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ON 3-MANIFOLD INVARIANTS ARISING FROM FINITE-DIMENSIONAL HOPF ALGEBRAS

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ABSTRACT. We reformulate Kauffman's method of defining invariants of 3-manifolds intrinsically in terms of right integrals on certain finite dimensional Hopf algebras and define a type of universal invariants of framed tangles and invariants of 3-manifolds.

1. INTRODUCTION

Hopf algebras arise from various settings in mathematics and theoretical physics. However, it has an interesting and profound connection with topology. One of the most intriguing area to be explored is perhaps the relationship between finite-dimensional Hopf algebras, or quantum groups, and invariants of knots, links and tangles and invariants of 3-manifolds. It appears that the deepest structural aspect of finite-dimensional Hopf algebras is closely related to the topological properties of the geometrical counterpart of Hopf algebras.

The purpose of this paper is to unify Kauffman's method of defining invariants of 3-manifolds in term of right integrals on certain Hopf algebras and Ohtsuki's method of defining universal invariants of links by using certain Hopf algebras [4]. Hennings [1] first pointed out a method to obtain invariants of 3-manifolds by directly using unimodular finite-dimensional Hopf algebras without representation theory involved. Kauffman and Radford [2] reformulated Hennings's method for unoriented links to define 3-manifold invariants. And Ohtsuki also reformulated Hennings' method by constructing universal invariants of links to obtain a similar type of 3-manifold invariants. In this paper, we first construct a regular isotopic invariants for homogeneous framed tangles using Kauffman's method, and then obtain a similar type of 3-manifold invariants. The paper is organized as follows. In section 2, we recall some basic concepts: quasitriangular Hopf algebras, ribbon Hopf algebras, right integrals on Hopf algebras and algebraic tensor products space $\Pi(H)$. In section 3, a comultiplication for homogeneous tangles is introduced, which is compatible

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with multiplication of homogeneous tangles. After a formal tensor space $\Pi(C)$ that corresponds to the set of homogeneous tangles is introduced, a map θ from the formal tensor space $\Pi(C)$ to $\Pi(H)$ is defined. This map will give isotopic universal invariant of homogeneous tangles. In section 4, we prove several properties of map θ, we then naturally arrive at a type of invariant of 3-manifolds. In paper [8], we will give a comprehensive comparison among these different types of invariants of links and that of 3-manifolds.

2. Ribbon Hopf algebras and right integrals

2.1. **Quasitriangular Hopf algebras.** Let $(H, \cdot, \Delta, \eta, \varepsilon, S)$ be a finite-dimensional Hopf algebra over a field k, with multiplication $\cdot$, comultiplication Δ, unit η, counit ε, and antipode S. For simplicity, we always say Hopf algebra H instead of $(H, \cdot, \Delta, \eta, \varepsilon, S)$. We take an element $R = \sum_i \alpha_i \otimes \beta_i \in H \otimes H$. For our purpose, a quasitriangular Hopf algebra is a pair (H, R), where R is invertible and obeys

$$\tau \circ \Delta(h) = R(\Delta(h))R^{-1}, \quad \forall h \in H \tag{1}$$

$$(\Delta \otimes id)R = R_{13} \otimes R_{23}, \quad (id \otimes \Delta)R = R_{13} \otimes R_{12} \tag{2}$$

The notation used here

$$R_{tl} = \sum_i 1 \otimes \cdots \otimes \alpha_t \otimes \cdots \otimes \beta_l \otimes \cdots \otimes 1,$$

is an element of $H \otimes \cdots \otimes H$, which is R in the $t - th$ and $l - th$ factors, and τ denotes the twist map.

Fundamental properties of finite-dimensional quasitriangular Hopf algebras (H, R) has been discussed by Majid [3]. Here we will need several of them. One is the inverse of R, that is

$$R^{-1} = (S \otimes id)(R) = (id \otimes S^{-1})(R). \tag{3}$$

Then, consequently we have

$$R = (S \otimes S)(R) = (S^{-1} \otimes S^{-1})(R), \tag{4}$$

where S is the antipode of H.

