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ON THE GEOMETRY OF TWO-STEP NILPOTENT GROUPS WITH LEFT INVARIANT FINSLER METRICS

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Abstract. We begin a systematic study of these spaces following Eberlein's comprehensive study in the Riemannian case. In particular, for some special groups (including Heisenberg groups) we give an explicit form for the Chern-Rund connection, study the curvature and the flag curvature, and derive the equations of relative geodesics.

1. Introduction

In this paper we study the differential geometry of simply connected, two-step, nilpotent Lie groups with left-invariant Finsler metric. There are some recent papers on invariant Finsler metrics on homogeneous manifolds (see e.g. [2, 4]), however the literature does not seem to contain much discussion of geometry of nilpotent Lie groups with a left invariant metric. The straight motivation of our present study is the first three sections of P. Eberlein's comprehensive work [3]. We generalize Eberlein's results (on curvature, sectional curvature, Ricci tensor and geodesics) to the Finsler setting, and this enumeration gives the outline of the present paper.

In possible connections determined by the Finslerian structure we consistently use the Chern-Rund connection. Following the usual approach, the Chern-Rund connection is a linear connection in the (splitted) pull-back bundle $\pi^*(\tau_N) = (TN \times_\pi TN, \mathrm{pr}_1, TN)$, i.e.

$$D\colon \mathfrak{X}(TN) \times \operatorname{Sec}\pi^*(\tau_N) \to \operatorname{Sec}\pi^*(\tau_N).$$

However, starting from this connection and a nowhere vanishing vector field W, it is possible to define a linear connection on the base manifold N by the

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following way (see e.g. [6]):

$$\nabla^W : \mathfrak{X}(N) \times \mathfrak{X}(N) \to \mathfrak{X}(N),$$
$$(X, Y) \mapsto \nabla_X Y = \mathrm{pr}_2 \circ D_{X^h} \mathrm{pr}_2^\sharp(Y) \circ W,$$

where $\sharp$ is the usual raising of a vector field on the base manifold to the pull-back bundle and the horizontal lift of a vector field is determined by the Finslerian structure (see Section 3.) This linear connection on the base manifold is called the *Chern-Rund connection*, too.

The discussion in the present work is restricted in the sense that the reference vector field is a (fixed) left-invariant vector field determined by a nonzero element of the *center* of the Lie algebra, moreover, our approach employs an additional condition, the so called *compatibility condition.* This compatibility condition formulates a relationship between the algebraic and geometric structure (see Definition 4). For example, this condition is true for the Heisenberg algebra ([3, Example 1]).

Generally, we assume that the two-step nilpotent Lie algebra $\mathcal{N}$ is equipped with a positive definite scalar product $\langle , \rangle$. Let $\mathcal{Z}$ denote the center of $\mathcal{N}$, and let $\mathcal{V}$ denote the orthogonal complement of the center with respect to $\langle , \rangle$. For each element $Z \in \mathcal{Z}$ we define a skew-symmetric linear transformation $j(Z) \colon \mathcal{V} \to \mathcal{V}$ by

$$j(Z)X = (\mathrm{ad}\, X)^* Z, \qquad X \in \mathcal{V}. \tag{1}$$

Note, for the Heisenberg algebra $j(Z)^2 = -\|Z\|^2 \,\mathrm{id}$. The transformations $\{j(Z) \mid Z \in \mathcal{Z}\}$ capture all the geometry of N equipped with the left invariant Riemannian metric determined by $\langle , \rangle$ ([3, 5]). The Finslerian structure on the manifold N determines a Riemannian structure with respect to the reference vector field W, thus in our approach the above skew-symmetric transformation depends on the reference vector field and it is denoted by j_W.

2. Preliminaries

This section is a quick review of notions of Finsler spaces with left-invariant metrics. For basic notions of Finsler geometry we refer to the monograph [1].

Left invariant Finsler metrics.

Definition 1. Let V be a finite dimensional vector space with the canonical differentiable structure. The function $f \colon V \to [0, \infty)$ is called a *Minkowski-functional* if it is smooth on $V \setminus \{0\}$, positively 1-homogeneous and for all $W \in V \setminus \{0\}$ the symmetric bilinear form

$$\langle , \rangle_W \colon V \times V \to \mathbf{R},$$
$$(X, Y) \mapsto \langle X, Y \rangle_W = \frac{1}{2} \frac{\partial^2 f^2(W + rX + sY)}{\partial r \partial s}\bigg|_{r=s=0} \tag{2}$$

is positive definite.

Definition 2. Let $\mathcal{N}$ be a Lie algebra and N is the simply connected Lie group with Lie algebra $\mathcal{N}$. The Finslerian fundamental function $F\colon TN \to [0,\infty)$ is called *left-invariant* when

$$\forall a \in N,\ \forall X \in T_eN:\ F((dL_a)_eX) = F(X), \tag{3}$$

where

$$L_a\colon N \to N,\ x \mapsto L_a(x) = a \cdot x$$

is the left translation and e is the unit element of the group.

