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SOME RIGIDITY THEOREMS FOR FINSLER MANIFOLDS

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Abstract. This is a survey article on global rigidity theorems for complete Finsler manifolds without boundary.

Introduction

Finsler geometry is actually the geometry of a simple integral and hence is differentiable metric geometry. Since the notion of Finsler manifolds is a generalization of Riemannian manifolds, it seems natural to consider the problem: How to distinguish Finsler manifolds from Riemannian manifolds? In this paper we will obtain some global rigidity properties in Finsler geometry.

A Finsler manifold M is locally *symmetric* if, for any $p \in M$, the geodesic reflection s_p is a local isometry of the Finsler metric, and called the geodesic symmetry relative to the point p. It is obvious that such s_p induces $-\mathrm{id}$ on the tangent space T_pM, therefore, complete locally symmetric Finsler manifolds have reversible metrics. Let us just mention that Busemann and Phadke proved (without differentiability assumptions) that, on the universal cover, the geodesic reflections extend to global isometries. Egloff [7] proves that Hilbert surfaces are symmetric if and only if they are Riemannian, hence hyperbolic. For surfaces the situation is completely resolves, whereas the higher dimensional case remains open. Due to recent result of Foulon [8], there are no compact example of genuine Finsler manifolds with parallel negative definite Jacobi endomorphism. The author [10] showed that any compact symmetric Finsler metrics with positive flag curvature must be Riemannian. In [6] Deng and Hou have independently proved the more general result (Corollary 8.4) in essentially the different manner.

In [21] Wang is proved that if a Finsler manifold M of $n(> 4)$-dimension admits group of isometries of dimension greater than $n(n-1)/2+1$ then M is a

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Riemannian manifold with constant sectional curvature. Under some additional topological conditions, we also have the rigidity result in the equality cases of the theorem of Wang.

Theorem A. *Let M be an $n(\neq 4)$-dimensional simply connected compact Finsler manifold and the dimension of isometry group of M is greater than or equal to $n(n-1)/2+1$. Then M is a Riemannian manifold with positive constant sectional curvature.*

A connected locally compact metric space (M, dist) is called *two-point homogeneous* if the group G of isometries of M is transitive on equidistant pairs of points. This mean that whenever $x_i, y_i \in M, i = 1, 2$, with distance $\mathrm{dist}(x_1, y_1) = \mathrm{dist}(x_2, y_2)$, there is an isometry $g \in G$ such that $g(x_1) = x_2$ and $g(y_1) = y_2$. The special case $x_i = y_i$ then proves M homogeneous; in particular M is complete. Two-point homogeneous Riemannian spaces have all been determined, all compact and the odd-dimensional non-compact spaces by Wang ([22]), the even-dimensional non-compact spaces by Tits([20]). Tits and Wang gave a classification of these spaces: It turns out, just from this list, that these spaces were symmetric. The following theorem gives a non-Riemannian Finsler manifold occupy too much symmetry.

Theorem B. *The two-point homogeneous Finsler spaces are Riemannian.*

The Finsler metric on M can be lifted to the Sasaki metric on unit tangent space SM in a natural way and define the Laplacian Δ of a scalar function φ on SM by

$$\Delta\varphi = \overline{\Delta}\varphi + \dot{\Delta}\varphi,\ \overline{\Delta}\varphi := -g^{ij} D_i D_j \varphi,\ \dot{\Delta}\varphi := -F^2 g^{ij} \partial_i \partial_j \varphi,$$

where D_i denotes the horizontal covariant differentiation in the connection and ∂_i denotes the ordinary vertical partial differentiation. We call $\overline{\Delta}$ is the *horizontal Laplacian* and $\dot{\Delta}$ the *vertical Laplacian.* In [1], Akbar-Zadeh have proved that on an n-dimensional Finsler manifold with Ricci curvature bounded blew by $(n-1)$ and vanishing vertical Laplacian, the first nonzero eigenvalue of the Laplacian of SM is greater than or equal to $n = \dim M$. We have the rigidity result in the equality cases.