Now, set $u = \sum_i S(\beta_i)\alpha_i$, then u is invertible and

$$u^{-1} = \sum_i \beta_i S^2(\alpha_i), \tag{5}$$

$$\Delta(u) = (u \otimes u)(\widetilde{R}R)^{-1} = (\widetilde{R}R)^{-1}(u \otimes u), \tag{6}$$

$$S^2(h) = uhu^{-1}, \quad \forall h \in H \tag{7}$$

$$\varepsilon(u) = 1, \tag{8}$$

where $\widetilde{R}$ is $\tau(R)$.

Since for all grouplike element g we have

$$S^2(g) = g, \quad \forall g \in G(H),$$

it follows by Equation (7) that u commutes with all grouplike elements of H.

2.2. **Ribbon Hopf algebras.** A ribbon Hopf algebras is a quasitriangular Hopf algebras with a designated element that has very special properties in connection with topology of links and 3-manifolds. We denote a finite-dimensional Ribbon Hopf algebra over K by a triple (H, R, v), where (H, R) is a finite-dimensional quasitriangular Hopf algebra over K and $v \in Z(H)$, the center of H, satisfies the following relations:

$$v^2 = uS(u), \tag{9}$$

$$S(v) = v, \tag{10}$$

$$\varepsilon(v) = 1, \tag{11}$$

and

$$\Delta(v) = (v \otimes v)(\widetilde{R}R)^{-1} = (\widetilde{R}R)^{-1}(v \otimes v). \tag{12}$$

Ribbon Hopf algebras were introduced and studied by Reshetikhin and Turaev in [6]. The element v is referred to as a special element or a ribbon element in the literatures. Because the element u and v are invertible, this notion can be formulated very simply in terms of grouplike elements in several different ways [2]. For example, if there exists a grouplike element G in a quasitriangular Hopf algebra (H, R), so that $G^{-1}u$ is in the center of H and $S(u) = G^{-2}u$, then $(H, R, v = G^{-2}u)$ is a ribbon Hopf algebra.

2.3. **Right integrals on Hopf algebra.** Let H be a Hopf algebra. We call $\lambda \in H^*$ is a right integral for H (see [7] for details), if for any $f \in H^*$, λ satisfies

$$\lambda f = f(1)\lambda$$

or equivalently

$$(\lambda \otimes id) \circ \Delta = \eta \circ \lambda$$

We include a theorem from [2] with a slightly modification for our use later, and also give a remark.

Theorem 2.1 ([2]). *Suppose that (H, R) is a unimodular finite-dimensional quasitriangular Ribbon Hopf algebra with antipode S over the field K, and that λ is a non-zero right integral for H. When $G \in G(H)$, the following two conditions are equivalent:*

$$S(u) = G^{-2}u, \quad G^{-1}u \in Z(H),$$

$$\mu_G = \lambda \cdot G \text{ is cocommutative and } \mu_G \circ S = \mu_G.$$

Furthermore, we have the equations

$$(u^{-1} \leftarrow \mu_G)u = \lambda(v^{-1})v,$$
$$(S(u) \leftarrow \mu_G)S(u^{-1}) = \lambda(v)v^{-1}$$

where v is $G^{-1}u$.

Remark 1.

$$f \cdot h(x) = f(hx), \quad for \ \ h, x \in H \ \ and \ \ f \in H^*$$
$$(h \leftarrow f) = \sum_{(h)} f(h_{(1)})h_{(2)} = (f \otimes id)\Delta(h).$$

2.4. **Algebraic tensor product space.** Let H be a ribbon Hopf algebra, we define a formal space which is a direct sum of the formal infinite tensor product of H and the formal infinite tensor product of quotients of H, and we denote this formal space by $\Pi(H)$, call it algebraic tensor product space. Specifically, $\Pi(H) = \otimes_{k=-\infty}^{-1} H_k/I \oplus \otimes_{n=1}^{\infty} H_n$, where $H_n = H_k = H$ for all n and k, I is the linear space spanned by $\alpha\beta - \beta\alpha$, $S(h) - h$ for any $\alpha, \beta, h \in H$.

Let S_n be the $n-th$ symmetric group. For each $\sigma \in S_n$, we have a natural map $\sigma \colon H^{\otimes n} \to H^{\otimes n}$,

$$\sigma(w) = w_{\sigma(1)} \otimes \cdots \otimes w_{\sigma(n)} \text{ if } \quad w = w_1 \otimes \cdots \otimes w_1.$$

Denote

$$S_\infty = S_1 \cup S_2 \cup \cdots \cup S_n \cup \cdots ,$$

the map σ could be viewed as an element of S_∞. Therefore, σ can be extended to a map on $\Pi(H)$.

