Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
24 (2008), 191–200
www.emis.de/journals
ISSN 1786-0091

THE SINGULAR DIRECTIONS OF SOLUTIONS OF SOME EQUATIONS

ZHAO-JUN WU AND DAO-CHUN SUN

Abstract. In this paper, we investigate the location of zeros and Borel direction of the solutions of the linear differential equation

$$f^{(n)} + A_{n-2}(z)f^{(n-2)} + \cdots + A_1(z)f' + A_0(z)f = 0,$$

where $A_0(z), \dots, A_{n-2}(z)$ are meromorphic functions. Results are obtained concerning the rays near which the exponent of convergence of zeros of the solutions attains its Borel direction, which improve some results given by S.J. Wu and other authors.

1. Introduction and statement of results

In this paper, by meromorphic functions we shall always mean meromorphic functions in complex plane $\mathbb{C}$, we shall assume that the reader is familiar with the standard notation of Nevanlinna theory and complex differential equation (see [7] or [10]). On the angular distribution of meromorphic function, 1919 Julia [9] introduced the concept of Julia direction and showed that every transcendental entire function has at least one Julia direction that is a refinement of Picard's theorem. In order to have a similar refinement for Borel's theorem, a more refined notion of Borel direction was introduced by Valiron [12] in 1928.

Suppose that $g(z)$ is a $\rho(0 < \rho \leq \infty)$ order meromorphic function. A ray $\arg z = \theta$ is called a Borel direction of order ρ for f if for every $0 < \varepsilon < \pi$,

$$\limsup_{r\to\infty} \frac{\log n(r, \theta, \varepsilon, a)}{\log r} = \rho,$$

holds for all a in $\mathbb{C}_\infty$ with at most two exceptions, where $n(r, \theta, \varepsilon, a)$ is the number of zeros of $f(z) - a$ in $\{z : \theta - \varepsilon < argz < \theta + \varepsilon\} \cap \{0 < |z| < r\}$,

2000 *Mathematics Subject Classification.* 30D35, 34A20.

Key words and phrases. Borel direction, the exponent of convergence of zeros, proximate order.

This work is supported by the NNSF of China (No: 10471048) and NSF of Xianning University (No: KZ0629, KY0718).

counting with multiplicities(see [13]). It's known that every $\rho(\rho > 0)$ order meromorphic function has at least one Borel direction (see [15]).

In this paper, we consider the connection of the the location of zeros and Borel direction of solutions of the linear differential equation

$$f^{(n)} + A_{n-2}(z)f^{(n-2)} + \cdots + A_1(z)f' + A_0(z)f = 0, \tag{1}$$

where $A_0(z), \ldots, A_{n-2}(z)$ are meromorphic functions of finite order. When every $A_j(z)$ is a polynomial, Zheng [17] proved the following

Theorem 1. *Let $f(z)$ be a transcendental solution of (1) have Stokes' rays $\arg z = \theta_j (j = 1, 2, \cdots, m)$ of order ρ. Then the number of zeros of $f(z)$ in $|z| \le r$, but outside the logarithmic strips $|\arg z - \theta_j| < A\frac{\log|z|}{|z|^{1/p}}$ for $\theta = \theta_1, \cdots, \theta_m$ (A a sufficiently large constant) is $O(r^{\rho-\varepsilon})$ for some $\varepsilon > 0$.*

A rays is called a Stokes ray of $f(z)$ of order ρ if and only if for arbitrary angular Ω contains the ray we have $n(r, \Omega, f = 0) = cr^\rho(1 + o(1)), c > 0$ (see e.g. [18]). Recently, Zheng [18] indicate that if $f(z)$ is a solution of (1) and has the exponent λ of convergence of zeros, then a ray is a Borel direction of $f(z)$ if and only if it is a Stokes ray of order λ with respect zeros. And above all, he also indicate that let $\{f_1, \cdots, f_n\}$ be a fundamental system of meromorphic solutions of (1), if $E = f_1 \cdots f_n$ is not a rational function, then its Borel directions are exactly stokes' rays of order λ with respect to its zeros, that is, its Borel directions are completely determined by the argument distribution of its zeros.

