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NONDOUBLING MEASURE ON VILENKIN GROUP

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Abstract. In this paper the authors introduce nondoubling measure on Vilenkin groups. The differentiation theorem is also obtained on this measure space.

1. Introduction

Let μ be a Radon measure on $\mathbb{R}^d$ which only satisfies the following growth condition

$$\mu(B(x,r)) \leq Cr^n \tag{1.1}$$

for all $x \in \mathbb{R}^d$ and $r > 0$, where C is independent of x and r. n is a fixed number satisfying $0 < n < d$. Here μ is not assumed to satisfy the doubling condition. Recall that μ is said to satisfy the doubling condition $\mu(B(x,2r)) \leq C\mu(B(x,r))$ in classical analysis. It is well known that the doubling condition is an essential assumption in the analysis on spaces of homogeneous type. Nowadays, it seems that this doubling condition can be removed and many results of Harmonic Analysis are still true without it. We can find some details in many paper, see [1, 7].

Motivated by their work, it is natural to consider what will be happen on locally compact Vilenkin group [4, 5]?

Let G be a bounded locally compact Vilenkin group, that is, G is locally compact Abelian group containing a strictly decreasing sequence of compact open subgroups $\{G_n\}_{n=-\infty}^{\infty}$ such that

(a) $\cup_{n=-\infty}^{\infty} G_n = G$ and $\cap_{n=-\infty}^{\infty} G_n = \{0\}$;
(b) $\sup\{$ order $(G_n/G_{n+1}) : n \in \mathbb{Z}\} = B < \infty$.

We usually choose Haar measure dx on G so that $|G_0| = 1$. Let $|G_n| = \dfrac{1}{m_n}$ for each $n \in \mathbb{Z}$. We know that $2m_n \leq m_{n+1} \leq Bm_n$.

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Now, if we choose another Radon measure μ on G, and it is not satisfied the doubling condition, that is, there does not exist a constant C, such that $\mu(G_n) \leq C\mu(G_{n+1})$ or $\mu(G_{n+1}) \leq C\mu(G_n)$. Then what will be happen on G? It is our main question to answer in this paper.

2. Main results and proofs

Firstly, we need to make out the connection of nondoubling measure μ and Lebesgue measure ν on $\mathbb{R}$.

Lemma 1. *Nondoubling measure μ is absolute continuous to Lebesgue measure ν.*

Proof. Let $E = \cup_{n=1}^{\infty} E_n$, E_n is open set and $\nu(E) = 0$, then

$$0 \leq \mu(E) = \mu(\cup_{n=1}^{\infty} E_n) \leq C \sum_{n=1}^{\infty} \mu(E_n) \leq C\nu(E) = 0.$$

This implies $\mu(E) = 0$. That is, $\mu \ll \nu$. □

Therefore, by Radon-Nikodym Theorem, there exists a negative measurable function f, such that $\mu(A) = \int_A f d\nu = \int_A f dx$.

Similar to this condition on μ on $\mathbb{R}$, suppose that a negative Radon measure μ on Vilenkin group G is absolute continuous to its Haar measure dx, and μ no longer satisfied the doubled condition, but satisfying the following growth condition: $\mu(G_k) \leq C_0|G_k|$.

In [7], X. Tolsa defined the coefficients $K_{Q,R}$ to characterize the closeness of two cubes Q and R in $\mathbb{R}^n$. We will consider the same question on Vilenkin groups.

Definition 1. The coefficient K_{G_k,G_j} measures how close G_k is to G_j, in some sense. Given two sets $G_k \subset G_j, (k \geq j)$ in G. Set

$$K_{G_k,G_j} = 1 + \sum_{i=1}^{k-j} \frac{\mu(G_{k-i})}{|G_{k-i}|},$$

where $|G_{k-i}|$ denotes G_{k-i}'s Haar measure.

In the following lemmas we will show some of the properties of the coefficients K_{G_k,G_j}.