Theorem C. *Let (M, F) be an n-dimensional reversible Finsler manifold with Ricci curvature bounded blew by $(n-1)$ and vanishing vertical Laplacian. If the first nonzero eigenvalue of the Laplacian of SM is equal to $n = \lambda_1(\mathbb{S}^n)$, then M is isometric to the standard Riemannian sphere $\mathbb{S}^n$ of constant sectional curvature one.*

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1. Preliminaries

In this section, we shall recall some well-known facts about Finsler geometry. See [1, 2, 11], for more details. Let M be an n-dimensional smooth manifold without boundary and TM denote its tangent bundle. A *Finsler structure* on a manifold M is a map $F\colon TM \to [0,\infty)$ which has the following properties:

- F is smooth on $\widetilde{TM} := TM \setminus \{0\}$;
- $F(tv) = tF(v)$, for all $t > 0,\ v \in T_xM$;
- F^2 is strongly convex, i.e.,
$$g_{ij}(x,y) := \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}(x,y)$$
is positive definite for all $(x,y) \in \widetilde{TM}$.

A Finsler structure F is called *reversible* if $F(-v) = F(v)$ for all $v \in T_xM$. A *Minkowski space* is a finite dimensional real vector space V that has a Finsler metric independent of x, $F(x,y) = F(y)$. Let F_x denote the restriction of F onto T_xM. When F is Riemannian, (T_xM, F_x) are all isometric to the Euclidean space $\mathbb{R}^n$. For a general Finsler metric F, however, the Minkowski space (T_xM, F_x) may not be isometric to each other.

The Finsler structure F induces a distance d_F on $M \times M$ by
$$d_F(p,q) := \inf_{\gamma} \int_0^1 F(\dot\gamma(t))dt,$$
where the infimum is taken over all Lipschitz continuous curves $\gamma\colon [0,1] \to M$ with $\gamma(0) = p$ and $\gamma(1) = q$. It is easy to verify that for all $p, q, r \in M$
$$d_F(p,r) \le d_F(p,q) + d_F(q,r).$$
At any point $x \in M$, there are an open neighborhood U of x, a constant $C \ge 1$ and a diffeomorphism $\psi\colon U \to \mathbb{B} \subset \mathbb{R}^n$ such that
$$|u-v|_{\mathbb{R}^n}/C \le d_F(\psi^{-1}(u), \psi^{-1}(v)) \le C \cdot |u-v|_{\mathbb{R}^n},\ u,v \in \mathbb{B}.$$
Thus $d_F(p,q) = 0$ if and only if $p = q$. We conclude that (M, d_F) is a metric space and the Finsler manifold topology coincides with metric topology. A diffeomorphism is an isometry on a Finsler manifold M if it preserves this metric. By the classical van Dantzing and van der Waerden Theorem and Montgomery-Zippin Theorem, the group of isometries on a Finsler manifold form a Lie group (see [12, Chapter 1, Theorem 4.6]).

In Euclidean geometry the group of isometries plays a fundamental role and intervenes in the introduction of notions as well as in powerful techniques such as the method of moving frames. In Minkowski geometry the group of rigid motions plays a modest role, nevertheless it is important to understand this role and to study the ways in which different Minkowski spaces can be distinguished. The first easy remark on the group of isometries of a Minkowski space is that contains all affine transformation. A more detailed study of the group of isometries is possible thanks to a beautiful theorem due to Loewner and Berhrend. For the sake of completeness we sketch the proof.