By left translations, for every Minkowski functional on $\mathcal{N}$ we can define a left invariant Finsler metric on N:

$$F((dL_a)(X_e)) = f(X_e), \quad (X_e \in T_eN,\ a \in N).$$

Our starting point in this paper is a two-step nilpotent Lie algebra $\mathcal{N}$ with a Minkowski functional $f\colon \mathcal{N} \to [0,\infty)$ which generates a Finsler space with left invariant Finsler metric (N,F) as written above.

Osculating objects. In the sequel we fix a nowhere vanishing vector field, the so called reference vector field. Generally such a vector field does not exist globally and we arrange that all objects live on an open subset $\mathcal{U} \subset N$, where the reference vector field exists. However, in our notations we do not distinguish between $\mathcal{U}$ and N.

Definition 3. Let W be a nowhere vanishing vector field on N. We refer to W as *reference vector field*. The *osculating Riemann metric* $\langle , \rangle_W$ is determined by the Finslerian fundamental function F and by the reference vector field W in the following way:

$$\langle X_p, Y_p\rangle_W = \frac{1}{2}\left.\frac{\partial^2 F^2(W_p + sX_p + tY_p)}{\partial s\partial t}\right|_{s,t=0}, \quad p \in N,\ X,Y \in \mathfrak{X}(N). \tag{4}$$

Moreover for $X,Y,Z \in \mathfrak{X}(N)$,

$$\mathcal{C}_\flat(X_p,Y_p,Z_p)_W = \frac{1}{4}\left.\frac{\partial^3}{\partial r\partial s\partial t}\right|_{r,s,t=0} F^2\left(W_p + rX_p + sY_p + tZ_p\right)$$

is the *(osculating) Cartan tensor*, its $(1,2)$-type version is defined by

$$\mathcal{C}_W\colon \mathfrak{X}(\mathcal{N}) \times \mathfrak{X}(\mathcal{N}) \to \mathfrak{X}(\mathcal{N}),\ \langle \mathcal{C}(X,Y)_W, Z\rangle_W = \mathcal{C}_\flat(X,Y,Z)_W.$$

For the Cartan tensor we have

$$\mathcal{C}_\flat(W,X,Y)_W = \mathcal{C}_\flat(X,W,Y)_W = \mathcal{C}_\flat(X,Y,W)_W = 0. \tag{5}$$

It is easy to see that if F is a left-invariant Finsler fundamental function, moreover the reference vector field is left invariant also, then the osculating Riemann metric and the Cartan tensor are left invariant objects. Thus we may regard the osculating Riemannian metric as a positive definite scalar product, the Cartan tensor as a trilinear form (its $(1,2)$ version as a bilinear vector-valued form) on $\mathcal{N}$.

The osculating scalar product $\langle , \rangle_W$ determines the orthogonal complement of the center which is denoted simply by $\mathcal{V}$ (without any indication of the reference vector field) and the skew-symmetric linear transformation

$$j_W\colon \mathcal{V} \to \mathcal{V}, \quad j_W(Z)X = (\operatorname{ad} X)^* Z, \tag{6}$$

where $*$ refers to $\langle , \rangle_W$, see (1).

Definition 4. The left invariant Finsler metric is *compatible* with $\mathcal{N}$, if $\forall X, Y \in \mathcal{N}$, $\forall Z \in \mathcal{Z}$:

$$\mathcal{C}_W(X,Y)^{\mathcal{Z}} = 0, \ \mathcal{C}_W(Z,Y)^{\mathcal{V}} = 0, \tag{7}$$

where $\mathcal{C}_W(,) = \mathcal{C}_W(,)^{\mathcal{Z}} + \mathcal{C}_W(,)^{\mathcal{V}}$ is the orthogonal decomposition of the Cartan tensor.

We note that if $W \in \mathcal{Z}$ then by (5) the condition (7) is true for a Lie algebra with one-dimensional center.

Lastly, we introduce the following notation: $\mathcal{D}_W\colon \mathcal{V} \times \mathcal{V} \to \mathcal{V}$,

$$\begin{aligned} \mathcal{D}_W(X,Y) = {} & \mathcal{C}_W(j_W(W)X, Y) + \mathcal{C}_W(j_W(W)Y, X) \\ & + j_W(W)\mathcal{C}_W(X,Y)^{\mathcal{V}}. \end{aligned} \tag{8}$$

