$\pi_{11}(\omega) = \tau_1\left[\frac{1}{\mu_1} + \frac{\omega^2(\epsilon-1)+1}{|\Delta|^2}\right], \tag{3.8}$$

$$\pi_{22}(\omega) = \tau_2\left[\frac{1}{\mu_2} - \frac{\omega^2(\omega^2+\epsilon-1)}{|\Delta|^2}\right], \tag{3.9}$$

$$\pi_{33}(\omega) = \tau_3\left[\frac{1}{\mu_3} - \frac{\omega^2(\omega^2(\epsilon-1)+1)}{|\Delta|^2}\right], \tag{3.10}$$

$$\pi_{12}(\omega) = \frac{\tau_1\bar{\Delta} - i\omega\tau_2\Delta}{|\Delta|^2} = \bar{\pi}_{21}(\omega), \tag{3.11}$$

$$\pi_{13} = \frac{\left(\bar{\Delta} - \omega^2\tau_3\Delta\right)}{2|\Delta|^2} = \bar{\pi}_{13}, \tag{3.12}$$

$$\pi_{23} = \frac{\left(i\omega\tau_2\bar{\Delta} - \omega^2\tau_3\Delta\right)}{2|\Delta|^2} = \bar{\pi}_{32}. \tag{3.13}$$

For us to show that the inequality (3.7) is valid, it suffices to use the Sylvester's criteria which demand that the principal minors of $\pi(\omega)$ in (3.7) be strictly positive. We shall now prove these in a series of lemma.

Lemma 1. *For all ω in $\mathbb{R}$, $\pi_{ii}(\omega) > 0$, $(i = 1, 2, 3)$.*

Proof. For $i = 1$, $\pi_{11}(\omega)$ in the equation (3.8) will be positive for all ω in $\mathbb{R}$ if

$$\mu_1 < -v(\epsilon - 1) + 1 - \frac{v(v + \epsilon - 1)^2}{v(\epsilon - 1) + 1}, \quad \omega^2 = v. \tag{3.14}$$

Let

$$S_1(v) = -v(\epsilon - 1) + 1 - \frac{v(v + \epsilon - 1)^2}{v(\epsilon - 1) + 1},$$

then $v_1 = 0$, $v_2 = 1 - \epsilon$ and $v_3 = -\frac{1}{\epsilon - 1}$ are the roots of the equations $v(v + \epsilon - 1)^2 = 0$ and $(v(\epsilon - 1) + 1)^2 = 0$ respectively. We shall denote by m_1, the minimum of $S_1(v)$ and let this minimum be attained at say, $v = v_0$. Then $S_1^{'}(v_0) = 0$. Thus, $S_1^{'}(v_0)$ can be zero in the interval $[v_1, v_2]$. Obviously the points v_1 and v_2 are the minimum points of $S_1(v)$. If we substitute $v = v_1$ and $v = v_2$ respectively in the inequality (3.14), we shall have $S_1(v_1) = 1$ and $S_1(v_2) = \epsilon^2 - 2\epsilon + 2$. We note that there is an asymptote at $v = v_3$ for $S_1(v)$. Thus the minimum value of $S_1(v) = m_1$ and it is attainable in the interval $[v_1, v_2]$. Hence $\pi_{11}(\omega) > 0$.

Next, for $i = 2$, $\pi_{22}(\omega)$ in the equation (3.9) will be positive for all ω in $\mathbb{R}$, if we can show that

$$\mu_2 < v(v + \epsilon - 1) + \frac{(v(\epsilon - 1) + 1)^2}{v(v + \epsilon - 1)}. \tag{3.15}$$

Let

$$S_2(v) = \mu_2 < v(v + \epsilon - 1) + \frac{(v(\epsilon - 1) + 1)^2}{v(v + \epsilon - 1)}.$$

We note that there are asymptotes at $S_2(v_1)$ and $S_2(v_2)$. The maximum that $S_2(v)$ can attain is at say $v = v_0$. Let $m_2 = S_2(v_0)$ be this maximum which is attainable at $v = v_3$, and given by

$$S_2(v_3) = \frac{\epsilon(1 - (\epsilon - 1))}{(\epsilon - 1)^2}. \tag{3.16}$$

Thus $\pi_{22}(\omega) > 0$.