Definition 2.1. For any $a, b \in \Pi(H)$, write them out as

$$a = w_{-m} \otimes \cdots \otimes w_{-1} \oplus w_1 \otimes \cdots \otimes w_n,$$
$$b = v_{-l} \otimes \cdots \otimes v_{-1} \oplus v_1 \otimes \cdots \otimes w_t,$$

when $t = n$, for any $\sigma \in S_n$, we can define a multiplication

$$a \underset{\sigma}{\cdot} b = w_{-m} \otimes \cdots \otimes w_{-1} \otimes v_{-l} \otimes \cdots \otimes v_{-1} \oplus w_1 v_{\sigma(1)} \otimes w_2 v_{\sigma(2)} \cdots \otimes w_n v_{\sigma(n)},$$

and define a comultiplication

$$\Delta a = \sum w_{\underset{(1)}{-m}} \otimes w_{\underset{(2)}{-m}} \otimes \cdots \otimes w_{\underset{(1)}{-1}} \otimes w_{\underset{(2)}{-1}} \oplus w_{\underset{(1)}{1}} \otimes w_{\underset{(2)}{1}} \cdots \otimes w_{\underset{(1)}{n}} \otimes w_{\underset{(2)}{n}}.$$

3. Regular Isotopic Invariants of Links (Tangles)

3.1. **Homogeneous tangles.** A homogeneous tangle is a finite set of segments and circles which are embedded in $R^2 \times [0,1]$, so that one end of each segment is in $R \times \{0\} \times \{0\}$ and another in $R \times \{0\} \times \{1\}$. A diagram is a regular projection of a tangle to $R \times [0,1]$. At $R \times \{0\}$ of a diagram of a homogeneous tangle T, we denote each component $C_1, C_2, \cdots, C_n$, from left to right. If T has m loops, we represent them with $C_{-1}, \cdots, C_{-m}$. So we may give a formal tensor $C_{-m} \otimes \cdots \otimes C_{-1} \oplus C_1 \otimes \cdots \otimes C_n$ to represent T. For example, Figure 1 show a homogeneous tangle with one loop and three segments. We will denote the set of all homogeneous tangles by Π.

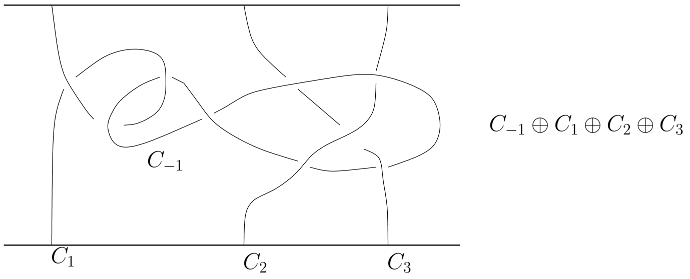


FIGURE 1. An example of homogeneous tangle

Definition 3.1. Let T_1, T_2 be homogeneous tangles and write them as

$$T_1 = C_{-m_1} \otimes \cdots \otimes C_{-1} \oplus C_1 \otimes \cdots \otimes C_{n_1}$$
$$T_2 = d_{-m_2} \otimes \cdots \otimes d_{-1} \oplus d_1 \otimes \cdots \otimes d_{n_2}$$

When $n_1 = n_2$, we can define tangle multiplication $T_1 \cdot T_2$ as a homogeneous tangle. It has $n_1(= n_2)$ components with free ends and $m_1 + m_2$ loops, and is obtained by connecting upper end of T_1 with lower end of T_2 from left to right. That is,

$$T_1 \cdot T_2 = C_{-m_1} \otimes \cdots \otimes C_{-1} \otimes d_{-m_1} \otimes \cdots \otimes d_{-1} \oplus C_1 d_{\sigma(1)} \otimes \cdots \otimes C_{n_1} d_{\sigma(n_1)}$$

where $\sigma \in S_n$ is determined by T_1.