3. The Chern-Rund connection

Definition 5 ([8])**.** The Chern-Rund connection

$$\nabla^W\colon \mathfrak{X}(N) \times \mathfrak{X}(N) \to \mathfrak{X}(N)$$

(with respect to the reference vector field W) is defined by

$$\begin{aligned} 2\left\langle \nabla^W_X Y, Z \right\rangle_W = {} & X\langle Y,Z\rangle_W + Y\langle Z,X\rangle_W - Z\langle X,Y\rangle_W + \\ & + \langle [X,Y],Z\rangle_W - \langle [Y,Z],X\rangle_W + \langle [Z,X],Y\rangle_W - \\ & - 2\mathcal{C}_\flat(\nabla^W_X W, Y, Z)_W - 2\mathcal{C}_\flat(\nabla^W_Y W, Z, X)_W + \\ & + 2\mathcal{C}_\flat(\nabla^W_Z W, X, Y)_W. \end{aligned} \tag{9}$$

The Chern-Rund connection is torsion-free, that is,

$$\nabla^W_X Y - \nabla^W_Y X - [X,Y] = 0, \tag{10}$$

and almost metric, that is,

$$X\langle Y,Z\rangle_W = \left\langle \nabla^W_X Y, Z\right\rangle_W + \left\langle Y, \nabla^W_X Z\right\rangle_W + 2\mathcal{C}_\flat(\nabla^W_X W, Y, Z)_W. \tag{11}$$

For left invariant vector fields X, Y, Z, W, the first three terms of the right hand side of (9) vanish:

$$\begin{aligned} 2\left\langle\nabla^W_X Y, Z\right\rangle_W = &\langle [X,Y], Z\rangle_W - \langle [Y,Z], X\rangle_W + \langle [Z,X], Y\rangle_W - \\ &- 2\mathcal{C}_\flat(\nabla^W_X W, Y, Z)_W - 2\mathcal{C}_\flat(\nabla^W_Y W, Z, X)_W + \\ &+ 2\mathcal{C}_\flat(\nabla^W_Z W, X, Y)_W. \end{aligned} \tag{12}$$

Proposition 6. *If the reference vector field W is from the center of the two-step nilpotent Lie algebra $\mathcal{N}$, then $\forall X, Y, Z \in \mathcal{N}$ we have*

$$\begin{aligned} 2\nabla^W_X Y = &[X,Y] - (\operatorname{ad} Y)^* X - (\operatorname{ad} X)^* Y \\ &+ \mathcal{C}_W((\operatorname{ad} X)^* W, Y) + \mathcal{C}_W((\operatorname{ad} Y)^* W, X) \\ &+ (\operatorname{ad} \mathcal{C}_W(X,Y))^* W, \end{aligned} \tag{13}$$

Proof. At first, we substitute $X = Y = W \in \mathcal{Z}$ into (12). Considering (5) we get $\nabla^W_W W = 0$.

Now, for any $Y = W \in \mathcal{Z}$ and $Z \in \mathcal{N}$,

$$2\left\langle\nabla^W_X W, Z\right\rangle_W = \langle [Z,X], W\rangle_W = -\left\langle Z, (\operatorname{ad} X)^* W\right\rangle_W, \tag{14}$$

i.e. $\nabla^W_X W = -\frac{1}{2}(\operatorname{ad} X)^* W$. Thus the right hand side of (12) can be expressed without any explicit reference to the Chern-Rund connection:

$$\begin{aligned} 2\left\langle\nabla^W_X Y, Z\right\rangle_W = &\langle [X,Y], Z\rangle_W - \langle Z, (\operatorname{ad} Y)^* X\rangle_W - \langle Z, (\operatorname{ad} X)^* Y\rangle_W \\ &- \langle \mathcal{C}_W((\operatorname{ad} X)^* W, Y), Z\rangle_W + \langle \mathcal{C}_W((\operatorname{ad} Y)^* W, X), Z\rangle_W \\ &- \langle \mathcal{C}_W(X,Y), (\operatorname{ad} Z)^* W\rangle_W. \end{aligned}$$

Here the last term:

$$\begin{aligned} -\left\langle \mathcal{C}_W(X,Y), (\operatorname{ad} Z)^* W\right\rangle_W &= -\left\langle [Z, \mathcal{C}_W(X,Y)], W\right\rangle_W \\ &= \left\langle Z, (\operatorname{ad} \mathcal{C}_W(X,Y))^* W\right\rangle_W, \end{aligned}$$

which gives the result. □

As a consequence, we have the following theorem and corollaries.

Theorem 7. *If the reference vector field W is from the center of the two-step nilpotent Lie algebra $\mathcal{N}$ and $X, Y \in \mathcal{N}$ then $\nabla^W_X Y \in \mathcal{N}$.*

Proposition 8.