Next, for $i = 2$, $\pi_{33}(\omega)$ in the equation (3.10) will be positive for all ω in $\mathbb{R}$ if we can show that

$$\mu_3 < \frac{v(\epsilon - 1) + 1}{v} + \frac{(v + \epsilon - 1)}{v(\epsilon - 1) + 1}. \tag{3.17}$$

This is possible if we let

$$S_3(v) = \frac{v(\epsilon - 1) + 1}{v} + \frac{(v + \epsilon - 1)}{v(\epsilon - 1) + 1},$$

and show that it has a maximum which is always negative. Let $v = v_0$ be the point at which this maximum (denoted by m_3) is attained. Observe that at $S_3(v_1)$ and $S_3(v_3)$, there are asymptotes, hence the maximum is not attainable at these points. At the point $v = v_2$,

$$S_3(v_2) = \frac{\epsilon(2-\epsilon)}{1-\epsilon}, \tag{3.18}$$

and the maximum is attainable there. Thus $\pi_{33}(\omega) > 0$.

Lemma 2. *For all ω in $\mathbb{R}$,*

$$\pi_{ii}(\omega)\pi_{jj}(\omega) - |\pi_{ij}(\omega)|^2 > 0, \qquad (i \neq j;\ i, j, = 1, 2, 3).$$

Proof. For $i = 1$ and $j = 2$, we derive from the inequality (3.7),

$$\pi_a(\omega) = \begin{pmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{pmatrix} > 0. \tag{3.19}$$

For us to show that the inequality (3.19) is satisfied, it suffices to use Sylvester's criteria which assert that the principal minors of the matrix in (3.19) and $\det \pi_a(\omega)$ be positive definite. It has already been shown in the proof of the Lemma 1 that both $\pi_{11}(\omega)$ and $\pi_{22}(\omega)$ are positive. It remains to show that $\det \pi_a(\omega)$ in the inequality (3.19) is positive definite.

On simplifying the inequality (3.19), we have

$$\begin{aligned} \pi_a(\omega) &= \pi_{11}\pi_{22} - |\pi_{12}|^2 \\ &= \frac{\tau_1\tau_2}{\mu_1\mu_2} - \omega^2 \frac{\tau_1\tau_2(\omega^2+\epsilon-1)}{\mu_1|\Delta|^2} + \frac{\tau_1\tau_2(\omega^2+(\epsilon-1)+1)}{\mu_2|\Delta|^2} \\ &\quad - \frac{\tau_1^2+\omega^2{\tau_2}^2}{4|\Delta|^2} > 0. \end{aligned} \tag{3.20}$$

Further simplifications give

$$\begin{aligned} \pi_a(\omega) = \frac{\tau_1\tau_2}{\mu_1\mu_2} &\left(\omega^6+\omega^4\right)\left[(\epsilon-1)^2 + 2(\epsilon-1) - \mu_2\right] \\ &+ \omega^2 \left[(\epsilon-1)^2 + (2+\mu_1-\mu_2)(\epsilon-1) - \frac{\tau_2\mu_1\mu_2}{4\tau_1}\right. \\ &\left. + \left(\mu_1 + 1 - \frac{\tau_2\mu_1\mu_2}{4\tau_1}\right)\right] > 0. \end{aligned}$$

This will be true if

$$(\mu_1-\mu_2)(\epsilon-1) - \frac{\tau_2}{4\tau_1}\mu_1\mu_2$$

and

$$\frac{4(1+\mu_1)}{\mu_1\mu_2} > \frac{\tau_1}{\tau_2}.$$

Hence we have

$$\frac{\mu_1\mu_2}{4(1+\mu_1)} < \frac{\tau_2}{\tau_1} < \frac{4(\mu_1-\mu_2)}{\mu_1\mu_2},$$

which imply that $(\mu_1\mu_2)^2 < 16(\mu_1 - \mu_2)(1 + \mu_1)$. Thus $\pi_a(\omega) > 0$.

Next is to show that

$$\pi_b(\omega) = \begin{pmatrix} \pi_{11} & \pi_{13} \\ \pi_{31} & \pi_{33} \end{pmatrix} > 0. \tag{3.21}$$

On following the above arguments, we only need to show that $\det \pi_b(\omega)$ in (3.21) is positive definite so as to satisfy Sylvester's criteria, since, it has already been proved in the Lemma 1 that π_{11} and π_{13} are respectively positive. On simplifying the inequality (3.21), we obtain

