Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
24 (2008), 125–133
www.emis.de/journals
ISSN 1786-0091

PROJECTIVE EINSTEIN FINSLER METRICS

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Abstract. In the present paper, we investigate the necessary and sufficient condition of a given Finsler metric to be Einstein. The considered Einstein Finsler metric in the study describes all different kinds of Einstein metrics which are pointwise projective to the given one.

1. Introduction

A Finsler metric on an open subset in R^n is called a projective flat Finsler metric if it is pointwise projective to Euclidean metric. The problem of characterizing and studying projective Finsler metrics is known as Hilbert's fourth problem. More general, it can be assumed a Finsler metric on a manifold whose geodesics coincide with the geodesics of the given one as set points.

There are some quantities in the projective Finsler geometry which are projective invariant. One most important of them is the Weyl curvature. The Finsler metrics with $W^i_k = 0$ are called Weyl metrics. It is well-known that a Finsler metric is a Weyl metric if and only if it is of scalar flag curvature. The Ricci curvature plays an important role in the projective geometry of Riemannian–Finsler manifolds. The well-known Ricci tensor was introduced in 1904 by G. Ricci. Nine years later Ricci's work was used to formulate the Einstein's theory of gravitation [5].

The Ricci curvature is defined as the trace of the Riemann curvature where the Riemann curvature in direction $y \in T_xM$ is a linear transformation $R_y\colon T_xM \to T_xM$. A Finsler metric is Einstein if the Ricci scalar Ric is a function of x alone. Equivalently $\mathrm{Ric}_{ij} = \mathrm{Ric}(x)g_{ij}$ In Riemannian space, if g and $\bar{g}$ are pointwise projectively related Riemannian metrics on manifolds of dimensional $n \geq 3$, then g is of constant curvature if and only if $\bar{g}$ is of constant curvature. The same statement is also true for Einstein metrics. More precisely, it can be said:

2000 *Mathematics Subject Classification.* 53C60, 53C25.

Key words and phrases. Projectively related Finsler metrics, projectively flat, Einstein-Finsler metric.

Theorem ([9, 10])**.** *Let (M, g) be an n-dimensional Riemannian space and $\bar{g}$ another Riemannian metric pointwise projective to g. Suppose that g is Einstein, then $\bar{g}$ must be Einstein.*

The paper focuses on the Einstein Finsler metrics which are projectively related to other Einstein Finsler metrics. The classification of these metrics on the compact manifold is investigated.

These metrics are studied in the two different categories including: Finsler metrics pointwise projectively related to Einstein isotropic Finsler metric and the non-isotropic one. A Finsler metric F is said *isotropic* if it is of scalar flag curvature and this metric is said *non-isotropic* if it is not of scalar curvature. The main proposed theorem is as follows:

Theorem. *Let F be a Finsler metric $(n > 2)$ projectively related to $\bar{F}$ where $\bar{F}$ is an Einstein non-isotropic Finsler metric, then F is Einstein if and only if it is a constant multiple of $\bar{F}$.*

The question that can be raised in the situations where a Finsler metric is projectively related to the Einstein one is as: When is a Finsler metric is Einstein? The contribution of the paper is to give an answer to this question.

2. Preliminaries

Let M be a n-dimensional C^∞ manifold. Denote by T_xM as the tangent space at $x \in M$, and by $TM = \cup_{x\in M} T_xM$ as the tangent bundle of M. Each element of TM has the form (x, y), where $x \in M$ and $y \in T_xM$. Let $TM_0 = TM\{0\}$ and the natural projection $\pi : TM \to M$ is given by $\pi(x, y) = x$. The *pull-back tangent bundle* π^*TM is a vector bundle over TM_0 whose fiber $\pi^*_\nu TM$ at $\nu \in TM_0$ is just T_xM, where $\pi(\nu) = x$. Then

$$\pi^*TM = \{(x, y, \nu) \mid y \in T_xM_0,\ \nu \in T_xM\}.$$

A *Finsler metric* on a manifold M is a function $F : TM \to [0, \infty)$ which has the following properties:

(i) F is C^∞ on TM_0;
(ii) $F(x, \lambda y) = \lambda F(x, y)$ $\lambda > 0$;
(iii) For any tangent vector $y \in T_xM$, the vertical Hessian of $\frac{F^2}{2}$ given by

$$g_{ij}(x, y) = \left[\frac{1}{2}F^2\right]_{y^iy^j}$$

is positive definite.