(a) $\nabla^W_X Y = \frac{1}{2}[X,Y] + \frac{1}{2}\mathcal{D}_W(X,Y)$ *for all $X, Y \in \mathcal{V}$;*

(b) $\nabla^W_X Z = -\frac{1}{2} j_W(Z) X + \frac{1}{2}\mathcal{C}_W(j_W(W)X, Z) + \frac{1}{2} j_W(W)\mathcal{C}_W(X,Z)^{\mathcal{V}}$, *and* $\nabla^W_X Z = \nabla^W_Z X$ *for all $X \in \mathcal{V}, Z \in \mathcal{Z}$;*

(c) $\nabla^W_Z Z^* = \frac{1}{2} j_W(W)\mathcal{C}_W(Z, Z^*)^{\mathcal{V}}$ *for all $Z, Z^* \in \mathcal{Z}$.*

Proof. If $X, Y \in \mathcal{V}$ and $Z \in \mathcal{N}$ then

$$\langle (\operatorname{ad} Y)^* X, Z \rangle_W = \langle X, (\operatorname{ad} Y) Z \rangle_W = \langle X, [Y, Z] \rangle_W = 0,$$

which implies $(\operatorname{ad} Y)^* X = 0$. Moreover, if $X \in \mathcal{V}$ then $(\operatorname{ad} W)^* X = j_W(W) X$. This observation with (8) gives (a).

By the identity $(\operatorname{ad} X)^* Z = 0$ ($X \in \mathcal{Z}$) we see that the second and the fifth terms of the right hand side of (13) vanish. The remaining terms give statement (b).

The proof of (c) follows from the same fact as the proof of (b). $\square$

For the compatibility case, one routinely obtains from the above proposition the following corollary.

Corollary 9. *If the left invariant Finsler metric is compatible with the Lie algebra, then for the Chern-Rund connection we have*

(A) $\nabla^W_X Y = \frac{1}{2}[X, Y] + \frac{1}{2}\mathcal{D}_W(X, Y)$ *for all* $X, Y \in \mathcal{V}$,

(B) $\nabla^W_X Z = \nabla^W_Z X = -\frac{1}{2} j_W(Z) X$ *for all* $X \in \mathcal{V}, Z \in \mathcal{Z}$,

(C) $\nabla^W_Z Z^* = 0$ *for all* $Z, Z^* \in \mathcal{Z}$.

4. Curvature

Recall that if X, Y, Z are vector fields on N then the curvature tensor is given by

$$R(X, Y) Z = \nabla^W_X \nabla^W_Y Z - \nabla^W_Y \nabla^W_X Z - \nabla^W_{[X,Y]} Z.$$

If X, Y, Z are left invariant vector fields, then $R(X, Y) Z$ is also left invariant, and we may regard R as a trilinear map $\mathcal{N} \times \mathcal{N} \times \mathcal{N} \to \mathcal{N}$.

Proposition 10. *If the left invariant Finsler metric is compatible with the Lie algebra, then for the curvature of the Chern-Rund connection we have*

(a)

$$R(Z_1, Z_2) Z_3 = 0$$

for all $Z_1, Z_2, Z_3 \in \mathcal{Z}$.

(b)

$$R(X, Z_1) Z_2 = -\frac{1}{4} j_W(Z_1)(j_W(Z_2) X),$$
$$R(Z_1, Z_2) X = \frac{1}{4} j_W(Z_2)(j_W(Z_1)(X) - \frac{1}{4} j_W(Z_2)(j_W(Z_1) X)$$

for all $X \in \mathcal{V}$, *all* $Z_1, Z_2 \in \mathcal{Z}$.

(c)

$$\begin{aligned}R(X,Y)Z &= -\frac{1}{4}[X, j_W(Z)Y] + \frac{1}{4}[Y, j_W(Z)X]-\\ &\quad -\frac{1}{4}\mathcal{D}_W(X, j_W(Z)Y) + \frac{1}{4}\mathcal{D}_W(Y, j(Z)X),\\ R(X,Z)Y &= -\frac{1}{4}[X, j_W(Z)Y] - \frac{1}{4}\mathcal{D}_W(X, j_W(Z)Y)\\ &\quad + \frac{1}{4}j_W(Z)(\mathcal{D}_W(X,Y))\end{aligned}$$

for all $X, Y \in \mathcal{V}$, all $Z \in \mathcal{Z}$.

(d)

$$\begin{aligned}R(X,Y)X^* = &-\frac{1}{4}j_W([Y,X^*])X + \frac{1}{4}j_W([X,X^*])Y + \frac{1}{2}j_W([X,Y])X^*+\\ &+\frac{1}{4}[X, \mathcal{D}_W(Y,X^*)] + \frac{1}{4}\mathcal{D}_W(X, \mathcal{D}_W(Y,X^*))-\\ &-\frac{1}{4}[Y, \mathcal{D}_W(X,X^*)] - \frac{1}{4}\mathcal{D}_W(Y, \mathcal{D}_W(X,X^*))\end{aligned}$$

for all $X, Y, X^* \in \mathcal{V}$.

Proof. Assertion (a) follows directly from part (C) of Corollary 9.