When there is at least one transcendental coefficient in (1), we pose the following question,

Question 1. Suppose that there is at least one transcendental meromorphic coefficients in (1), we ask whether a ray is a infinity order Borel direction of E if the exponent of convergence of zeros of E in any angle containing the ray is infinite.

For the case $n = 2$, S.J. Wu [14] have confirmed the Question 1 in the case of entire coefficients. The present authors have confirmed the Question 1 in the case of entire coefficients and $n \ge 2$ in [15, 16]. In order to state their results, we need the following definitions (see e.g. [13]). Let $f(z)$ be a meromorphic function in the plane and let $\arg z = \theta \in \mathbb{R}$ be a ray, we denote, for each $\varepsilon > 0$, the exponent of convergence of zero-sequence of $f(z)$ in the angular region $\{z : \theta - \varepsilon < \arg z < \theta + \varepsilon, |z| > 0\}$ by $\lambda_{\theta,\varepsilon}(f)$ and by $\lambda_\theta(f) = \lim_{\varepsilon\to 0} \lambda_{\theta,\varepsilon}(f)$. In [14] S.J. Wu proved the following result.

Theorem 2. *Let $A(z)$ be a transcendental entire function of finite order in the plane and let f_1, f_2 be two linearly independent solutions of $f'' + A(z)f = 0$. Set $E = f_1 f_2$, then $\lambda_\theta(E) = \infty$, if and only if $\arg z = \theta$ is an infinity order Borel direction of E.*

In the following, we shall prove the Question 1 is true for $n \geq 2$. In order to state our results, we need give some definitions yet.

Definition 1. Let $f(z)$ be a meromorphic function of infinite order. A real function $\rho(r)$ is called a proximate order of $f(z)$, if $\rho(r)$ has the following properties:

1) $\rho(r)$is continuous and nondecreasing for $r \geq r_0 > 0$ and tends to $+\infty$ as $r \to \infty$.
2) the function $U(r) = r^{\rho(r)}(r \geq r_0)$ satisfies the condition

$$\lim_{r\to\infty} \frac{\log U(R)}{\log U(r)} = 1, \quad R = r + \frac{r}{\log U(r)}.$$

3) $\limsup\limits_{r\to\infty} \frac{\log T(r,f)}{\rho(r)\log r} = 1$.

This definition is duo to K.L. Hiong [8]. A simple proof of the existence of $\rho(r)$ was given by C.T. Chuang [4]. A ray $\arg z = \theta$ is called a $\rho(r)$ order Borel direction of $\rho(r)$ order meromorphic function f, if no matter how small the positive number $0 < \varepsilon < \pi/2$ is, for each value a in $\mathbb{C}_\infty$

$$\limsup_{r\to\infty} \frac{\log n(r,\theta,\varepsilon,a)}{\rho(r)\log r} = 1,$$

with at most two exceptional values a (see [2]).

Now, we are in the position to state our main results.

Theorem 3. *Let $A_0(z), \ldots, A_{n-2}(z)$ be meromorphic functions (at lees one of them is transcendental) of finite order and satisfy* $\max\{\sigma(A_i(z)) : 1 \leq i \leq n-2\} < \sigma(A_0(z)) := \sigma$ *and* $\max\{\lambda(\frac{1}{A_i(z)}) : 0 \leq i \leq n-2\} < \sigma$. *Suppose that the equation (1) posses a solution base $\{f_1, \cdots, f_n\}$ and $\sigma(E) = \infty$, here $E = f_1 \cdots f_n$. If $\rho(r)$ is a proximate order of E, then a ray $\arg z = \theta$ is a $\rho(r)$ order Borel direction of E, if and only if*

$$\limsup_{\varepsilon\to 0}\limsup_{r\to\infty} \frac{\log n(r,\theta,\varepsilon,E=0)}{\rho(r)\log r} = 1.$$

2. Proof of Theorem 3

Our proof requires the Nevanlinna Characteristic for an angel (see e.g. [3, 6, 11]). In convenient, we introduce the following Nevanlinna notations on angular domains (see [2]). Let $f(z)$ be a meromorphic function, consider a direction $L : \arg z = \theta_0$ and an angle $\alpha = \theta_0 - \eta \leq \arg z \leq \theta_0 + \eta = \beta$,

