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A FRAMED $f(3,1)-$ STRUCTURE ON TANGENT MANIFOLDS

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Abstract. A tangent manifold is a pair (M, J) with J a tangent structure ($J^2 = 0$, $\ker J = \operatorname{im} J$) on the manifold M. A systematic study of tangent manifolds was done by I. Vaisman in [5]. One denotes by HM any complement of $\operatorname{im} J := TV$. Using the projections h and v on the two terms in the decomposition $TM = HM \oplus TV$ one naturally defines an almost complex structure F on M. Adding to the pair (M, J) a Riemannian metric g in the bundle TV one obtains what we call a $GL-$tangent manifold. We assumes that the $GL-$tangent manifold (M, J, g) is of bundle-type, that is M posses a globally defined Euler or Liouville vector field. This data allow us to deform F to a framed $f(3,1)-$structure $\mathcal{F}$. The later kind of structures have origin in the paper [6] by K. Yano. Then we show that $\mathcal{F}$ restricted to a submanifold that is similar to the indicatrix bundle in Finsler geometry, provides a Riemannian almost contact structure on the said submanifold.

The present results extend to the framework of tangent manifolds our previous results on framed structures of the tangent bundles of Finsler or Lagrange manifolds, see [1], [2].

1. Bundle-type tangent manifolds

Let M be a smooth i.e. C^∞ manifold. We denote by $\mathcal{F}(M)$ the ring of smooth functions on M, by TM the tangent bundle and by $\mathcal{X}(M) = \Gamma TM$ the $\mathcal{F}(M)-$module of vector fields on M (sections in tangent bundle).

Definition 1.1. An *almost tangent structure* on M is a tensor field J of type $(1,1)$ on M i.e. $J \in \Gamma\mathrm{End}(TM)$ such that

$$J^2 = 0,\ \operatorname{im} J = \ker J. \tag{1.1}$$

It follows that the dimension of M must be even, say $2n$ and $\operatorname{rank} J = n$.

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Definition 1.2. An almost tangent structure J is called a **tangent structure** if there exists an atlas on M with local coordinates (x^i, y^i), $i, j, k \ldots = 1, 2, \ldots, n$, such that

$$J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i},\ J\left(\frac{\partial}{\partial y^i}\right) = 0. \tag{1.2}$$

A pair (M, J) is called a *tangent manifold*.

For the geometry of tangent manifolds we refer to I. Vaisman's paper [5]. The basic example is tangent bundle TM, cf. the book [4].

Let (M, J) be a tangent manifold. The distribution $\operatorname{im} J$ is integrable. It defines a *vertical* foliation V with $TV = \operatorname{im} J$. Let us choose and fix a complement bundle HM called also the horizontal bundle such that

$$TM = HM \oplus TV. \tag{1.3}$$

In the following we shall use bases adapted to the decomposition (1.3):

$$\left(\delta_i = \partial_i - N_i^j(x,y)\dot{\partial}_j, \dot{\partial}_i = \frac{\partial}{\partial y^i}\right),\ \partial_i := \frac{\partial}{\partial x^i},$$

such that $TV = \operatorname{span}\{\dot{\partial}_j\}$, $HM = \operatorname{span}\{\delta_i\}$.

The dual cobase is $(dx^i, \delta y^i = dy^i + N_j^i(x,y)dx^j)$, that is $(HM)^* = \operatorname{span}\{dx^i\}$ and $(TV)^* = \operatorname{span}\{\delta y^i\}$. Here $(N_i^j(x,y))$ are local functions. Notice that $J(\delta_i) = \dot{\partial}_i$, $J(\dot{\partial}_i) = 0$.

Let be another atlas on M with local coordinates $(\tilde{x}^i, \tilde{y}^i)$ in which (1.2) also holds. Then necessarily one has

$$\tilde{x}^i = \tilde{x}^i(x),\ \tilde{y}^i = \frac{\partial \tilde{x}^i}{\partial x^j}(x)y^j + b^i(x), \tag{1.4}$$

$$\frac{\partial}{\partial x^i} = \frac{\partial}{\partial \tilde{x}^j}\frac{\partial \tilde{x}^j}{\partial x^i} + \frac{\partial}{\partial \tilde{y}^j}\left(\frac{\partial^2 \tilde{x}^j}{\partial x^k \partial x^i}y^k + \frac{\partial b^j}{\partial x^i}\right), \frac{\partial}{\partial y^i} = \frac{\partial \tilde{x}^j}{\partial x^i}\frac{\partial}{\partial \tilde{y}^j} \tag{1.4'}$$