We can also define tangle comultiplication $\overline{\Delta}$. Let T be a homogeneous tangle, $T = a_{-m} \otimes \cdots \otimes a_{-1} \oplus a_1 \otimes \cdots \otimes a_n$. $\overline{\Delta}(T)$ is also a homogeneous tangle that has $2n$ components with free ends and $2m$ loops, obtained from T by taking $2-$parallel on each component with blackboard framing of T. Or, formally,

$$\overline{\Delta}(T) = a_{-m} \otimes a_{-m'} \otimes \cdots \otimes a_{-1} \otimes a_{-1'} \oplus a_1 \otimes a_{1'} \otimes \cdots \otimes a_n \otimes a_{n'}.$$

We then have a proposition as follows. It seems obvious.

Proposition 3.1. *$\overline{\Delta}$ as a map from the set of all homogeneous tangles, Π to Π, has a property that*

$$\overline{\Delta}(T_1 \cdot T_2) = \overline{\Delta}(T_1) \cdot \overline{\Delta}(T_2) \tag{13}$$

where $T_1, T_2 \in \Pi$.

Definition 3.2. Define a formal tensor space $\Pi(C)$ as $\otimes_{n=-\infty}^{-1} C_n \oplus \otimes_{n=1}^{\infty} C_n$, and an element of $\Pi(C)$ has at most finite non-vanity terms. Every homogeneous tangle diagram can be viewed as an element of $\Pi(C)$ if we regard each homogeneous tangle as its formal tensor representation. $\Pi(C)$ possesses a multiplication and a comultiplication as Π does.

3.2. **A map from $\Pi(C)$ to $\Pi(H)$.** Given any homogeneous tangle $T \in \Pi$, equip it with a Morse function h, so that its diagram $D \in \Pi(C)$ consists entirely of crossings, minima, maxima and vertical arcs:

$$D = C_{-m} \otimes \cdots \otimes C_{-1} \oplus C_1 \otimes \cdots \otimes C_n.$$

Given a ribbon Hopf algebra $(H, R, v = G^{-1}u)$, we can express universal element R and its inverse in terms of basis of H as

$$R = \sum_{i \in \Lambda} \alpha_i \otimes \beta_i, \qquad R^{-1} = \sum_{i \in \Lambda} \alpha_i^{'} \otimes \beta_i^{'}$$

where Λ is an index set.

Call a map

$$\rho \colon \{\text{crossings of } D\} \to \Lambda$$

a state. For each state we attach an element of H to strings of D at crossings, as shown in Figure2:

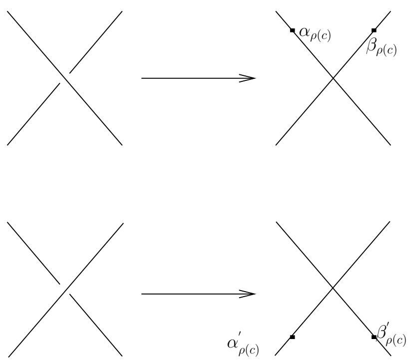


FIGURE 2. The assignments

Mark a point on the vertical arc of each component of $C_{-m}, \cdots, C_{-1}$ as a base point, and each component of $C_1, \cdots, C_n$ has a natural base point which is its down end. We define a weight $W(\rho) \in H^{m+n}$ as follows: in each component, we move algebraic elements in turn to this component's base point. When algebraic elements slid across a maxima or minima, they are replaced by the application of antipode to them if the motion is anti-clockwise, and replaced by the application of the inverse of antipode to them if the motion is clockwise. The algebraic elements W_k is obtained by multiplying those elements which slide to the base point of $k-th$ component. Let d_k be the Whitney degree of the $k-th$ component that is obtained by traversing $k-th$ component upward from the base point. This is the total turns of the tangent vector to the component as one traverse it in the upward direction from the base point, and this is $+1$ if the traverse is once clockwise and -1 if anti-clockwise. So the $k-th$ component of $w(\rho)$ is defined $W_k(\rho)G^{d_k}$. Now, let's define a map as follows.

Define

$$\theta\colon \Pi(C) \to \Pi(H)$$
$$\theta(T,D) = (\pi \otimes \cdots \otimes \pi \otimes id \otimes \cdots \otimes id)(\sum_{\rho} w(\rho)),$$

where the sum is taken over all states. I is the vector subspace in H spanned by $\alpha\beta - \beta\alpha$, $S(h) - h$ for any α, β, $h \in H$ and $\pi\colon H \to H/I$ is the projection.