Theorem 1.1. *For a unit disk D on a Minkowski space, there is just one Euclidean ball of minimal volume contains D.*

Proof. It is clear that there exists at least a Euclidean ball of minimal volume contains D. If $\mathbb{B}_1$, $\mathbb{B}_2$ is two such Euclidean balls we must show they coincide by contradiction. They key idea is to define the Euclidean ball

$$\mathbb{B}_3 := \frac{1}{2}\left(\mathbb{B}_1 + \mathbb{B}_2\right),$$

to notice that if $v \in \partial D$, then $v \in \mathbb{B}_3^c$ and the volume of $\mathbb{B}_3$ is strictly smaller that of $\mathbb{B}_1$ and $\mathbb{B}_2$ unless these two Euclidean balls are coincide. $\square$

The Chern connection on a Finsler manifold M is defined by the unique set of local 1-forms $\{\omega_j{}^i\}_{1\leq i,j\leq n}$ on $\widetilde{TM}$ such that

$$\begin{aligned}
&d\omega^i = \omega^j \wedge \omega_j{}^i,\\
&dg_{ij} = g_{kj}\omega_i{}^k + g_{ik}\omega_j{}^k + 2A_{ijk}\omega_n{}^k, \text{ where } A_{ijk} = \frac{\partial g_{ij}}{\partial y^k}.
\end{aligned}$$

Define the set of local curvature forms $\Omega_j{}^i$ by

$$\Omega_j{}^i := d\omega_j{}^i - \omega_j{}^k \wedge \omega_k{}^i.$$

Then one can write

$$\Omega_j{}^i = \frac{1}{2}R_j{}^i{}_{kl}\ \omega^k \wedge \omega^l + P_j{}^i{}_{kl}\ \omega^k \wedge \omega^{n+l}.$$

Define the curvature tensor R by $R(U,V)W = u^k v^l w^j R_j{}^i{}_{kl} E_i$, where $U = u^i E_i, V = v^i E_i, W = w^i E_i$ are vectors in the pull-back bundle π^*TM of TM by $\pi\colon \widetilde{TM} \to M$. For a fixed $v \in T_xM$ let γ_v be the geodesic from x with $\dot\gamma_v(0) = v$. Along γ_v, we have the osculating Riemannian metrics $g^{\dot\gamma_v(t)} := g\big(\gamma_v(t), \dot\gamma_v(t)\big)$ in $T_{\gamma_v(t)}M$. Define the *flag curvature* $R^{\dot\gamma_v(t)}\big(u(t)\big)\colon T_{\gamma_v(t)}M \to T_{\gamma_v(t)}M$ by

$$R^{\dot\gamma_v(t)}\big(u(t)\big) := R\big(U(t), V(t)\big)V(t),$$

where $U(t) = (\hat\gamma_v(t); u(t)), V(t) = (\hat\gamma_v(t); \gamma_v(t)) \in \pi^*TM$. The flag curvature is independent of connections, that is, the term appears in the second variation of arc length, thus is of particular interest to us. We remark that if F is Riemannian, then the flag curvature coincides with the sectional curvature. Then the Ricci curvature is defined by

$$\mathrm{Ric}(v) := \sum_{i=1}^{n} g^v\big(R^v(e_i), e_i\big), v \in T_xM,$$

where $\{e_i\}_{i=1}^n$ is a g^v-orthonormal basis for T_xM.

Let $\{\frac{\partial}{\partial x^i}\}_{i=1}^n$ be a local basis for TM and $\{dx^i\}_{i=1}^n$ be its dual basis for T^*M. Put $S_xM := \{y \in T_xM : F(x,y) = 1\}$. Let $\alpha(n-1)$ be the volume of the unit $(n-1)$-sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$. The volume form dv on M is defined by

$$dv(x) := \frac{\alpha(n-1)}{\mathrm{vol}(S_xM)}\ dx^1 \wedge \cdots \wedge dx^n := \sigma(x)\ dx,$$

where $\mathrm{vol}(A)$ denotes the volume of a subset A with respect to the standard Euclidean structure on $\mathbb{R}^n$. Busemann proved that for any bounded open subset $U \subset M$, $\mathrm{vol}_F(U) := \int_U dv(x) = \mathcal{H}_{d_F}(U)$, where $\mathcal{H}_{d_F}(U)$ denotes the Hausdorff measure of U for the metric d_F on M.