$$\delta_i = \frac{\partial \tilde{x}^j}{\partial x^i}\delta_j. \tag{1.4''}$$

By (1.4") the functions $(N_j^i(x,y))$ change to the functions $(\widetilde{N}_h^k(\tilde{x},\tilde{y}))$ given by

$$\tilde{N}_k^h \frac{\partial \tilde{x}^k}{\partial x^i} = \frac{\partial \tilde{x}^h}{\partial x^k}N_i^k - \left(\frac{\partial b^h}{\partial x^i} + \frac{\partial^2 \tilde{x}^h}{\partial x^k \partial x^i}y^k\right). \tag{1.5}$$

The projections on the two terms in (1.3) will be denoted by h and v, respectively. Then $P = h - v$ is an almost product tensor structure that has the horizontal and vertical distribution as +1 (-1)-eigen distributions, respectively.

It is obvious that $J|_{HM}$ is an isomorphism $j\colon HM \to TV$ and $J = j \oplus 0$. Then $J' = 0 \oplus j^{-1}$ is an almost tangent structure, $Q = J' + J$ is an almost product structure and $F = J' - J$ is an almost complex structure.

In the adapted bases $(\delta_i, \dot{\partial}_i)$ we have:

$$J(\delta_i) = \dot{\partial}_i,\ J(\dot{\partial}_i) = 0, \tag{1.6$_1$}$$

$$J'(\delta_i) = 0,\ J'(\dot{\partial}_i) = \delta_i, \tag{1.6$_2$}$$

$$P(\delta_i) = \delta_i,\ P(\dot{\partial}_i) = -\dot{\partial}_i, \tag{1.6$_3$}$$

$$Q(\delta_i) = \dot{\partial}_i,\ Q(\dot{\partial}_i) = \delta_i, \tag{1.6$_4$}$$

$$F(\delta_i) = -\dot{\partial}_i,\ F(\dot{\partial}_i) = \delta_i. \tag{1.6$_5$}$$

Moreover, we have

$$PF = -FP = Q. \tag{1.7}$$

2. GL-tangent manifolds

Let (M, J) be a tangent manifold.

Definition 2.1. A pseudo-Riemannian structure g in the vertical subbundle $TV = \operatorname{im} J$ will be called a *generalized Lagrange* $(GL)-$structure on M. We will say that g is a $GL-$metric and (M, J, g) will be called a $GL-$tangent manifold.

Remark 2.1. The notion of GL-metric for tangent bundle TM was defined by R. Miron. Properties of various classes of GL-metrics have been established in the monograph [4].

The GL-metric g is determined by the local coefficients $g_{ij}(x,y) = g(\dot{\partial}_i, \dot{\partial}_j)$ with $\det(g_{ij}) \neq 0$ and the quadratic form $g_{ij}\xi^i\xi^j$, $(\xi \in \mathbb{R}^n)$, of constant signature. Using $(g_{ij}(x,y))$ we define a pseudo-Riemannian metric G on M by

$$G(x,y) = g_{ij}(x,y)dx^i dx^j + g_{ij}(x,y)\delta y^i \delta y^j. \tag{2.1}$$

It is clear that the subbundles HM and TV are orthogonal with respect to G.

From now on we assume that the GL-tangent manifold (M, J, g) is of *bundle-type*, that is $C = y^i\dot{\partial}_i$ is a global vector field called Liouville or Euler vector field. Then in (1.4) one has $b^i \equiv 0$.

3. A framed $f(3,1)-$structure on a GL-tangent manifold of bundle-type

Let (M, J, g) be a GL-tangent manifold of bundle-type such that g is a Riemannian metric in $TV = \operatorname{im} J$.

We call L defined by $L^2 = g_{ij}(x,y)y^iy^j$ a Lagrangian on M and if the matrix with the entries $\left(\frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2\right)$ is nonsingular, L will be called a regular Lagrangian.

The condition "bundle-type" assures that the subset $O = \{(x^i, y^i) \mid y^i = 0\}$ is a closed submanifold of M.We restrict our considerations to the open submanifold $\tilde{M} = M \backslash O$ of M and we keep the same notations for the geometrical objects involved.We notice that $(\tilde{M}, J, g)$ is a GL-tangent manifold of bundle-type.