Theorem 3.1. *$\theta(T,D)$ does not depend on the choice of a diagram and base points of each component.*

Proof. The proof is to verify invariance of algebraic results under Reidemeister moves and crossing maxima or minima. We verify them in two steps.

(i) It is sufficient to check invariance under the following moves as show in Figure 3, 4, 5 and 6.

I. Figure 3 shows the invariance under Reidemeister move II.

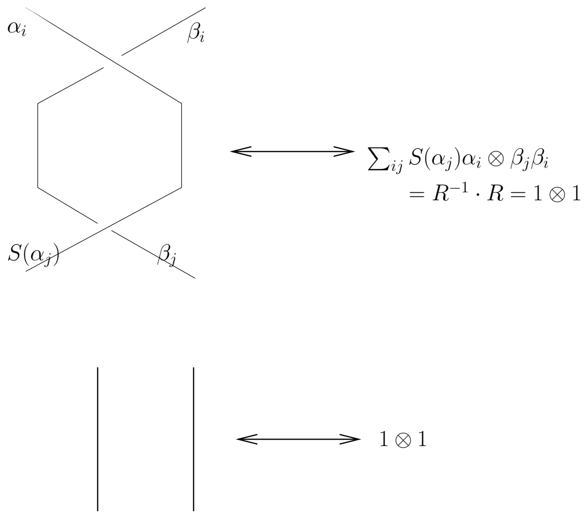


Figure 3. Sliding over

II. Figure 4 and Figure 5 show the invariance under Reidemeister move III. where $\widetilde{R} = \tau(R)$.

By Equation (1) and (2), we have

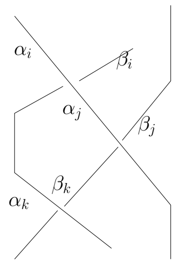
$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

therefore,

$$\widetilde{R}_{12}\widetilde{R}_{13}\widetilde{R}_{23} = \widetilde{R}_{23}\widetilde{R}_{13}\widetilde{R}_{12}.$$

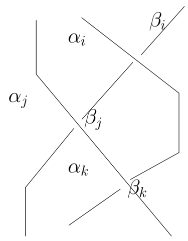
III. Figure 6 shows the invariance under deformed crossings.

(ii) It is sufficient to check invariance under one base point across one maxima or minima and only in one component $C_k, k < 0$. So, we just check them as show in Figure 7 and 8.



$$\longleftrightarrow \quad \begin{aligned} &\sum \beta_k\beta_j \otimes \alpha_k\beta_i \otimes \alpha_j\alpha_i \\ &= \sum \beta_k \otimes \alpha_k \otimes 1 \cdot \beta_j \otimes 1 \otimes \alpha_j \cdot 1 \otimes \beta_i \otimes \alpha_i \\ &= \widetilde{R}_{12}\widetilde{R}_{13}\widetilde{R}_{23} \end{aligned}$$

FIGURE 4. Yung-Baxter relation



$$\longleftrightarrow \quad \begin{aligned} &\sum \beta_j\beta_i \otimes \beta_k\alpha_i \otimes \alpha_k\alpha_j \\ &= \sum 1 \otimes \beta_k \otimes \alpha_k \cdot \beta_j \otimes 1 \otimes \alpha_j \cdot \beta_i \otimes \alpha_i \otimes 1 \\ &= \widetilde{R}_{23}\widetilde{R}_{13}\widetilde{R}_{12} \end{aligned}$$