For a tangent vector $v = (x,y) \in \widetilde{TM}$, define the *mean distortion* ρ by

$$\rho(v) := \frac{\sigma(x)}{\sqrt{\det(g_{ij}^v)}} = \frac{\alpha(n-1)}{\mathrm{vol}(S_xM)} \cdot \frac{1}{\sqrt{\det(g_{ij}^v)}} = \frac{\alpha(n-1)}{\mathrm{vol}_{g^v}(S_xM)},$$

and the *mean tangent curvature* $S\colon \widetilde{TM} \to \mathbb{R}$ is defined by

$$S(v) := \frac{d}{dt}\bigg|_{t=0} \Big\{ \ln \rho\big(\dot{\gamma_v}(t)\big) \Big\}.$$

The mean tangent curvature measures the rate of changes of Minkowski tangent spaces over a Finsler manifold. An important property is that $S = 0$ for Finsler manifolds modeled on a single Minkowski space. In particular, $S = 0$ for Berwald spaces. Locally Minkowski spaces and Riemannian spaces are all Berwald spaces.

2. Proof of Theorem A

In view of [21], it is natural to ask which Finsler manifold of dimension n admits a group of isometries of dimension $n(n-1)/2 + 1$. In the Riemannian cases, Kuiper [13] and Obata [14] has classified all such groups together with their actions, and in the non-Riemannian Finsler cases, Szabó [18] also determines. A local version of Szabó result is essentially due to Tashiro [19, Theorem 6.3] although he excluded the case $n = 4$ from consideration.

Theorem 2.1. *Let M be an $n\,(\neq 4)$-dimensional simply connected compact Finsler manifold and the dimension of isometry group of M is greater than or equal to $n(n-1)/2+1$. Then M is a Riemannian manifold with positive constant sectional curvature.*

Proof. First let us consider $n = 2$ and the dimension of isometry group is equal to two. Then M is diffeomorphic to two-dimensional sphere $\mathbb{S}^2$ and the isometry group is compact and hence torus $\mathbb{S}^1 \times \mathbb{S}^1$. Since no $\mathbb{S}^1 \times \mathbb{S}^1$ actions on $\mathbb{S}^2$, the isometry group is three-dimensional, and hence M is a Riemannian manifold with positive constant sectional curvature.

In the three-dimensional case, four-dimensional group of isometries acts on a three-dimensional Finsler manifold, this action is transitive. Thus M has an osculating Riemannian metric g^* and satisfies that the isometry group of Finsler manifold (M, F) is a closed subgroup of isometry group of the osculating Riemannian manifold (M, g^*). So if the isometry group is four-dimensional, then by the theorem of Obata [14], the Riemannian manifold (M, g^*) must be one of the following:

- $\mathbb{R} \times \Sigma^2$ and $\mathbb{S}^1 \times \Sigma^2$, where Σ^2 is the two-dimensional Riemannian manifold with constant curvature.
- $\mathbb{H}^3$ is the hyperbolic space.

Since the above all spaces are not compact simply connected, the dimension of isometry group is larger that four. Because neither can a M admit a group of isometries of dimension five, we have proved.

In the $n > 4$ cases, with a standard argument we can assume that the dimension of isometry group is $n(n-1)/2 + 1$ or $n(n+1)/2$. In the last cases by Wang's argument [21] we have proved and in the other cases, the lists of classification of Kuiper [13], Obata [14] and Szabó [18] are not contained the compact simply connected manifold. Thus the group of isometries of M is $n(n+1)/2$-dimensional. □