We obtain a symmetric tensor $\mathbf{C}$ defined by

$$\mathbf{C}(U, V, W) = C_{ijk}(y)U^iV^jW^k,$$

where $U = U^i\frac{\partial}{\partial x^i}$, $V = V^i\frac{\partial}{\partial x^i}$, $W = W^i\frac{\partial}{\partial x^i}$ and $C_{ijk} = \frac{1}{4}[F^2]_{y^iy^jy^k}(y)$.

$\mathbf{C}$ is called the *Cartan* tensor. It is well known that $\mathbf{C} = \mathbf{0}$ if and only if F is Riemannian. Every Finsler metric F induces a spray $G = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}$ by

$$G^i(x,y) := \frac{1}{4} g^{il}(x,y)\left\{2\frac{\partial g_{jl}}{\partial x^k}(x,y) - \frac{\partial g_{jk}}{\partial x^l}(x,y)\right\} y^j y^k.$$

The Riemann curvature $\mathbf{R}_y = R^i_k dx^k \oplus \frac{\partial}{\partial x^i}|_p : T_pM$ is defined by

$$R^i_k(y) := 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k} y^j + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}. \tag{2.1}$$

The Riemann curvature has the following properties. For any non-zero vector $\mathbf{y} \in T_pM$,

$$\mathbf{R}_y(\mathbf{y}) = 0, \; g_y(\mathbf{R}_y(\mathbf{u}), \mathbf{v}) = g_y(\mathbf{u}, \mathbf{R}_y(\mathbf{v})), \quad \mathbf{u}, \mathbf{v} \in T_pM,$$

and

$$R^i_{kl} = \frac{1}{3}\left\{\frac{\partial R^i_k}{\partial y_l} - \frac{\partial R^i_l}{\partial y_k}\right\}. \tag{2.2}$$

For a two-dimensional plane $P \subset T_pM$ and a non-zero vector $\mathbf{y} \in T_pM$, the *flag curvature* $\mathbf{K}(P,y)$ is defined by [16]

$$\mathbf{K}(P,\mathbf{y}) := \frac{g_y(\mathbf{u}, \mathbf{R}_y(\mathbf{u}))}{g_y(\mathbf{y},\mathbf{y})g_y(\mathbf{u},\mathbf{u}) - g_y(\mathbf{y},\mathbf{u})^2},$$

where $P = \mathrm{span}\{\mathbf{y}, \mathbf{u}\}$. F is said to be of *scalar curvature* $\mathbf{K} = \lambda(y)$ if for any $\mathbf{y} \in T_pM$, the flag curvature $\mathbf{K}(P,\mathbf{y}) = \lambda(\mathbf{y})$ is independent of P containing $\mathbf{y} \in T_pM$, that is equivalent to the following system in a local coordinate system (x^i, y^i) in TM,

$$R^i_k = \lambda F^2\{\delta^i_k - F^{-1}F_{y^k} y^i\}.$$

If λ is a constant, then F is said to be of *constant curvature.* The Ricci scalar function of F is given by

$$\rho := \frac{1}{F^2} R^i_i$$

therefore, the Ricci scalar function is positive homogeneous of degree 0 in y. This means that $\rho(x,y)$ depends on the direction of the flag pole y but not its length. The Ricci tensor of a Finsler metric F is defined by

$$\mathrm{Ric}_{ij} := \left\{\frac{1}{2} R^k_k\right\}_{y^i y^j}$$

Ricci-flat manifolds are Riemannian manifolds whose Ricci tensor vanishes. In physics, they are important because they represent vacuum solutions to Einstein's equations.

Definition ([13])**.** A Finsler metric is said to be an *Einstein metric* if the Ricci scalar function is a function of alone, equivalently

$$\mathrm{Ric} = \rho(x) g_{ij}, \quad \text{or} \quad \mathrm{Ric}_{00} = \rho(x) F^2.$$

Ricci-flat manifolds are special cases of Einstein manifolds. We now consider projectively related Finsler metrics on M i.e. the metrics having the same geodesics as the point sets.