Lemma 2. *If $G_k \subset G_j \subset G_\nu$, then $K_{G_k,G_j} \leq K_{G_k,G_\nu}$, $K_{G_j,G_\nu} \leq CK_{G_k,G_\nu}$ and $K_{G_k,G_\nu} \leq K_{G_k,G_j} + K_{G_j,G_\nu}$.*

Proof. Let $G_k \subset G_j \subset G_\nu$, then $k > j > \nu$ and $k - j < k - \nu$

$$K_{G_k,G_j} = 1 + \sum_{i=1}^{k-j} \frac{\mu(G_{k-i})}{|G_{k-i}|} \leq 1 + \sum_{i=1}^{k-\nu} \frac{\mu(G_{k-i})}{|G_{k-i}|} = K_{G_k,G_\nu}$$

and

$$\begin{aligned}
K_{G_j,G_\nu} &= 1+\sum_{k=1}^{j-\nu}\frac{\mu(G_{j-k})}{|G_{j-k}|} = 1+\sum_{k=1}^{j-\nu}\frac{\mu(G_{j-k-(i-j)})}{|G_{i-k-(i-j)}|}\\
&= 1+\sum_{k'=1+i-j}^{i-\nu}\frac{\mu(G_{i-k'})}{|G_{i-k'}|}\\
&\le 1+\sum_{k'=1}^{i-\nu}\frac{\mu(G_{i-k'})}{|G_{i-k'}|}\\
&= K_{G_i,G_\nu},
\end{aligned}$$

where $k'=k+i-j$ and $i-j<i-\nu$.

Furthermore, we have

$$\begin{aligned}
K_{G_i,G_\nu} &= 1+\sum_{k=1}^{i-\nu}\frac{\mu(G_{i-k})}{|G_{i-k}|} = 1+\sum_{k=1}^{i-j}\frac{\mu(G_{i-k})}{|G_{i-k}|}+\sum_{k=i-j+1}^{i-k}\frac{\mu(G_{i-k})}{|G_{i-k}|}\\
&= K_{G_i,G_j}+\frac{\mu(G_{j-1})}{|G_{j-1}|}+\frac{\mu(G_{j-2})}{|G_{j-2}|}+\cdots+\frac{\mu(G_\nu)}{|G_\nu|}\\
&\le K_{G_i,G_j}+1+\sum_{k=1}^{j-\nu}\frac{\mu(G_{j-k})}{|G_{j-k}|}\\
&= K_{G_i,G_j}+K_{G_j,G_\nu}.
\end{aligned}$$

□

We say that $G_k\subset G_{k-1}$ is (B,β)-doubling if $\mu(G_{k-1})\le\beta\mu(G_k)$.

Lemma 3. *If N is some positive integer and the cubes $G_{k-1},G_{k-2},\dots,G_{k-N}$ are non (B,β)-doubling (with $\beta>B$), then $K_{G_k,G_{k-N}}\le C$, with C depending on β and C_0.*

Proof. For $\beta>B$, suppose $G_{k-1},G_{k-2},\dots,G_{k-N}$ are non (B,β)-doubling. We have $\mu(G_{k-2})\ge\beta\mu(G_{k-1})$ and then

$$\mu(G_{k-N})\ge\beta\mu(G_{k-N+1})\ge\cdots\ge\beta^{N-i}\mu(G_{k-i})\quad(\forall i=1,\dots,N-1).$$

That is,

$$\mu(G_{k-i})\le\frac{\mu(G_{k-N})}{\beta^{N-i}}\quad(\forall i=1,\dots,N-1)$$

then

$$
\begin{aligned}
K_{G_k,G_{k-N}} &= 1+\sum_{i=1}^{N}\frac{\mu(G_{k-i})}{|G_{k-i}|} \\
&= 1+\frac{\mu(G_{k-N})}{|G_{k-N}|}+\sum_{i=1}^{N-1}\frac{\mu(G_{k-N})}{|G_{k-i}|\beta^{N-i}} \\
&= 1+\frac{\mu(G_{k-N})}{|G_{k-N}|}+\frac{\mu(G_{k-N})}{|G_{k-N}|}\sum_{i=1}^{N-1}\frac{|G_{k-N}|}{|G_{k-i}|}\frac{1}{\beta^{N-i}} \\
&\le 1+C_0+C_0\sum_{i=1}^{N-1}\left(\frac{B}{\beta}\right)^{N-i} \\
&\le 1+C_0+C_0\sum_{i=1}^{\infty}\left(\frac{B}{\beta}\right)^{i} \le C.
\end{aligned}
$$

□

Lemma 4. *If N is some positive integer and for some $\beta < 2$,*

$$\mu(G_{k-N}) \le \beta\mu(G_{k-N+1}) \le \ldots \le \beta^N\mu(G_k)$$

then $K_{G_k,G_{k-N}} \le C$, with C depending on β and C_0.