Remark 2.2. In the simply connected four-dimensional cases, Oh [16] proved that if M supports an effective action of a compact Lie group G, then G is one of the groups $\mathrm{SO}(5)$, $\mathrm{SU}(3)/\mathbb{Z}_3$, $\mathrm{SO}(3) \times \mathrm{SO}(3)$, $\mathrm{SO}(4)$, $\mathrm{SO}(3) \times \mathbb{S}^1$, $(\mathrm{SU}(2) \times \mathbb{S}^1)/D$, $\mathrm{SU}(2)$, $\mathrm{SO}(3)$, $\mathbb{S}^1 \times \mathbb{S}^1$, $\mathbb{S}^1$. By the restriction to the dimension of isometry group, the group G is either $\mathrm{SO}(5)$ or $\mathrm{SU}(3)/\mathbb{Z}_3$. If $G = \mathrm{SO}(5)$, then the Finsler metric on M is the canonical Riemannian metric on four-dimensional sphere with positive constant sectional curvature. In the case $G = \mathrm{SU}(3)/\mathbb{Z}_3$, Oh [16] also proved that M is diffeomorphic to a two-dimensional complex projective space.

3. Proof of Theorem B

In this section we prove Theorem B. Let G be a group of isometries of M and for $x \in M$, $G_x := \{g \in G : g(x) = x\}$ is the isotropy group of G at x. Then G_x acts on the tangent space T_xM and preserves the unit tangent sphere S_xM at x. M is called *isotropic* at x if G_x is transitive on the S_xM at x; it isotropic if it is isotropic at every point. The notion of transitive is easier to use than that of two-point homogeneity because it is formulated in group theoretic terms. But the two concepts are equivalent:

Proposition 3.1. *The Finsler manifold M is two-point homogeneous if and only if M is isotropic.*

Proof. Let M be two-point homogeneous, r be the radius of a normal coordinate neighborhood

$$U = \exp_x(\{v \in T_xM : F(v) < r\})$$

of x, and $y, z \in U$ be at a distance $r/2$ from x. Then there exist $g \in G$ with $g(x) = x$, $g(y) = z$. There are $v, w \in \frac{r}{2} S_x M$ with $\exp_x(v) = y, \exp_x(w) = z$, so $dg(v) = w$. Thus G_x is transitive on $\frac{r}{2} S_x M$, hence on $S_x M$.

Let M be isotropic, and $x_i, y_i \in M$ with $d_F(x_1, y_1) = d_F(x_2, y_2)$. By homogeneity, we have $g \in G$ with $g(x_2) = x_1$. Let $\exp_{x_1}(tv)$ be the minimal geodesic, with arc-length parameterization, from x_1 to y_1 and $\exp_{x_1}(tw)$ from x_1 to $g(y_2)$. Then we obtain

$$\begin{aligned} F(v) &= d_F(x_1, y_1) = d_F(x_2, y_2) \\ &= d_F\big(g(x_2), g(y_2)\big) = d_F\big(x_1, g(y_2)\big) = F(w). \end{aligned}$$

This yield $h \in G_{x_1}$ with $dh(w) = v$. Now hg sends x_2 to $hg(x_2) = h(x_1) = x_1$ and sends y_2 to

$$hg(y_2) = h \exp_{x_1}(w) = \exp_{hx_1}\big(dh(w)\big) = \exp_{x_1}(v) = y_1.$$

This proves that M is two-point homogeneous. □

Remark 3.2. The Banach-Mazur rotation problem asks whether a separable isotropic Banach space is isometrically isomorphic to a Hilbert space. As well as we know, that question remains open to date. As we have just commented, the answer is negative if the assumption of separability is removed (see [3]). On the other hand, it is worth to mention that problem has an affirmative answer if the assumption of separable Banach spaces is strengthened to Minkowski spaces.

Now we are ready to prove Theorem B.