By properties (C) and (B) of Corollary 9 it follows that

$$\begin{aligned}R(X,Z_1)Z_2 &= -\nabla^W_{Z_1}\nabla^W_X Z_2 = \frac{1}{2}\nabla^W_{Z_1}(j_W(Z_2)X)\\ &= -\frac{1}{4}j_W(Z_1)(j_W(Z_2)X)\end{aligned}$$

and

$$\begin{aligned}R(Z_1,Z_2)X &= \nabla^W_{Z_1}\nabla^W_{Z_2}X - \nabla^W_{Z_2}\nabla^W_{Z_1}X\\ &= -\frac{1}{2}\nabla^W_{Z_1}(j_W(Z_2)X) + \frac{1}{2}\nabla^W_{Z_2}(j_W(Z_1)X)\\ &= \frac{1}{4}j_W(Z_1)(j_W(Z_2)X) - \frac{1}{4}j_W(Z_2)(j_W(Z_1)X)\end{aligned}$$

which gives part (b) of the proposition.

Assertion (c) follows routinely from (C), (B) and (A):

$$\begin{aligned}R(X,Y)Z &= \nabla^W_X\nabla^W_Y Z - \nabla^W_Y\nabla^W_X Z\\ &= -\frac{1}{2}\nabla^W_X(j_W(Z)Y) + \frac{1}{2}\nabla^W_Y(j_W(Z)X)\\ &= -\frac{1}{4}[X, j_W(Z)Y] - \frac{1}{4}\mathcal{D}_W(X, j_W(Z)Y)\\ &\quad + \frac{1}{4}[Y, j_W(Z)X] + \frac{1}{4}\mathcal{D}_W(Y, j_W(Z)X).\end{aligned}$$

Similarly,

$$\begin{aligned}
R(X,Z)Y &= \nabla^W_X \nabla^W_Z Y - \nabla^W_Z \nabla^W_X Y \\
&= -\frac{1}{2}\nabla^W_X (j_W(Z)Y) - \frac{1}{2}\nabla^W_Z \left([X,Y] + \mathcal{D}_W(X,Y)\right) \\
&= -\frac{1}{2}\nabla^W_X (j_W(Z)Y) - \frac{1}{2}\underbrace{\nabla^W_Z [X,Y]}_{=0} - \frac{1}{2}\nabla^W_Z \mathcal{D}_W(X,Y) \\
&= \frac{1}{4}\big(- [X, j_W(Z)Y] - \mathcal{D}_W(X, j_W(Z)Y) \\
&\quad + j_W(Z)\mathcal{D}_W(X,Y)\big).
\end{aligned}$$

Finally,

$$\begin{aligned}
R(X,Y)X^* &= \nabla^W_X \nabla^W_Y X^* - \nabla^W_Y \nabla^W_X X^* - \nabla^W_{[X,Y]} X^* \\
&= \frac{1}{2}\nabla^W_X \left([Y,X^*] + \mathcal{D}_W(Y,X^*)\right) \\
&\quad - \frac{1}{2}\nabla^W_Y \left([X,X^*] + \mathcal{D}_W(X,X^*)\right) + \frac{1}{2} j_W([X,Y])X^*,
\end{aligned}$$

and applying Corollary 9 once more, we get statement (d). $\square$

5. Ricci tensor

For $X, Y \in \mathcal{N}$ the Ricci tensor of (N, F) is given by

$$\mathrm{Ric}(X,Y) = \mathrm{trace}\{\xi \mapsto R(\xi, X)Y \mid \xi \in \mathcal{N}\}.$$

Moreover, let

$$\begin{aligned}
d_1(X,Z) &= \mathrm{trace}\{\xi \mapsto \mathcal{D}_W(X, j_W(Z)\xi) \mid \xi \in \mathcal{V}\} \\
d_2(X,Z) &= \mathrm{trace}\{\xi \mapsto \mathcal{D}_W(\xi, j_W(Z)X) \mid \xi \in \mathcal{V}\}, \quad X \in \mathcal{V}, Z \in \mathcal{Z}.
\end{aligned}$$

Theorem 11. *Let the left invariant Finsler metric be compatible with $\mathcal{N}$.*

(a)

$$4\,\mathrm{Ric}(X,Z) = d_2(X,Z) - d_1(X,Z)$$

for all $X \in \mathcal{V}$, $Z \in \mathcal{Z}$. In particular,

$$\mathrm{Ric}(X,Z) = 0 \textit{ for all } X \in \mathcal{V}, Z \in \mathcal{E} = \{Z \in \mathcal{Z} \mid j_W(Z) = 0\}.$$

(b)

$$\mathrm{Ric}(Z,Z^*) = -\frac{1}{4}\,\mathrm{trace}\left\{j_W(Z) \circ j_W(Z^*)\right\}$$

for all $Z, Z^* \in \mathcal{Z}$.