On $\tilde{M}$ we have $L > 0$ and so we may consider the vector fields

$$\xi = \frac{1}{L} y^i \delta_i, \ \zeta = \frac{1}{L} y^i \dot{\partial}_i, \tag{3.1}$$

as well as the 1-forms

$$\omega = \frac{1}{L} y_i dx^i, \ \eta = \frac{1}{L} y_i \delta y^i, \tag{3.2}$$

where $y_i = g_{ij}(x, y) y^j$.

It is immediately that

$$\begin{aligned} \omega(\xi) = 1, \quad \omega(\zeta) = 0, \\ \eta(\xi) = 0, \quad \eta(\zeta) = 1. \end{aligned} \tag{3.3}$$

Moreover, if G is the metric given by (2.1), then

$$G(\xi, \xi) = 1, \ G(\xi, \zeta) = 0, \ G(\zeta, \zeta) = 1. \tag{3.4}$$

Recall that on $\tilde{M}$ we have the almost complex structure F.

From (1.6_5) it follows

$$F(\xi) = -\zeta, \ F(\zeta) = +\xi, \tag{3.5}$$

and one checks

Lemma 3.1. *$\omega \circ F = \eta$, $\eta \circ F = -\omega$.*

Then (3.3) and (3.4) yield

Lemma 3.2. *$\omega(X) = G(X, \xi), \eta(X) = G(X, \zeta)$, $\forall X \in \mathcal{X}(\tilde{M})$.*

Now, we set

$$\mathcal{F} = F + \omega \otimes \zeta - \eta \otimes \xi. \tag{3.6}$$

Theorem 3.1. *The triple $F = (\mathcal{F}, (\xi, \zeta), (\omega, \eta))$ is a framed $f(3, 1)$-structure, that is*

$$\begin{aligned} \mathcal{F}(\xi) = \mathcal{F}(\zeta) = 0, \ \omega \circ \mathcal{F} = \eta \circ \mathcal{F} = 0, \\ \mathcal{F}^2 = -I + \omega \otimes \xi + \eta \otimes \zeta, \end{aligned} \tag{3.7}$$

where I is the Kronecker tensor field.

Proof. A direct calculation using (3.3), (3.5) and Lemma 3.1. □

Theorem 3.2. *The tensor field $\mathcal{F}$ is of rank $2n - 2$ and satisfies*

$$\mathcal{F}^3 + \mathcal{F} = 0. \tag{3.8}$$

Proof. The equation (3.8) easily follows from (3.7). We show that $\ker \mathcal{F}$ is spanned by ξ and ζ, that is $\ker \mathcal{F} = \operatorname{span}\{\xi, \zeta\}$. The inclusion "$\supset$" follows from (3.7). For proving the inclusion "$\subset$" let be $Z = X^i\delta_i + Y^i\dot{\partial}_i \in \ker \mathcal{F}$. Then by (3.6),

$$\mathcal{F}(Z) = -X^i\dot{\partial}_i + Y^i\delta_i - (\omega_i X^i)\zeta + (\eta_i Y^i)\xi$$

and $\mathcal{F}(Z) = 0$ gives $X^i = \frac{1}{L}(\omega_i X^i)y^i$ and $Y^i = \frac{1}{L}(\eta_i Y^i)y^i$ and so

$$Z = (\omega_i X^i)\xi + (\eta_i Y^i)\zeta.$$

Hence, $Z \in \operatorname{span}\{\xi, \zeta\}$. □

The study of structures on manifold defined by tensor field f satisfying $f^3 \pm f = 0$ has the origin in a paper by K. Yano, [6]. Later on, these structures have generically called $f-$structures. They have been extended and can be encountered under various names. We refer to the book [3].

Theorem 3.3. *The Riemannian metric G defined by (2.1) satisfies*

$$G(\mathcal{F}X, \mathcal{F}Y) = G(X,Y) - \omega(X)\omega(Y) - \eta(X)\eta(Y), \forall X, Y \in \mathcal{X}(\tilde{M}). \tag{3.9}$$

Proof. First, we notice that from the Lemmas 3.1 and 3.2 it follows $G(FX, \xi) = \eta(X)$ and $G(FX, \zeta) = -\omega(X)$
for all $X \in \mathcal{X}(\tilde{M})$. Then we have

$$\begin{aligned}
&G(FX + \omega(X)\zeta - \eta(X)\xi, FY + \omega(Y)\zeta - \eta(Y)\xi) \\
&= G(FX, FY) + \omega(Y)G(FX,\zeta) - \eta(Y)G(FX,\xi) + \omega(X)G(FY,\zeta) \\
&\quad + \omega(X)\omega(Y) - \eta(X)G(FY,\xi) + \eta(X)\eta(Y) \\
&= G(X,Y) - \omega(X)\omega(Y) - \eta(X)\eta(Y),
\end{aligned}$$

because of $G(FX, FY) = G(X,Y)$. □

Theorem 3.3 says that $(\mathcal{F}, G)$ is a Riemannian framed $f(3,1)-$ structure on $\tilde{M}$.