FIGURE 5. Yung-Baxter relation

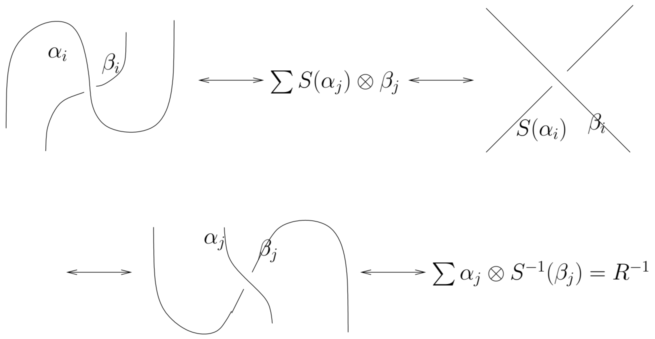


FIGURE 6. Crossings

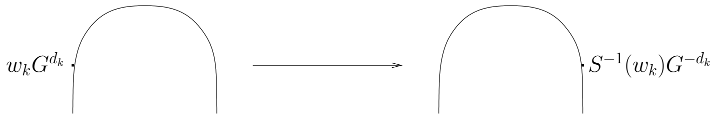


FIGURE 7. Maxima

Algebraically, for maximum showed in Figure 7, we have

$$\pi(S^{-1}(w_k)G^{-d_k}) = \pi(S^{-1}(G^{d_k}w_k)) = \pi(G^{d_k}w_k) = \pi(w_kG^{d_k}).$$

Algebraically, for minimum showed in Figure 8, we have

$$\pi(S(w_k)G^{-d_k}) = \pi(S(G^{d_k}w_k)) = \pi(G^{d_k}w_k) = \pi(w_kG^{d_k}).$$

By these two steps, we finish the proof. $\square$

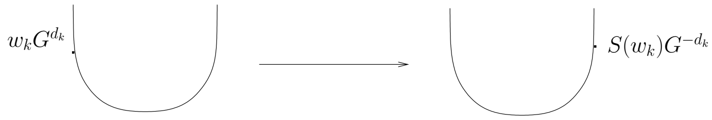


FIGURE 8. Minima

Definition 3.3. We denote $\theta(T, D)$ by $\theta(T)$. Then $\theta(T)$ is a regular isotopic invariant of homogeneous tangle, we call it universal invariant.

Ohtsuki [4] defined so-called universal invariants for oriented framed links. We here use Kauffman's method to obtain universal invariants for unoriented tangles. In a sense, we unify these two methods. However, we will point out the difference between these invariants in article [8].

4. INVARIANTS OF 3-MANIFOLDS DERIVED FROM $\theta(T)$

Proposition 4.1. *Let T_1, T_2 be homogeneous tangles, if they have the same number of free ends, then*

$$\theta(T_1 \cdot T_2) = \theta(T_1) \underset{\sigma}{\cdot} \theta(T_2)$$

where σ is determined by T_1.

Proof. It is sufficient to prove the case $m_1 = m_2 = 0$ and $n = 1$, because the product of algebraic elements is taken along one component. Let's suppose that

$$\theta(T_1) = w_1 G^{d_1}, \qquad \theta(T_2) = w_2 G^{d_2},$$

then the product is

$$\theta(T_1) \cdot \theta(T_2) = w_1 G^{d_1} \cdot w_2 G^{d_2}.$$

By the definition of the Whitney degree d and the action of the antipode, when w_2 crosses d_1 curls, it becomes $S^{2d_1}(w_2)$. Therefore,

$$\theta(T_1 \cdot T_2) = w_1 S^{2d_1}(w_2) G^{d_1+d_2}.$$

By the definition of ribbon Hopf algebras

$$\begin{aligned} S^2(h) &= GhG^{-1} \\ Gh &= S^2(h)G \\ G^2h &= G(Gh) = S^2(S^2(h)G)G = S^4(h)G^2. \end{aligned} \tag{14}$$

Inductively, for any positive integer d, we have

$$G^d h = S^{2d}(h) G^d.$$

And by 14, we have

$$\begin{aligned} S(h) &= S^{-1}(G^{-1}) S^{-1}(h) S^{-1}(G) = G S^{-1}(h) G^{-1}, \\ h &= G S^{-2}(h) G^{-1}. \end{aligned}$$

Therefore, we get

$$G^{-1}h = S^{-2}(h)G^{-1}.$$

Similarly, for any positive integer d, we have

$$G^{-d}h = S^{-2d}(h)G^{-d}.$$

In a word, for any integer d, we have

$$G^{d}h = S^{2d}(h)G^{d}.$$

Hence,

$$\theta(T_1)\cdot\theta(T_2) = w_1G^{d_1}\cdot w_2G^{d_2} = w_1S^{2d_1}(w_2)G^{d_1+d_2} = \theta(T_1\cdot T_2).$$

The proof is thus completed. □

Proposition 4.2. *Given T be a homogeneous tangle, then*

$$\theta(\overline{\Delta}(T)) = \Delta(\theta(T))$$

The proof is too long and too tedious, but basically is verifications on many cases. It is not given in here.