$0<\eta<\frac{\pi}{2}$. For $r>1$, we define $k=\frac{\pi}{\beta-\alpha}$ and

$$A_{\alpha\beta}(r,f)=\frac{k}{\pi}\int_1^r(\frac{1}{t^k}-\frac{t^k}{r^{2k}})\{\log^+|f(te^{i\alpha})|+\log^+|f(te^{i\beta})|\}\frac{dt}{t};$$
$$B_{\alpha\beta}(r,f)=\frac{2k}{\pi r^k}\int_\alpha^\beta \log^+|f(te^{i\alpha})|\sin k(\theta-\alpha)d\theta;$$
$$C_{\alpha\beta}(r,f)=2\sum_{b\in\triangle}(\frac{1}{|b_v|^k}-\frac{|b_v|^k}{r^{2k}})\sin k(\beta_v-\alpha),$$

where the summation $\sum\limits_{b\in\triangle}$ is taken over all poles $b=|b|e^{i\theta}$ of the function $f(z)$ in the sector $\triangle: 1<|z|<r, \quad \alpha<\arg z<\beta$, each pole b occurs in the sum $\sum\limits_{b\in\triangle}$ as many times as it's order, when pole b occurs in the sum $\sum\limits_{b\in\triangle}$ only once, we denote it $\overline{C}(r,f)$. Furthermore, for $r>1$, we define

$$D_{\alpha\beta}(r,f)=A_{\alpha\beta}(r,f)+B_{\alpha\beta}(r,f), \quad S(r,f)=S_{\alpha\beta}(r,f)=C_{\alpha\beta}(r,f)+D_{\alpha\beta}(r,f).$$

In order to prove Theorem 3, we need the following Lemmas.

Lemma 1 ([2])**.** *With the above notations, in order that the direction L : $\arg z=\theta_0$ is a $\rho(r)$ order Borel direction of the function $f(z)$ of order $\rho(r)$, it is necessary and sufficient that for each number $\eta(0<\eta<\pi/2)$, we have*

$$\limsup_{r\to\infty}\frac{\log S(r,f)}{\log U(r)}=1, \quad U(r)=r^{\rho(r)}.$$

Lemma 2 ([6])**.** *With the above notations, let $g(z)$ be a nonconstant meromorphic function and $\Omega(\alpha,\beta)$ be a sector, where $0<\beta-\alpha\le 2\pi$, then, for any $r<R$,*

$$A_{\alpha\beta}(r,\frac{g'}{g})\le K\{(\frac{R}{r})^k\int_1^R\frac{logT(t,g)}{t^{1+k}}dt+log\frac{r}{R-r}+log\frac{R}{r}+1\},$$

$$B_{\alpha\beta}(r,\frac{g'}{g})\le\frac{4k}{r^k}m(r,\frac{g'}{g}).$$

Now, we are in the position to prove the Theorem 3.

Proof of Theorem 3. Suppose that $L:\arg z=\theta$ is a $\rho(r)$ order Borel direction of E. Apply Lemma 1, we have for each number $\eta(0<\eta<\pi/2)$,

$$\limsup_{r\to\infty}\frac{\log S_{\theta-\eta,\theta+\eta}(r,E)}{\log U(r)}=1, \quad U(r)=r^{\rho(r)}. \tag{2}$$

Let $f(z)$ be a nontrivial solution of (1), it follows from Theorem 1 in [1] that the order of $\log T(r,f)$ is at most σ. Hence, the order of $\log T(r,E)$ is at most

σ. The Wronskian determinant $W(f_1, f_2, \ldots, f_n)$ of the fundamental system of solutions $\{f_1, f_2, \ldots, f_n\}$ is given as follows

$$\frac{W}{E} = \frac{W(f_1, f_2, \ldots, f_n)}{E} = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \frac{f_1'}{f_1} & \frac{f_2'}{f_2} & \cdots & \frac{f_n'}{f_n} \\ & \cdots & \cdots & \\ \frac{f_1^{(n-1)}}{f_1} & \frac{f_2^{(n-1)}}{f_2} & \cdots & \frac{f_n^{(n-1)}}{f_n} \end{vmatrix}$$