Definition. [15] A Finsler space F^n is projective to another Finsler space $\bar{F}^n$, if and only if there exist a one-positive homogeneous scalar field $P(x,y)$ on TM satisfying

$$\bar{G}^i(x,y) = G^i(x,y) + P(x,y) y^i.$$

Let G^i and $\bar{G}^i = G^i + P y^i$ be sprays on n-manifold M. The Riemann curvatures are related by [3]

$$\bar{R}^i_k = R^i_k + E\delta^i_k + \tau_k y^i, \tag{2.3}$$

where

$$E := P^2 - P_{|k} y^k,$$
$$\tau_k = 3(P_{|k} - P P_{y^k}) + E_{y^k}.$$

Definition ([15])**.** Let (M,G) be a spray space. Assume that a function P on TM is C^∞ on $TM \setminus \{0\}$ satisfying

$$P(\lambda y) = \lambda P(y), \quad \forall \lambda > 0,$$

(a) P is called a Funk function if it satisfies the following system of PDEs

$$P_{|k} = P P_{.k}.$$

(b) P is called a weak Funk function if it satisfies the following system of PDEs

$$y^k P_{|k} = P^2.$$

Lemma ([12])**.** *Let (M,F) be a Finsler space. A Finsler metric F is pointwise projective to $\tilde{F}$ if and only if*

$$\frac{\partial \tilde{F}_{|k}}{\partial y^l} y^k - \tilde{F}_{|l} = 0.$$

then

$$\tilde{G}^i = G^i + P y^i,$$

where

$$P = \frac{\tilde{F}_{|k} y^k}{2\tilde{F}}.$$

By above lemma an (α,β)-metric in the form of (1.1) *is pointwise projective to α if and only if*

$$\varphi''(\alpha_{.l}\beta - \alpha b_l)\beta_{|k} y^k = \alpha^2 \varphi'(\beta_{|k.l} y^k - \beta_{|l}). \tag{2.4}$$

Now, we are going to study the Weyl curvature of spray as an important projective invariant. The Weyl's projective invariant is constructed from the Riemann curvature. Define [3]

$$W^i_k(y) = R^i_k - K\delta^i_k - \frac{1}{n+1\partial y^m}(R^m_k - K\delta^m_k)y^i,$$

here $K := \frac{1}{n-1}\,\mathrm{Ric} = \frac{1}{n-1}R^m_m$. $W_y : T_xM \to T_xM$ *is a linear transformation satisfying* $W_y(y) = 0$. *We call* $W := W_{y_{y\in TM_0}}$ *the Weyl curvature.* W *is a projective invariant under projective transformations* [14].

Theorem ([15])**.** *A Finsler metric is of scalar curvature if and only if* $W = 0$.

Theorem ([15])**.** *For a Riemannian metric* (M, g) *of dimension* $n > 2$, *the following conditions are equivalent.*

(a) $W = 0$
(b) g *is of scalar curvature.*
(c) g *is of constant curvature.*
(d) g *is locally projectively flat.*

3. Proof of the main Theorem

In the followings, we prove the main theorem.

Proposition. *Let* (M, F) *be a Finsler space of dimension* $n > 2$. F *is Einstein metric if and only if*

$$y_iV^i_k = -\frac{3(n-1)}{n+1}Ky_k,$$

Where $V^i_k = W^i_k - R^i_k - \frac{3}{n+1}\,\mathrm{Ric}_{0k}\, y^i$.

Proof. (i) Assume that F is Einstein. By definition of Weyl tensor, we have

$$y_iW^i_k - y_iR^i_k = -Ky_k - \frac{n-2}{n+1}K_{.k}F^2 + \frac{3F^2}{n+1}\,\mathrm{Ric}_{0k},$$

then

$$y_iV^i_k = -Ky_k - \frac{n-2}{n+1}K_{.k}F^2,$$

Since F is non-isotropic Einstein metric, it can be concluded

$$2Ky_k = K_{.k}F^2,$$

therefore

$$y_iV^i_k = -\frac{3(n-1)}{n+1}Ky_k,$$

this completes the proof (i).

(ii) Suppose

$$y_iV^i_k = -\frac{3(n-1)}{n+1}Ky_k,$$

by definition of Weyl tensor, we have

$$-y_iV^i_k = y_i\left(K\delta^i_k + \frac{n-2}{n+1}K_{.k}y^i\right),$$

it can be resulted in

$$Ky_k + \frac{n-2}{n+1}K_{.k}F^2 = \frac{3(n-1)}{n+1}Ky_k,$$

by a simple computation and $n > 2$, it is concluded that,

$$2Ky_k = K_{.k}F^2,$$

this implies

$$\left(\frac{K}{F^2}\right)_{.k} = 0,$$

therefore F is Einstein. $\square$

Proposition. *Let F be projectively related to $\bar{F}$ with projective factor P (of dimension $n > 2$) which $\bar{F}$ is an Einstein Finsler metric. If F be Einstein, then $\left(\frac{E}{F^2}\right)$ where $E = P^2 - P_{|k}y^k$.*