In order to obtain invariants of 3-manifolds, we seek a map $\varphi\colon H/I \to K$, so that $\varphi^{\otimes m}(\theta(T))$ is unchangeable under Kirby moves, where T is a link with m-components. We have local Kirby moves showed in Figure 9 and 10.

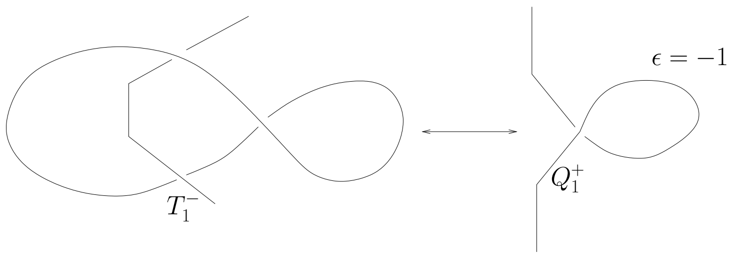


FIGURE 9. Kirby move with framing -1

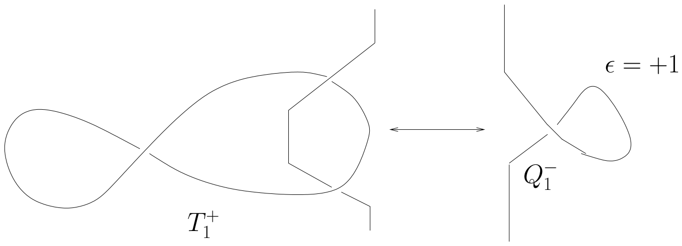


FIGURE 10. Kirby move with framing +1

We request that

$$\varphi(\theta(T_\circ^{\mp}) = C_{\mp},$$

$$(\varphi \otimes id)(\theta(T_1^{\mp})) = C_{\mp}\theta(Q_1^{\pm}),$$
$$(\varphi \otimes id^{\otimes l})(\theta(T_l^{\mp})) = C_{\mp}\theta(Q_l^{\pm}),$$
$$(id \otimes \Delta^{(l-1)})(T_l^{\mp}) = T_l^{\mp} \text{ (naturally)}.$$

By the Theorem 2.1, we read out a map $\varphi = \mu(= \lambda \cdot G)$, and now check its property

1.
$$\varphi\theta(T_\circ^-) = \varphi(v^{-1}G^{-1}) = \lambda(Gv^{-1}G^{-1}) = \lambda(v^{-1});$$
$$\varphi\theta(T_\circ^+) = \varphi(S(v)G^{-1})) = \lambda(GS(v)G^{-1}) = \lambda(v).$$

2.
$$\begin{aligned}(\varphi \otimes id)(\theta(T_1^-)) &= (\varphi \otimes id)(e_j^{'}v^{-1}e_iG^{-1} \otimes e_je_i^{'}) \\ &= (\varphi \otimes id)(e_j^{'} \otimes e_j \cdot e_i \otimes e_i^{'} \cdot v^{-1}G^{-1} \otimes 1) \\ &= (\varphi \otimes id)(R_{21}R_{12})(u^{-1} \otimes u^{-1})(1 \otimes u) \\ &= (\varphi \otimes id)(\Delta(u^{-1}))u \\ &= \lambda(v^{-1})v = \lambda(v^{-1})\theta(Q_1^+);\end{aligned}$$

$$\begin{aligned}(\varphi \otimes id)(\theta(T_1^+)) &= (\varphi \otimes id)(S(e_j)e_i^{'}vG^{-1} \otimes e_j^{'}S(e_i)) \\ &= (\varphi \otimes id)(R_{21}R_{12})^{-1}(uG^{-2} \otimes uG^{-2})(1 \otimes G^2u^{-1}) \\ &= (\varphi \otimes id)(\Delta(S(u)) \cdot (1 \otimes G^2u^{-1}) \\ &= (\varphi \otimes id)\Delta(S(u) \cdot S(u^{-1}) \\ &= \lambda(v)v^{-1} = \lambda(v)\theta(Q_1^-).\end{aligned}$$