Remark. Note, in the Riemannian case $\mathrm{Ric}(X,Z) = 0$ for all $X \in \mathcal{V}$, $Z \in \mathcal{Z}$.

Proof. (a) Let $(V_1,\dots,V_n)$ and $(Z_1,\dots,Z_m)$ be orthonormal bases of $\mathcal{V}$ and $\mathcal{Z}$ respectively. By definition,

$$\mathrm{Ric}(X,Z)=\sum_{i=1}^{n}\langle R(V_i,X)Z,V_i\rangle_W+\sum_{\alpha=1}^{m}\langle R(Z_\alpha,X)Z,Z_\alpha\rangle_W$$

$R(Z_\alpha,X)Z\in\mathcal{V}$ (see part (b) of Proposition 10) thus the second term is zero. Concerning the first term, we apply part (c) of Proposition 10:

$$\begin{aligned}&=-\sum_{i=1}^{n}\frac{1}{4}\langle\mathcal{D}_W(V_i,j_W(Z)X),V_i\rangle_W\\&\quad+\sum_{i=1}^{n}\frac{1}{4}\langle\mathcal{D}_W(X,j_W(Z)V_i),V_i\rangle_W\,.\end{aligned}$$

(b) In a similar way, applying parts (a) and (b) of Proposition 10 we get

$$\begin{aligned}\mathrm{Ric}(Z,Z^*)&=\sum_{i=1}^{n}\langle R(V_i,Z)Z^*),V_i\rangle_W+\sum_{\alpha=1}^{m}\langle\underbrace{R(Z_\alpha,Z)Z^*}_{0},Z_\alpha\rangle_W\\&=-\frac{1}{4}\langle j_W(Z)(j_W(Z^*))V_i,V_i\rangle_W\\&=-\frac{1}{4}\,\mathrm{trace}\,\{j_W(Z)\circ j_W(Z^*)\}\,.\end{aligned}$$

□

Corollary 12. *For the Heisenberg algebra*

$$\mathrm{Ric}(Z,Z)=\frac{n}{4}\|Z\|_W^2,\quad Z\in\mathcal{Z}\tag{15}$$

where $n=\dim\mathcal{V}$.

Proof. By (b), if $Z=Z^*$ then

$$\mathrm{Ric}(Z,Z)=-\frac{1}{4}\,\mathrm{trace}\{j_W(Z)^2\}.$$

For the Heisenberg algebra $j_W(Z)^2=-\|Z\|_W^2\,\mathrm{id}$ on $\mathcal{V}$ which gives the statement. □

6. Flag curvature

The flag curvature in Finsler geometry is a natural extension of the sectional curvature in Riemannian geometry. In general, the flag curvature $K(\Pi,V)$ depends not only on the 2-plane Π, but also the *pole* vector $V\in\Pi$.

Definition 13. Let $\Pi\subset T_xN$, $\Pi=\mathrm{span}(Y,V)$ where (Y,V) is an orthonormal pair with respect to $\langle,\rangle_V$. The *flag curvature* of (Π,V) is defined by the formula

$$K(\Pi,V)=\langle R(Y,V)V,Y\rangle_V\,.$$

In our approach the pole vector coincides with the reference vector W, but contrary to the previous sections we do not suppose that $W \in \mathcal{Z}$, we suppose only the condition that the reference vector is a geodesic vector, i.e. $\nabla^W_W W = 0$.

Remark. A $W \in \mathcal{N}$ non-zero vector is a geodesic vector if and only if

$$\forall Z \in N: \quad \langle [Z, W], W \rangle_W = 0, \tag{16}$$

in particular, a non-zero vector of the center is automatically a geodesic vector. Indeed, from (12) we have

$$2 \left\langle \nabla^W_W W, Z \right\rangle_W = -\langle [W, Z], W \rangle_W + \langle [Z, W], W \rangle_W = 2 \langle [Z, W], W \rangle_W .$$

Theorem 14. *Let $\mathcal{N}$ be a two-step nilpotent Lie algebra and N is the simply connected two-step nilpotent Lie group with Lie algebra $\mathcal{N}$. Let W be a unit-length geodesic vector of the Finsler manifold N with left invariant Finsler metric.*

a) *If $Z \in \mathcal{Z}$, $\Pi = \operatorname{span}(Z, W)$ and (Z, W) is an orthonormal pair with respect to $\langle , \rangle_W$ then*

$$K(\Pi, W) = \frac{1}{4} \|(\operatorname{ad} W)^* Z\|^2_W.$$

In particular

$$K(\Pi, W) = 0, \ \textit{if}\ W \in \mathcal{Z}.$$

b) *If $V \in \mathcal{V}$, $\Pi = \operatorname{span}(V, W)$ and (V, W) is an orthonormal pair with respect to $\langle , \rangle_W$ then*

$$K(\Pi, W) = -\frac{3}{4} \|[V, W]\|^2_W + \frac{1}{4} \|(\operatorname{ad} V)^* W\|^2_W.$$