Theorem 3.3. *The two-point homogeneous (but not necessary reversible) Finsler spaces are Riemannian.*

Proof. We will assert that for all $x \in M$, the Minkowski space $(T_x M, F_x)$ is Euclidean. Let $\mathbb{B}$ be the Euclidean ball of minimal volume which contains the unit tangent disk $D_x M$ on $(T_x M, F_x)$. Since the volume of $\mathbb{B}$ is minimal, the boundary $\partial\mathbb{B}$ of the Euclidean ball $\mathbb{B}$ contains at least one point v of $S_x M$. For a given point w of $S_x M$, by hypothesis and Proposition 3.1 there is affinity $g \in G_x, g(v) = w$, which maps $S_x M$ on itself; it leaves volumes unchanged, hence it maps $\mathbb{B}$ on a Euclidean ball of the same volume which contains $D_x M$. By Theorem 1.1 it must coincide with $\mathbb{B}$. Since $g \in G_x$ maps $\mathbb{B}$ on itself with $g(v) = w$, we obtains that the point w lies on $\partial\mathbb{B}$, hence $S_x M = \partial\mathbb{B}$. □

4. Proof of Theorem C

Throughout this section M is a compact Finsler manifold without boundary. Before proving Theorem C, we need a simple but frequently useful theorem.

Theorem 4.1 ([9]). *Any reversible Finsler metrics with positive constant flag curvature must be Riemannian.*

In [1] Akbar-Zadeh improved Obata's theorem [15] to Finsler cases. For a Finsler manifold M with Ricci curvature bounded blew by $(n-1)$ and vanishing vertical Laplacian, the first nonzero eigenvalue λ_1 of the Laplacian of SM is equal to $n = \dim M$ if and only if M has constant flag curvature one. Thus by Theorem 4.1, we have:

Theorem 4.2. *Let (M, F) be an n-dimensional reversible Finsler manifold with Ricci curvature bounded blew by $(n-1)$ and vanishing vertical Laplacian. If the first nonzero eigenvalue λ_1 is equal to $n = \dim M$, then M is isometric to the unit Riemannian sphere $\mathbb{S}^n$.*

In order to prove Cheng's maximal diameter theorem on Riemannian manifold M, Cheng [4] obtained an upper bound on the first eigenvalue of Laplacian operator on M and showed that the equality holds if and only if M is isometric to the standard Riemannian sphere $\mathbb{S}^n$ of constant sectional curvature one. Shen [17] also obtained an upper bound the first eigenvalue of Laplacian operator on Finsler manifolds with Ricci curvature bounded below. However in his argument the equality does not guarantee the rigidity property on Finsler manifolds with vanishing mean tangent curvature. Thus in order to extend Cheng's maximal diameter theorem to Finsler manifolds, the author and Yim [11] have adopted a well-known technique in Riemannian geometry and we have the following result;

Theorem 4.3 ([11, Corollary 1])**.** *Let (M, F) be an n-dimensional reversible Finsler manifold with Ricci curvature bounded blew by $(n-1)$ and mean tangent curvature $S = 0$. If the diameter of M is equal to π, then M is isometric to the unit Riemannian sphere $\mathbb{S}^n$.*

Recall that for an n-dimensional Riemannian manifold M whose Ricci curvature $\geq n-1$, Obata ([15]) showed that the first nonzero eigenvalue λ_1 can only be n if M is the unit Riemannian sphere $\mathbb{S}^n$. Cheng ([4]) have proved that if the diameter of M is close to π, then λ_1 is close to n. Coupling this with Obata's result shows that the diameter of M is equal to π implies $\lambda_1 = n$, and therefore M is the unit sphere. Croke ([5]) showed a converse to Cheng's result, namely, that if λ_1 is close to n, then the diameter of M is close to π.

In Finsler geometry, the first nonzero eigenvalue of Laplacian on $M(SM$, resp.) has a close relationship with Ricci curvature and mean tangent curvature (vertical Laplacian, resp.) but the relation between the vertical Laplacian and the mean tangent curvature is not understood.

Problem 4.4. Is it true that the vertical Laplacian is vanishing if and only if the mean tangent curvature is zero?

The answer to question is known to be affirmative if Finsler manifolds are Berwald.

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