Apply Abel Lemma ([10, p.16]), we can derive that W is a positive constant and denote it by C. Hence

$$\frac{1}{E} = \frac{1}{C}\frac{W}{E} = \frac{1}{C}\sum_{1\le i_l \ne i_l \le n} (-1)^{\tau} \prod_{l=1}^{n-1} \frac{f_{i_l}^{(l)}}{f_{i_l}}.$$

Using the lemma 2 in which $R = 2r$, for sufficiently small ε, we have for any f_i,

$$A_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{f_i'}{f_i}) = O(\int_1^{2r} \frac{\log^+ T(t, f_i)}{t^{1+\frac{\pi}{2\varepsilon}}}dt) = O(\int_1^{2r} \frac{t^{\sigma+1}}{t^{1+\frac{\pi}{2\varepsilon}}}dt) = O(1). \tag{3}$$

Since,

$$m(r, \frac{f_i'}{f_i}) = O(\log rT(r, f_i)) = O(r^{\sigma+1}).$$

We deduce from lemma 2 that

$$B_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{f_i'}{f_i}) \le \frac{4k}{r^k} m(r, \frac{f_i'}{f_i}) = O(r^{\sigma+1-\frac{\pi}{2\varepsilon}}) = O(1). \tag{4}$$

Hence,

$$D_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{f_i'}{f_i}) = A_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{f_i'}{f_i}) + B_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{f_i'}{f_i}) = O(1) \quad i = 1, 2, \ldots, n.$$

Similarly, we have

$$\begin{array}{l} D_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{f_i^{(h)}}{f_i}) \le \sum_{i=1}^{h} D_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{f_i^{(l)}}{f_i^{(l-1)}}) + O(1) = O(1) \\ i = 1, 2, \ldots, n; \quad h = 2, 3, \ldots, n-1. \end{array} \tag{5}$$

Therefore, we have

$$D_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{E}) \le D_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{C}) + D_{\theta-\varepsilon,\theta+\varepsilon}(r, \sum_{1\le i_l \ne i_l \le n} (-1)^{\tau} \prod_{l=1}^{n-1} \frac{f_{i_l}^{(l)}}{f_{i_l}}) = O(1).$$

It's known that

$$S_{\theta-\varepsilon,\theta+\varepsilon}(r, E) = S_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{E}) + O(1) = D_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{E}) + C_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{E}) + O(1).$$

It follows from above argument that

$$S_{\theta-\varepsilon,\theta+\varepsilon}(r, E) = C_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{E}) + O(1). \tag{6}$$

From (2) and (6), we have

$$\limsup_{r\to\infty}\frac{\log C_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{E})}{\rho(r)\log r}=1.$$

Since,

$$C_{\theta-\varepsilon,\theta+\varepsilon}(r,a)\leq 2n(r,\theta,\varepsilon,f=a)$$

(see [5]), we deduce

$$\limsup_{r\to\infty}\frac{\log n(r,\theta,\varepsilon,E=0)}{\rho(r)\log r}\geq 1.$$

Hence,

$$\limsup_{\varepsilon\to 0}\limsup_{r\to\infty}\frac{\log n(r,\theta,\varepsilon,E=0)}{\rho(r)\log r}\geq 1.$$

On the other hand, for any $r>0$, we have

$$n(r,\theta,\varepsilon,E=0)\leq n(r,E=0)\leq N(R,E=0)\log\frac{r}{R}\leq T(R,f)\log\frac{r}{R},$$

where $R=r+\frac{r}{\log U(r)}$. Hence

$$\limsup_{\varepsilon\to 0}\limsup_{r\to\infty}\frac{\log n(r,\theta,\varepsilon,E=0)}{\rho(r)\log r}\leq 1.$$

It remains to show that if

$$\limsup_{\varepsilon\to 0}\limsup_{r\to\infty}\frac{\log n(r,\theta,\varepsilon,E=0)}{\rho(r)\log r}=1,$$

then $L:\arg z=\theta$ is a $\rho(r)$ order Borel direction of E. Apply the Lemma 1, it is sufficient to prove that for any $\varepsilon>0$,

$$\limsup_{r\to\infty}\frac{\log S_{\theta-\varepsilon,\theta+\varepsilon}(r,E)}{\rho(r)\log r}=1.$$