In particular

$$K(\Pi, W) = \frac{1}{4} \|j_W(W) V\|^2_W \ \textit{if}\ W \in \mathcal{Z}.$$

Proof. If W is a geodesic vector, i.e. $\nabla^W_W W = 0$ then for the Chern-Rund connection one obtains

$$2 \left\langle \nabla^W_X W, Z \right\rangle_W = \langle [X, W], Z \rangle_W - \langle [W, Z], X \rangle_W + \langle [Z, X], W \rangle_W ,$$

see (12). Thus

$$\nabla^W_X W = \frac{1}{2} \left\{ [X, W] - (\operatorname{ad} W)^* X - (\operatorname{ad} X)^* W \right\},$$

where the adjoint operator $*$ refers to the scalar product $\langle , \rangle_W$. The Chern-Rund connection is torsion-free, thus

$$\nabla^W_W X = \frac{1}{2} \left\{ [W, X] - (\operatorname{ad} W)^* X - (\operatorname{ad} X)^* W \right\}.$$

In particular

$$\nabla^W_Z W = -\frac{1}{2} (\operatorname{ad} W)^* Z = \nabla^W_W Z \text{ if } Z \in \mathcal{Z}, \tag{17}$$

$$\nabla_V^W W = \frac{1}{2}\left\{[V,W] - (\operatorname{ad} V)^* W\right\} \text{ if } V \in \mathcal{V}, \tag{18}$$

$$\nabla_W^W V = \frac{1}{2}\left\{[W,V] - (\operatorname{ad} V)^* W\right\} \text{ if } V \in \mathcal{V}. \tag{19}$$

Let $Z \in \mathcal{Z}$.

$$\begin{aligned} R(Z,W)W &= \nabla_Z^W \nabla_W^W W - \nabla_W^W \nabla_Z^W W - \nabla_{[Z,W]}^W W \\ &= -\nabla_W^W \nabla_Z^W W = \frac{1}{2}\nabla_W^W (\operatorname{ad} W)^* Z. \end{aligned}$$

$(\operatorname{ad} W)^* Z \in \mathcal{V}$, thus applying (19) we immediately obtain

$$R(Z,W)W = \frac{1}{4}\left\{[W,(\operatorname{ad} W)^* Z] - [\operatorname{ad}((\operatorname{ad} W)^* Z)]^* W\right\}$$

and

$$\begin{aligned} \langle R(Z,W)W, Z\rangle_W &= \frac{1}{4}\left\{\langle (\operatorname{ad} W)^* Z, (\operatorname{ad} W)^* Z\rangle_W - \langle W, [(\operatorname{ad} W)^* Z, Z]\rangle_W\right\} \\ &= \frac{1}{4}\|(\operatorname{ad} W)^* Z\|_W^2. \end{aligned}$$

Let $V \in \mathcal{V}$. From (17)–(19) we obtain

$$\begin{aligned} R(V,W)W &= \nabla_V^W \nabla_W^W W - \nabla_W^W \nabla_V^W W - \nabla_{[V,W]}^W W \\ &= -\frac{1}{2}\nabla_W^W \left([V,W] - (\operatorname{ad} V)^* W\right) + \frac{1}{2}(\operatorname{ad} W)^*[V,W] \\ &= \frac{1}{4}(\operatorname{ad} W)^*[V,W] + \frac{1}{4}[W,(\operatorname{ad} V)^* W] \\ &\quad - \frac{1}{4}[\operatorname{ad}((\operatorname{ad} V)^* W)]^* W + \frac{1}{2}(\operatorname{ad} W)^*[V,W]. \end{aligned}$$

Observe that

$$\begin{aligned} \langle R(V,W)W, V\rangle_W &= -\frac{3}{4}\langle [V,W],[V,W]\rangle_W + \frac{1}{4}\langle [W,(\operatorname{ad} V)^* W], V\rangle_W \\ &\quad - \frac{1}{4}\langle W, [(\operatorname{ad} V)^* W, V]\rangle_W \\ &= -\frac{3}{4}\|[V,W]\|_W^2 + \frac{1}{4}\langle (\operatorname{ad} V)^* W, (\operatorname{ad} V)^* W\rangle_W \\ &= -\frac{3}{4}\|[V,W]\|_W^2 + \frac{1}{4}\|(\operatorname{ad} V)^* W\|_W^2. \end{aligned}$$

□

7. Relative geodesics

Definition 15. The curve $\gamma\colon \mathcal{I} \to N$ is called a *relative geodesic* of the Chern-Rund connection when $\nabla_{\gamma'}^W \gamma' = 0$ (see e.g. [7]).