4. A Riemannian almost contact structure

Let be $IL = \{(x,y) \in \tilde{M} \mid L(x,y) = 1\}$. This set is a $(2n-1)-$ dimensional submanifold of $\tilde{M}$. It will be called the indicatrix of L. We are interested to study the restriction of the Riemannian framed $f(3,1)-$ structure to IL.

We shall see that in certain hypothesis on L, the said restriction is a Riemannian almost contact structure.

We consider $\tilde{M}$ endowed with the Riemannian metric G given by (2.1) and we try to find a unit normal vector field to IL.

Let be

$$x^i = x^i(u^\alpha),$$
$$y^i = y^i(u^\alpha), \tag{4.1}$$
$$\operatorname{rank}\left(\frac{\partial x^i}{\partial u^\alpha}, \frac{\partial y^i}{\partial u^\alpha}\right) = 2n-1,\ \ \alpha = 1, 2, \ldots, 2n-1,$$

a parameterization of the submanifold IL.

The local vector fields $\left(\dfrac{\partial}{\partial u^\alpha}\right)$ that form a base of the tangent space to IL, take the form

$$\frac{\partial}{\partial u^\alpha} = \frac{\partial x^i}{\partial u^\alpha}\delta_i + \left(\frac{\partial y^i}{\partial u^\alpha} + N^i_j \frac{\partial x^j}{\partial u^\alpha}\right)\dot{\partial}_i \tag{4.2}$$

and it comes out that ζ is normal to IL if and only if

$$G\left(\frac{\partial}{\partial u^\alpha}, \zeta\right) = \frac{1}{L}\left(\frac{\partial y^i}{\partial u^\alpha} + N^i_j \frac{\partial x^j}{\partial u^\alpha}\right) y_i = 0. \tag{4.3}$$

We derive the identity $L^2(x(u^\alpha), y(u^\alpha)) \equiv 1$ with respect to u^α and we obtain

$$(\delta_i L^2)\frac{\partial x^i}{\partial u^\alpha} + \left(\frac{\partial y^i}{\partial u^\alpha} + N^i_j \frac{\partial x^i}{\partial u^\alpha}\right)(\dot{\partial}_i L^2) \equiv 0. \tag{4.4}$$

Looking at (4.4) and (4.3) it comes out that (4.3) holds if L satisfies the following two conditions:

(H_1) $\delta_i L^2 = 0$,
(H_2) $\dot{\partial}_i L^2 = f y_i$, for $f \neq 0$ any smooth function on $\tilde{M}$.

If (H_1) and (H_2) hold, then ζ is the unit normal vector to IL. We restrict to IL the element from **F** and we point out this by a bar over those elements.

Thus we have:

- $\bar{\xi} = \xi$ since ξ is tangent to IL,
- $\bar{\eta} = 0$ since $\eta(X) = G(X, \zeta) = 0$ for any vector field tangent to IL,
- $\bar{\mathcal{F}} = \mathcal{F} + \omega \otimes \zeta$, because of

$$G(\bar{\mathcal{F}}X, \zeta) = G(FX, \zeta) + \omega(X)G(\zeta, \zeta) = -\omega(X) + \omega(X) = 0,$$

for any vector field X tangent to IL.

Now, we state

Theorem 4.1. *The triple $(\bar{\mathcal{F}}, \bar{\xi}, \bar{\omega})$ defines a Riemannian almost contact structure on IL, that is*

(i) $\bar{\omega}(\bar{\xi}) = 1$, $\bar{\mathcal{F}}(\bar{\xi}) = 0$, $\bar{\omega} \circ \bar{\mathcal{F}} = 0$,
(ii) $\bar{\mathcal{F}}^2 = -I + \bar{\omega} \otimes \bar{\xi}$ *on IL,*
(iii) $G(\bar{\mathcal{F}}X, \bar{\mathcal{F}}Y) = G(X, Y) - \bar{\omega}(X)\bar{\omega}(Y)$ *for any vector fields tangent to IL.*

Moreover, we have

(iv) $\bar{\mathcal{F}}^3 - \bar{\mathcal{F}} = 0$ *and* $\operatorname{rank} \bar{\mathcal{F}} = 2n - 1$.