$$\begin{aligned} \pi_b(\omega) &= \pi_{11}\pi_{33} - |\pi_{13}|^2 \\ &= \frac{\tau_1\tau_3}{\mu_1\mu_3} - \frac{\omega^2\tau_1\tau_3(\omega^2(\epsilon-1)+1)}{\mu_1|\Delta|^2} + \frac{\tau_1\tau_3(\omega^2(\epsilon-1)+1)}{\mu_3|\Delta|^2} \\ &\quad - \frac{{\tau_1}^2 + \omega^4{\tau_3}^2}{4|\Delta|^2} + \omega^2\frac{\tau_1\tau_3(\omega^2(\epsilon-1)+1)^2 - 2\omega^4\tau_1\tau_3(\omega^2+\epsilon-1)}{|\Delta|^4} > 0, \end{aligned}$$

from which we have

$$\begin{aligned} &|\Delta|^4 + |\Delta|^2\left[(\omega^2(\epsilon-1)+1)(\mu_1-\mu_3)\frac{1}{4}\left(\frac{\tau_1}{\tau_3} + \omega^2\frac{4\tau_3}{\tau_1}\right)\right]\mu_1\mu_3 \\ &\quad > \left[2\omega^4(\omega^2+\epsilon-1)^2 - \omega^2(\omega^2(\epsilon-1)+1)^2\right]\mu_1\mu_3. \end{aligned} \tag{3.22}$$

For the inequality (3.22) to be valid, it suffices to show that the minimum of its left hand side is positive and is greater than the maximum of the right hand side. The maximum of the right hand side is $-2\mu_1\mu_3$ and the minimum of the left hand side is $1 + \mu_1 - \frac{\tau_1\mu_1\mu_3}{4\tau_3}$. Thus we have

$$1 + \mu_1 - \frac{\tau_1\mu_1\mu_3}{4\tau_3} > -2\mu_1\mu_3,$$

which implies that

$$\frac{4(1+\mu_1)}{\mu_1\mu_3} + 8 > \frac{\tau_1}{\tau_3}.$$

Hence $\pi_b(\omega) > 0$.

At last, we shall show that

$$\pi_c(\omega) = \begin{pmatrix} \pi_{22} & \pi_{23} \\ \pi_{32} & \pi_{33} \end{pmatrix} > 0. \tag{3.23}$$

On using the preceding arguments, we have

$$\begin{aligned} \pi_c(\omega) &= \pi_{22}\pi_{33} - |\pi_{23}|^2 \\ &= \frac{\tau_2\tau_3}{\mu_2\mu_3} - \frac{\omega^2\tau_2\tau_3(\omega^2(\epsilon-1)+1)}{\mu_2|\Delta|^2} - \frac{\omega^2\tau_2\tau_3(\omega^2+\epsilon-1)}{\mu_3|\Delta|^2} \\ &\quad - \frac{\omega^2({\tau_2}^2 + \omega^2{\tau_3}^2)}{4|\Delta|^2} > 0, \end{aligned}$$

which reduces to

$$\begin{aligned}(3.24)\qquad &\omega^6+\omega^4[(\epsilon-1)^2+(\epsilon-1)-\mu_2-\frac{\tau_3\mu_2\mu_3}{4\tau_2}]\\ &+\omega^2[(\epsilon-1)^2+(2-\mu_2)(\epsilon-1)-\mu_3-\frac{\tau_2\mu_2\mu_3}{4\tau_3}]+1>0.\end{aligned}$$

The inequality (3.24) holds if

$$\frac{(\epsilon-1)-\mu_2}{\mu_2\mu_3}>\frac{\tau_3}{4\tau_2}$$

and

$$\frac{(2-\mu_2)(\epsilon-1)-\mu_3}{\mu_2\mu_3}>\frac{\tau_2}{4\tau_3}.$$

This is possible since

$$\frac{\mu_2\mu_3}{4((\epsilon-1)-\mu_3)}<\frac{\tau_2}{\tau_3}<4\left(\frac{(2-\mu_2)(\epsilon-1)-\mu_3}{\mu_2\mu_3}\right),$$

from which we have $(\mu_2\mu_3)^2<16[((2-\mu_2)(\epsilon-1)-\mu_3)((\epsilon-1)-\mu_2)$. □

4. Conclusion to the Proof of the Main Result

It now remains to show that from the inequality (3.7),

$$\begin{aligned}(4.1)\qquad \det\pi(\omega)&=\pi_{11}\left(\pi_{22}\pi_{33}-|\pi_{23}|^2\right)+\pi_{22}\left(\pi_{11}\pi_{33}-|\pi_{31}|^2\right)\\ &\quad+\pi_{33}\left(\pi_{11}\pi_{22}-|\pi_{12}|^2\right)-2\pi_{11}\pi_{22}\pi_{33}\\ &\quad+2\,\mathrm{Re}(\pi_{12}\pi_{23}\pi_{31})>0.\end{aligned}$$