For this, if $0<\eta<\frac{\pi}{2}$ is sufficiently small, we have [3],

$$\limsup_{r\to\infty}\frac{\log S_{\theta-\eta,\theta+\eta}(r,E)}{\rho(r)\log r}\leq 1.$$

Suppose that for any $0<\eta<\frac{\pi}{2}$, $\limsup\limits_{r\to\infty}\frac{\log S_{\theta-\eta,\theta+\eta}(r,E)}{\rho(r)\log r}<1$. Then for any $a\in\mathbb{C}$, we have $\limsup\limits_{r\to\infty}\frac{\log n(r,\theta,\frac{\eta}{3},E=a)}{\rho(r)\log r}<1$.

Suppose that the argument does not hold. Then there exists $0<\varepsilon<\frac{\pi}{2}$, such that

$$\limsup\lim_{r\to\infty}\frac{\log S_{\theta-\varepsilon,\theta+\varepsilon}(r,E)}{\rho(r)\log r}<1,\quad \limsup_{r\to\infty}\frac{\log n(r,\theta,\frac{\varepsilon}{3},E=a)}{\rho(r)\log r}=1.$$

Apply definition 1, we have

$$\begin{aligned}\limsup_{r\to\infty}\frac{\log n(r,\theta,\frac{\varepsilon}{3},E=a)}{\log U(R)} &= \limsup_{r\to\infty}\frac{\log n(r,\theta,\frac{\varepsilon}{3},E=a)}{\log U(r)}\frac{\log U(r)}{\log U(R)}\\ &\geq \limsup_{r\to\infty}\frac{\log n(r,\theta,\frac{\varepsilon}{3},E=a)}{\log U(r)}\liminf_{r\to\infty}\frac{\log U(r)}{\log U(R)}\\ &\geq 1.\end{aligned}$$

Hence, for any $\tau > 0$ which satisfies $1-\tau > \limsup\limits_{r\to\infty}\frac{\log S_{\theta-\varepsilon,\theta+\varepsilon}(r,E)}{\rho(r)\log r}$, there exists $\{R_n = r_n + \frac{r_n}{\log U(r_n)}\}$, $R_n \to \infty (n\to\infty)$, such that

$$n(r_n) = n(r_n,\theta,\frac{\varepsilon}{3},E=a) \geq (U(R_n))^{1-\tau}.$$

Let $b_v = |b_v|e^{i\beta_v}$ $(v = 1,2,\ldots)$ is the root of $E = a$ in angular domain $\Omega(\theta-\frac{\varepsilon}{3},\theta+\frac{\varepsilon}{3})$, counting complicity. Since, $\theta-\frac{\varepsilon}{3} < \beta_v < \theta+\frac{\varepsilon}{3}$, $v = 1,2,\ldots,$ we deduce $\frac{\varepsilon}{6} < \beta_v - \theta + \frac{\varepsilon}{2} < \frac{5\varepsilon}{6}$. From the Nevanlinna theory it follows that

$$\begin{aligned}S_{\theta-\varepsilon,\theta+\varepsilon}(R_n,E) &\geq C_{\theta-\varepsilon,\theta+\varepsilon}(R_n,a)+O(1) \geq C_{\theta-\frac{\varepsilon}{2},\theta+\frac{\varepsilon}{2}}(R_n,a)+O(1)\\ &\geq 2\sum_{\substack{1<|b_v|<r_n\\ \theta-\frac{\varepsilon}{2}<\beta_v<\theta+\frac{\varepsilon}{2}}}(\frac{1}{|b_v|^k}-\frac{|b_v|^k}{(R_n)^{2k}})\sin\frac{\pi}{\varepsilon}(\beta_v-\theta+\frac{\varepsilon}{2})+O(1)\\ &\geq 2\sum_{\substack{1<|b_v|<r_n\\ \theta-\frac{\varepsilon}{3}<\beta_v<\theta+\frac{\varepsilon}{3}}}(\frac{1}{|b_v|^k}-\frac{|b_v|^k}{(R_n)^{2k}})\sin\frac{\pi}{\varepsilon}(\beta_v-\theta+\frac{\varepsilon}{2})+O(1)\\ &\geq \sum_{\substack{1<|b_v|<r_n\\ \theta-\frac{\varepsilon}{3}<\beta_v<\theta+\frac{\varepsilon}{3}}}(\frac{1}{|b_v|^k}-\frac{|b_v|^k}{(R_n)^{2k}})+O(1),\end{aligned}$$