Proof. Let W and $\overline{W}$ be the Weyl curvature of F and $\bar{F}$. For Einstein Finsler metric F, we have

$$y_iV^i_k = \frac{-3(n-1)}{n+1}Ky_k,$$

therefore

$$y_iW^i_k = y_i\left(R^i_k + \frac{3\,\mathrm{Ric}_{0k}}{n+1}y^i\right) - \frac{3(n-1)}{n+1}Ky_k,$$

but $\overline{W}^i_k$ is invariant under projective transformation, then

$$y_iW^i_k = y\overline{W}^i_k = y_i\left(\bar{R}^i_k + \frac{3\overline{\mathrm{Ric}}_{0k}}{n+1}y^i\right) - \frac{3(n-1)}{n+1}\bar{K}y_k,$$

therefore

$$y_i(R^i_k - \bar{R}^i_k) + \frac{3y_i}{n+1}\left(\mathrm{Ric}_{0k}\,y^i - \overline{\mathrm{Ric}}_{0k}y^i\right) - \frac{3(n-1)}{n+1}(K-\bar{K})y_k = 0, \tag{3.1}$$

but we know that

$$\mathrm{Ric} = \overline{\mathrm{Ric}} + (n-1)E,$$

and it implies that

$$K = \bar{K} + E, \tag{3.2}$$

By settling (3.1) in (2.2) one can conclude

$$3R^i_{kl} = 3\bar{R}^i_{kl} + (E_{.l} - \tau_l)\delta^i_k - (E_{.k} - \tau_k)\delta^i_l + (\tau_{k.l} - \tau_{l.k})y^i,$$

therefore

$$3\,\mathrm{Ric}_{0l} = 3\overline{\mathrm{Ric}}_{0l} + (n-2)E_{.l} - (n+1)\tau_l,$$

by substituting the above in (3.1), it is concluded:

$$Ey_k + \tau_k F^2 + \frac{3y_i}{3(n+1)}((n-2)E_{.k}y^i - (n+1)\tau_k y^i) - \frac{3(n-1)}{n+1}Ey_k$$
$$= -\frac{2(n-2)}{n+1}Ey_k + \frac{(n-2)F^2}{n+1}E_{.k} = 0.$$

thus

$$\frac{n-2}{n+1}(E_{.k}F^2 - 2Ey_k) = 0.$$

Since $n > 2$, then $\left(\frac{E}{F^2}\right)_{.k} = 0$. □

Proposition. *Let F be a Finsler metric $(n > 2)$ projectively related to $\bar{F}$, where $\bar{F}$ is an Einstein non-isotropic Finsler metric, then F is Einstein if and only if*

$$\left(\frac{F}{\bar{\bar{F}}}\right)_{.k} = 0.$$

Proof. Assume F is Einstein, by definition we have $\left(\frac{K}{F^2}\right)_{.k} = 0$ and by above Lemma $\left(\frac{E}{F^2}\right)_{.k} = 0$, then there exist a function $\xi(x)$ where $\frac{K-E}{F^2} = \xi(x)$. F is projectively related to $\bar{F}$, then by (3.2) we have $\frac{K}{F^2} = \frac{\bar{K}}{F^2} + \frac{E}{F^2}$. But $\bar{F}$ is Einstein non-isotropic Finsler metric then there is a non-zero function $\lambda(x)$ such that $\bar{K} = \lambda(x)\bar{F}$. It can be concluded that $\left(\frac{F}{\bar{\bar{F}}}\right)_{.k} = 0$. It is clear, conversely. □

Proof of Theorem. F is projectively related to $\bar{F}$ then it can be said that $G^i = \bar{G}^i + Py^i$ where $P = \frac{F_{|k}y^k}{2F}$. By above proposition, there is a function of x only, where $F = f(x)\bar{F}$ then $P = \frac{f_{|k}y^k}{2f}$.
By using the formula of G^i told in previous, it can be concluded

$$G^i = \bar{G}^i + \frac{f_{;x^k}y^k}{2f}y^i - \frac{\bar{F}^2}{4f}\bar{g}^{il}f_{;x^l} = \bar{G}^i + \frac{f_{;x^k}y^k}{2f}y^i.$$

Then one can conclude that $f_{;x^l} = 0$ and therefore f is constant. □

Corollary. *A Finsler metric F $(n > 2)$ projectively related to a Riemannian metric $\bar{F}$ of constant sectional curvature is Einstein if and only if α is of constant flag curvature.*

Proof. Let F be projectively related to Riemannian metric α of constant sectional curvature. This Riemannian metric is of constant sectional curvature, then $\overline{W} = 0$ and since F is projectively related to α, then $W = 0$. This resulted in F is of scalar curvature. Then it is Einstein if and only if it is of constant flag curvature [13]. □

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