This is equivalent on further simplifications to

$$\begin{aligned}(4.2)\qquad \det\pi(\omega)=&\frac{\tau_1\tau_2\tau_3}{\mu_1\mu_2\mu_3}-\frac{1}{|\Delta|^2}\left(\frac{\omega^2}{\mu_1\mu_2}\tau_1\tau_2\tau_3(\omega^2(\epsilon-1)+1)\right.\\ &+\frac{\omega^2\tau_1}{4\mu_1}({\tau_2}^2+\omega^2{\tau_3}^2)+\frac{\tau_2}{4\mu_2}({\tau_1}^2+\omega^4{\tau_3}^2)\\ &+\frac{\tau_1}{4\mu_3}({\tau_1}^2+\omega^2{\tau_2}^2)-\frac{\omega^2\tau_1\tau_2\tau_3}{\mu_1\mu_3}(\omega^2+\epsilon-1)\\ &\left.+\frac{\tau_1\tau_2\tau_3}{\mu_2\mu_3}(\omega^2(\epsilon-1)+1)\right)+\frac{\omega^2\tau_1\tau_2\tau_3}{\mu_2|\Delta|^4}\left(\omega^2(\epsilon-1)+1)^2\right.\\ &\left.-2\omega^2(\omega^2+\epsilon-1)^2+\frac{\mu_2}{4\tau_1\tau_3}({\tau_1}^2+\omega^4{\tau_3}^2)(\omega^2+\epsilon-1)\right)\\ &\frac{\omega^4}{|\Delta|^6}(\omega^2+\epsilon-1)\left(2\omega^2(\omega^2+\epsilon-1)^2\right.\\ &\left.-3(\omega^2(\epsilon-1)+1)^2\right)\tau_1\tau_2\tau_3>0.\end{aligned}$$

For $\omega^2 = 0$, $\det \pi(\omega)$ is positive if

$$1 - \mu_1 > \frac{\mu_1 \tau_1}{4} \left[\frac{\mu_3}{\tau_3} + \frac{\mu_2}{\tau_2} \right].$$

For $\omega^2 \neq 0$, $\det \pi(\omega)$ will be positive if we can show that

$$\begin{aligned}
&|\Delta|^6 - |\Delta|^4 \Bigg(\omega^2 \mu_3 [\omega^2(\epsilon - 1) + 1] + \omega^2 \frac{\mu_2 \mu_3}{4\tau_2\tau_3} ({\tau_2}^2 + \omega^2 {\tau_3}^2) \\
&+ \frac{\mu_1 \mu_3}{4\tau_1\tau_3} ({\tau_1}^2 + \omega^4 {\tau_3}^2) + \frac{\mu_1 \mu_2}{4\tau_1\tau_2} ({\tau_1}^2 + \omega^2 {\tau_2}^2) - \omega^2 \mu_2 (\omega^2 + \epsilon - 1) \\
&+ \mu_1 [\omega^2(\epsilon - 1) + 1] \Big) |\Delta|^2 \left(\omega^2 \mu_1 \mu_3 [\omega^2(\epsilon - 1) + 1]^2 \right. \\
&\left. - 2\omega^2 \mu_1 \mu_3 (\omega^2 + \epsilon - 1)^2 + \omega^2 ({\tau_1}^2 + \omega)(\omega^2 + \epsilon - 1) \mu_1 \mu_2 \mu_3 \right) \\
&> \omega^4 (\omega^2 + \epsilon - 1) \left(3[\omega^3(\epsilon - 1) + 1]^2 - 2\omega^2 (\omega^2 + \epsilon - 1)^2 \right) \mu_1 \mu_2 \mu_3.
\end{aligned} \tag{4.3}$$

The inequality (4.3) will hold if we can show that the minimum of its left hand side, is strictly greater than the maximum of its right hand side. The minimum of the left hand side of the inequality (4.3) is

$$1 - \mu_1 - \frac{\mu_1 \tau_1}{4} \left(\frac{\mu_3}{\tau_3} + \frac{\mu_2}{\tau_2} \right),$$

while the maximum of the right hand side is $6(\epsilon - 1)\mu_1\mu_2\mu_3$. Thus the inequality (4.3) holds if

$$(1 - \mu_1) - \frac{\mu_1 \tau_1}{4} \left(\frac{\mu_3}{\tau_3} + \frac{\mu_2}{\tau_2} \right) > 6(\epsilon - 1)\mu_1\mu_2\mu_3.$$

This is possible by using Lemmas 1 and 2. The conclusions to the proof follow from the generalized theorem of Yacubovich.

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