3.
$$\begin{aligned}&(\varphi \otimes id \otimes id)(\theta(T_2^-)) \\ &= (\varphi \otimes id \otimes id)(\theta(id \otimes \overline{\Delta})(T_1^-)) \\ &= (\varphi \otimes id \otimes id)(id \otimes \Delta)(\theta(T_1^-)) \text{ by proposition (4.2)} \\ &= (\varphi \otimes \Delta)(\theta(T_1^-)) \\ &= (id \otimes \Delta)(\varphi \otimes id)(\theta(T_1^-)) \\ &= (id \otimes \Delta)(\lambda(v^{-1})\theta(Q_1^+)) \text{ by step 2} \\ &= \lambda(v^{-1})\Delta(\theta(Q_1^+)) \\ &= \lambda(v^{-1})\theta(\overline{\Delta}(Q_1^+)) \text{ by proposition (4.2)} \\ &= \lambda(v^{-1})\theta(Q_2^+).\end{aligned}$$

Inductively,
$$(\varphi \otimes id^{\otimes l})(\theta(T_l^-)) = \lambda(v^{-1})\theta(Q_l^+).$$
Similarly, we can obtain
$$(\varphi \otimes id^{\otimes l})(\theta(T_l^+)) = \lambda(v)\theta(Q_l^-).$$

Now, we arrive at a theorem.

Theorem 4.1. *Let M be a 3-manifold obtained by surgery on S^3 along a framed link L. If a map $\varphi\colon H/I \to C$ is μ_G in the Theorem 2.1, then*

$$w(M) = (\lambda(v^{-1}))^{\sigma_+ - c}(\lambda(v))^{-\sigma_+}\varphi^{\otimes c}(\theta(L))$$

is a topological invariants of M, where c is the number of components of L, and σ_+ is the number of positive eigenvalues of linking matrix of L.

Proof. It is sufficient to check that $(\lambda(v^{-1}))^{\sigma_+ - c}(\lambda(v))^{-\sigma_+}\varphi^{\otimes c}(\theta(L))$ is a constant under Kirby moves.

Suppose that $L^{'}$ is the link that is obtained by Kirby moves which delete an unknotted component with framing $\varepsilon = 1$, then we have

$$\varphi^{\otimes c}(\theta(L)) = \varphi^{\otimes (c-1)}(\theta(L^{'}))\lambda(v),$$

and

$$\sigma_+(L^{'}) = \sigma_+(L) - 1.$$

So

$$\begin{aligned}
&\varphi^{\otimes(c-1)}(\theta(L^{'}) \cdot (\lambda(v^{-1}))^{\sigma_+(L^{'})-(c-1)}(\lambda(v))^{-\sigma_+(L^{'})} \\
&= (\lambda(v^{-1}))^{\sigma_+ - c}(\lambda(v))^{-\sigma_+} \cdot \lambda(v)\varphi^{\otimes(c-1)}(\theta(L^{'})) \\
&= (\lambda(v^{-1}))^{\sigma_+ - c}(\lambda(v))^{-\sigma_+}\varphi^{\otimes c}(\theta(L)).
\end{aligned}$$

Similarly, when

$$\begin{gathered}
\varepsilon = -1, \qquad \sigma_+(L^{'}) = \sigma_+(L), \\
\varphi^{\otimes(c-1)}(\theta(L^{'}) \cdot (\lambda(v^{-1})) = \varphi^{\otimes c}(\theta(L)),
\end{gathered}$$

we have

$$\begin{aligned}
&\varphi^{\otimes(c-1)}(\theta(L^{'})(\lambda(v^{-1}))^{\sigma_+ -(c-1)}(\lambda(v))^{-\sigma_+} \\
&= (\lambda(v^{-1}))^{\sigma_+ - c}(\lambda(v))^{-\sigma_+}\lambda(v^{-1})\varphi^{\otimes(c-1)}(\theta(L^{'})) \\
&= (\lambda(v^{-1}))^{\sigma_+ - c}(\lambda(v))^{-\sigma_+}\varphi^{\otimes c}(\theta(L)).
\end{aligned}$$

Thus we get the proof. □

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