where $k = \frac{\pi}{\varepsilon}$. We write above sum as a Stieltjes-integral and application the partial integration of this Stieltjes-integral now result in

$$\begin{aligned}
S_{\theta-\varepsilon,\theta+\varepsilon}(R_n,E) &\geq \int_1^{r_n}\frac{1}{t^k}dn(t)+\frac{1}{R_n^{2k}}\int_1^{r_n}t^k dn(t)+O(1)\\
&\geq k\int_1^{r_n}\frac{1}{t^{k+1}}dn(t)+\frac{n(r_n)}{r_n^k}-\frac{r_n^k n(r_n)}{R_n^{2k}}+\\
&\qquad +\frac{k}{R_n^{2k}}\int_1^{r_n}t^{k-1}dn(t)+O(1)\\
&\geq \frac{n(r_n)}{r_n^k}-\frac{r_n^k n(r_n)}{R_n^{2k}}+O(1)\\
&\geq \frac{n(r_n)}{r_n^k}-\frac{R_n^k n(r_n)}{R_n^{2k}}+O(1)\\
&\geq (\frac{1}{r_n^k}-\frac{1}{R_n^k})n(r_n)+O(1).
\end{aligned}$$

Hence,

$$\begin{aligned}
\limsup_{r\to\infty}\frac{\log S_{\theta-\varepsilon,\theta+\varepsilon}(r,E)}{\rho(r)\log r} &\geq \limsup_{r\to\infty}\frac{\log S_{\theta-\varepsilon,\theta+\varepsilon}(R_n,E)}{\rho(R_n)\log R_n}\\
&\geq \liminf_{r\to\infty}\frac{\log(\frac{1}{r_n^k}-\frac{1}{R_n^k})}{\rho(R_n)\log R_n}+\limsup_{r\to\infty}\frac{\log n(r_n)}{\rho(R_n)\log R_n}\\
&\geq 1-\tau+\liminf_{r\to\infty}\frac{\log(R_n^k-r_n^k)-k(\log R_n^k+\log r_n^k)}{\rho(R_n)\log R_n}\\
&= 1-\tau+\liminf_{r\to\infty}\frac{\log\{(r_n+\frac{r_n}{\log U(r_n)})^k-r_n^k\}}{\rho(R_n)\log R_n}\\
&= 1-\tau.
\end{aligned}$$

This contradicts with the hypothesis of τ. Hence for any $0<\eta<\frac{\pi}{2}$, if

$$\limsup_{r\to\infty}\frac{\log S_{\theta-\eta,\theta+\eta}(r,E)}{\rho(r)\log r}<1.$$

then for any $a\in\mathbb{C}$, we have $\limsup\limits_{r\to\infty}\frac{\log n(r,\theta,\frac{\eta}{3},E=a)}{\rho(r)\log r}<1$.

Put $a=0$, we have

$$\limsup_{r\to\infty}\frac{\log n(r,\theta,\frac{\eta}{3},E=0)}{\rho(r)\log r}<1.$$

Hence,

$$\limsup_{\varepsilon\to 0}\limsup_{r\to\infty}\frac{\log n(r,\theta,\varepsilon,E=0)}{\rho(r)\log r}<1.$$

This contradicts with the hypothesis and the Theorem follows. □

3. Acknowledgment

The authors would like to thank Prof. J. H. Zheng for his kind help.

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Received October 3, 2006.

Zhao-jun Wu,
Department of Mathematics,
Xianning University,
Xianning, Hubei 437100,
P. R. China
E-mail address: wuzj52@hotmail.com

Dao-chun Sun,
School of Mathematics,
South China Normal University,
Guangzhou, 510631,
P. R. China
E-mail address: sundch@scnu.edu.cn