Proof. For $\beta < 2$, we have

$$
\begin{aligned}
K_{G_k,G_{k-N}} &= 1+\sum_{i=1}^{N}\frac{\mu(G_{k-i})}{|G_{k-i}|} \\
&\le 1+\sum_{i=1}^{N}\frac{\beta^i\mu(G_k)}{|G_{k-i}|} \\
&= 1+\frac{\mu(G_k)}{|G_k|}\sum_{i=1}^{N}\frac{\beta^i|G_k|}{|G_{k-i}|} \\
&\le 1+C_0\sum_{i=1}^{N}\left(\frac{\beta}{2}\right)^{i} \\
&\le 1+C_0\sum_{i=1}^{\infty}\left(\frac{\beta}{2}\right)^{i} \le C.
\end{aligned}
$$

□

Notice that, the properties of Lemma 3 says that if the density of the measure μ grows much faster than the Haar measure of G_K, then the coefficients $K_{G_k,G_{k-N}}$, remain bounded while the Lemma 4 says that if the measure grows too slowly, then they also remain bounded.

Next theorem is the application of the differentiation theorem. The case on $\mathbb{R}^n$ can be seen in [2].

Theorem 1. *Let $f \in L^1_{loc}(\mu)$. If $\beta > B$, then for almost every x with respect to μ, there exists a sequence of (B,β)-doubling $G_j(x), (x \in G_j)$ with $\lim_{j\to\infty} |G_j| = 0$, such that*

$$\lim_{j\to\infty} \frac{1}{\mu(G_j)} \int_{G_j} f(y) d\mu(y) = f(x).$$

Proof. We will show that for almost every x with respect to μ there is (B,β)-doubling G_j contained x with $|G_j|$ as small as we wish. This fact combined with the differentiation theorem, completes the proof of the lemma.

We know that for almost every x with respect to μ

$$\lim_{j\to\infty} \frac{\mu(G_j)}{|G_j|} > 0. \tag{2.1}$$

Suppose for any $x \in G_j$ satisfying (2.1) and assume that none of $G_{j+k}, k \geq 1$, is (B,β)-doubling. Then, it is easy to see that

$$\mu(G_j) > \beta\mu(G_{j+1}) > \beta^2\mu(G_{j+2}) > \cdots > \beta^k\mu(G_{j+k}) \quad (\forall k \geq 1)$$

and then

$$\frac{\beta^k \mu(G_{j+k})}{|G_{j+k}|} < \frac{\mu(G_j)}{|G_j|} \cdot \frac{|G_j|}{|G_{j+k}|}.$$

Therefore,

$$\frac{\mu(G_{j+k})}{|G_{j+k}|} < \left(\frac{B}{\beta}\right)^k \cdot \frac{\mu(G_j)}{|G_j|} \leq C_0 \left(\frac{B}{\beta}\right)^k.$$

Note that $\beta > B$, the right hand side tends to zero for $k \to \infty$ and this makes contradiction. $\square$

References

[1] J. García-Cuerva and A. E. Gatto. Boundedness properties of fractional integral operators associated to non-doubling measures. *Studia Math.*, 162(3):245–261, 2004.

[2] J. García-Cuerva and A. E. Gatto. Lipschitz spaces and Calderón-Zygmund operators associated to non-doubling measures. *Publ. Mat.*, 49(2):285–296, 2005.

[3] J. García-Cuerva and J. M. Martell. Weighted inequalities and vector-valued Calderón-Zygmund operators on non-homogeneous spaces. *Publ. Mat.*, 44(2):613–640, 2000.

[4] G. Gát. Orthonormal systems on Vilenkin groups. *Acta Math. Hungar.*, 58(1-2):193–198, 1991.

[5] G. Gát. Investigations of certain operators with respect to the Vilenkin system. *Acta Math. Hungar.*, 61(1-2):131–149, 1993.

[6] J. Orobitg and C. Pérez. A_p weights for nondoubling measures in $\mathbf{R}^n$ and applications. *Trans. Amer. Math. Soc.*, 354(5):2013–2033 (electronic), 2002.

[7] X. Tolsa. BMO, H^1, and Calderón-Zygmund operators for non doubling measures. *Math. Ann.*, 319(1):89–149, 2001.

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