To describe relative geodesics of (N, F) it suffices to describe those geodesics that begin at the identity of N. Let γ be a curve with $\gamma(0) = e$ (identity of N), and let $\gamma'(0) = X_0 + Z_0 \in \mathcal{N}$, where $X_0 \in \mathcal{V}$, $Z_0 \in \mathcal{Z}$. In exponential coordinates we write

$$\gamma(t) = \exp(X(t) + Z(t)),\ X(t) \in \mathcal{V},\ Z(t) \in \mathcal{Z},$$

and

$$X'(0) = X_0,\ Z'(0) = Z_0.$$

It is easy to see that

$$\gamma'(t) = dL_{\gamma(t)}\left(X'(t) + Z'(t) + \frac{1}{2}[X'(t), X(t)]\right) \tag{20}$$

Let $S = X' + Z' + \frac{1}{2}[X', X]$.

Theorem 16. *Suppose that the left invariant Finsler metric is compatible with the 2-step nilpotent Lie algebra and $W \in \mathcal{Z}$. Then the curve γ is a relative geodesic of the Chern-Rund connection if and only if the following equations are satisfied:*

$$Z' + \frac{1}{2}[X', X] = Z_0,\ X'' - j_W(Z_0)X' + \frac{1}{2}\mathcal{D}_W(X', X') = 0.$$

Proof. Using the left-invariant property of the Chern-Rund connection and (20) we get

$$\nabla^W_{\gamma'}\gamma' = \nabla^W_S S = \nabla^W_S X' + \nabla^W_S (Z' + \frac{1}{2}[X', X]). \tag{21}$$

Let $(V_1, \dots, V_n)$ and $(Z_1, \dots, Z_m)$ be orthonormal bases of $\mathcal{V}$ and $\mathcal{Z}$ respectively and $X = v^i V_i,\ Z = u^\alpha Z_\alpha$. Moreover, the structure constants A^α_{ij} for the Lie algebra are given as follows:

$$[V_i, V_j] = A^\alpha_{ij} Z_\alpha.$$

At first we determine the first term of the right hand side of (21).

$$\begin{aligned}
\nabla^W_S X' &= \nabla^W_S (\dot{v}^i V_i) = \ddot{v}^i V_i + \dot{v}^i \nabla^W_S V_i \\
&= X'' + \dot{v}^i \nabla^W_{X'} V_i + \dot{v}^i \nabla^W_{Z' + \frac{1}{2}[X', X]} V_i \\
&= X'' + \frac{1}{2}\dot{v}^i [X', V_i] + \frac{1}{2}\dot{v}^i \mathcal{D}_W(X', V_i) \\
&\quad - \frac{1}{2}\dot{v}^i j_W(Z' + \frac{1}{2}[X', X]) V_i \\
&= X'' + \frac{1}{2}\mathcal{D}_W(X', X') - \frac{1}{2} j_W(Z' + \frac{1}{2}[X', X]) X'.
\end{aligned}$$

Now, calculate the second term of the right hand side of (21).

$$\begin{aligned}\nabla_S^W\left(Z' + \frac{1}{2}[X', X]\right) &= \nabla_S^W(\dot{u}^\alpha Z_\alpha) + \frac{1}{2}\nabla_S^W\left(\dot{v}^i v^j A_{ij}^\alpha Z_\alpha\right)\\ &= \ddot{u}^\alpha Z_\alpha + \dot{u}^\alpha \nabla_S^W Z_\alpha\\ &\quad + \frac{1}{2}\left(\ddot{v}^i v^j + \dot{v}^i \dot{v}^j\right) A_{ij}^\alpha Z_\alpha + \frac{1}{2}\dot{v}^i v^j A_{ij}^\alpha \nabla_S^W Z_\alpha\\ &= Z'' + \frac{1}{2}[X'', X] + \left(\dot{u}^\alpha + \frac{1}{2}\dot{v}^i v^j A_{ij}^\alpha\right)\nabla_{X'}^W Z_\alpha\\ &= Z'' + \frac{1}{2}[X'', X] - \frac{1}{2} j_W\left(Z' + \frac{1}{2}[X', X]\right)X'.\end{aligned}$$

Therefore

$$\begin{aligned}\nabla_{\gamma'}^W \gamma' = X'' + \frac{1}{2}\mathcal{D}_W(X', X') - j_W\left(Z' + \frac{1}{2}[X', X]\right)X'\\ + Z'' + \frac{1}{2}[X'', X].\end{aligned}$$

Relative geodesics are then characterized by the system

$$\begin{aligned}X'' + \frac{1}{2}\mathcal{D}_W(X', X') - j_W\left(Z' + \frac{1}{2}[X', X]\right)X' = 0\\ Z'' + \frac{1}{2}[X'', X] = 0.\end{aligned} \tag{22}$$

The second equation implies

$$Z' + \frac{1}{2}[X', X] = Z_0$$

and the first equation of (22) reduces to

$$X'' + \frac{1}{2}\mathcal{D}_W(X', X') - j_W(Z_0)X' = 0.$$

□

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