Proof. All assertions easily follow from Theorems 3.1 - 3.3. $\square$

We end with a discussion on the hypothesis (H_1) and (H_2). More precisely we show that these hypothesis can be replaced with a weaker one (H) that is referring to (g_{ij}) only.

(H) The functions $g_{ij}(x,y)$ are 0-homogeneous in (y^i) and the functions $C_{ijk} = \frac{1}{2}\dot{\partial}_k g_{ij}$ are symmetrical in the indices i,j,k.

First, (H) implies $C_{ijk}y^k = C_{ijk}y^i = C_{ijk}y^j = 0$. Using these we compute: $\dot{\partial}_j L^2 = 2C_{ijk}y^i y^j + 2g_{jk}y^k = 2y_j$. Thus (H) implies (H_2). A new derivation with respect to (y^i) gives $\frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2 = g_{ij}$.

In order to show that (H) implies also (H_1) we need to find a set of local coefficients $(N^i_j(x,y)$ depending only on (g_{ij}).

We denote by (g^{jk}) the inverse of the matrix (g_{ij}) and consider the functions $G^i(x,y)$ given by

$$4G^i(x,y) = g^{ik}[(\dot{\partial}_k\partial_h L^2)y^h - \partial_k L^2], \tag{4.5}$$

and define the local coefficients $N^i_j(x,y)$ as

$$N^i_j(x,y) = \frac{\partial G^i}{\partial y^j}. \tag{4.6}$$

When we replace the adapted coordinates (x^i, y^i) with the adapted coordinates $(\tilde{x}^i, \tilde{y}^i)$, a direct calculation shows that the new functions $\tilde{G}^i$ are related to G^i by

$$\tilde{G}^i(\tilde{x},\tilde{y}) = G^i(x,y) - \frac{1}{2}\frac{\partial\tilde{x}^i}{\partial x^k\partial x^h}y^k y^h. \tag{4.7}$$

As a consequence of (4.7) easily follows that the functions (N^i_j) are related to $(\tilde{N}^i_j)$ by (1.5) with $b^i \equiv 0$. We recall that $\tilde{M}$ is a tangent manifold of bundle-type.

Now, we are preparing for the computation of $\delta_i L^2$.

First, we write (4.5) in the form

$$\begin{aligned} 4g_{jk}G^k &= \partial_h(2y_j)y^h - \partial_j L^2 = 2(\partial_h g_{jk})y^k y^h - \partial_j L^2 = \\ &= (2\partial_h g_{jk} - \partial_j g_{kh})y^k y^h \end{aligned}$$

and derive the both members with respect to (y^i).

We get the equation

$$8C_{jki}G^k + 4g_{jk}N^k_i = 2(\partial_k g_{ij} + \partial_i g_{jk} - \partial_j g_{ik})y^k.$$

Equivalently,

$$N^h_i = \frac{1}{2}g^{hj}(\partial_k g_{ij} + \partial_i g_{kj} - \partial_j g_{ik})y^k$$

which, by a contraction with (y^j) yields

$$2y_k N_i^k = (\partial_i g_{jk}) y^j y^k. \tag{4.8}$$

We continue computing

$$\begin{aligned} \delta_i L^2 = \partial_i L^2 - N_i^k \dot{\partial}_k L^2 &= \partial_i (g_{jk}) y^j y^k - 2N_i^k y^k \\ &\overset{(4.8)}{=} \partial_i (g_{jk}) y^j y^k - (\partial_i g_{jk}) y^j y^k = 0. \end{aligned}$$

Thus, (H) implies (H_1), too.

A simple case when the hypothesis (H) holds is when the functions (g_{ij}) depend on x only. Then g_{ij} are homogeneous of any degree in (y^i) and $C_{ijk} \equiv 0$.

Remark 4.1. It is well-known that a tangent bundle is a bundle-type tangent manifold. The results of this paper generalize those from our papers [1], [2] first from Finsler setting to $GL-$metrics and then from tangent bundles framework to bundle-type tangent manifolds.

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