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IDEALLY FACTORED ALGEBRAS

M. AMYARI AND M. MIRZAVAZIRI

Abstract. A complex algebra $\mathcal{A}$ is called ideally factored if $\mathcal{I}_a = \mathbb{C}a$ is a left ideal of $\mathcal{A}$ for all $a \in \mathcal{A}$. In this article, we investigate some interesting properties of ideally factored algebras and show that these algebras are always Arens regular but never amenable. In addition, we investigate ρ-homomorphisms and (ρ, τ)-derivations on ideally factored algebra.

1. Introduction and Preliminaries

Suppose $\mathcal{A}$ is a complex normed algebra with an identity and $a \in \mathcal{A}$. Recall that the spectrum $\mathrm{sp}_{\mathcal{A}}(a)$ of a is defined as

$$\mathrm{sp}_{\mathcal{A}}(a) = \{\lambda \in \mathbb{C} : a - \lambda I \text{ is not invertible}\},$$

and the spectral radius $r_{\mathcal{A}}(a)$ of a is defined by

$$r_{\mathcal{A}}(a) = \sup\{|\lambda| : \lambda \in \mathrm{sp}_{\mathcal{A}}(a)\} = \lim_n \|a^n\|^{1/n}.$$

If $\mathcal{A}$ has no identity, then the spectrum of $a \in \mathcal{A}$ is determined by its spectrum in the unitization $\mathcal{A}^+ = \mathcal{A} \oplus \mathbb{C}$ of $\mathcal{A}$.

If $\mathcal{A}$ is a normed algebra and $\mathcal{X}$ is a normed space, then $\mathcal{X}$ is said to be a normed left $\mathcal{A}$-module if $\mathcal{X}$ is a left $\mathcal{A}$-module and there exists a positive constant k such that $\|ax\| \leq k\|a\|\|x\|$ $(a \in \mathcal{A}, x \in \mathcal{X})$. The notion of a normed right $\mathcal{A}$-module is similarly defined. A space $\mathcal{X}$ is said to be an $\mathcal{A}$-bimodule, if it is both a normed left $\mathcal{A}$-module and a normed right $\mathcal{A}$-module and $(ax)b = a(xb)$ $(a, b \in \mathcal{A}, x \in \mathcal{X})$. Furthermore, if $\mathcal{X}$ is a complete space, then it is called a Banach module.

Example 1.1. $\mathcal{A}$ is an $\mathcal{A}$-bimodule when the left and right actions are defined by the multiplication of $\mathcal{A}$ in a usual manner.

Example 1.2. If $\mathcal{X}$ is an $\mathcal{A}$-bimodule then the dual $\mathcal{X}^*$ of $\mathcal{X}$ is also an $\mathcal{A}$-bimodule under the actions $(af)(x) = f(xa), (fa)(x) = f(ax)$ $(a \in \mathcal{A}, x \in \mathcal{X})$.

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Suppose that $\mathcal{A}$ is a Banach algebra and $\mathcal{X}$ is a Banach $\mathcal{A}$-bimodule. A linear map $\delta\colon \mathcal{A} \to \mathcal{X}$ is called a derivation if

$$\delta(ab) = a\delta(b) + \delta(a)b \qquad (a, b \in \mathcal{A}).$$

Given $x \in \mathcal{X}$, the map $\delta_x(a) = ax - xa$ is a derivation on $\mathcal{A}$ which is called an inner derivation. The set of all bounded derivations and inner derivations are denoted by $Z^1(\mathcal{A}, \mathcal{X})$ and $B^1(\mathcal{A}, \mathcal{X})$, respectively. $H^1(\mathcal{A}, \mathcal{X}) = Z^1(\mathcal{A}, \mathcal{X})/H^1(\mathcal{A}, \mathcal{X})$ is called the first cohomology of $\mathcal{A}$ with coefficients in $\mathcal{X}$. A complex Banach algebra $\mathcal{A}$ is called amenable if $H^1(\mathcal{A}, \mathcal{X}^*) = \{0\}$ for every Banach $\mathcal{A}$-bimodule $\mathcal{X}$.

One may be referred to [2, 3] for undefined notions and notations.

Let $\mathcal{A}$ be a complex algebra. If $\mathcal{I} = \mathbb{C}a$ is a left ideal of $\mathcal{A}$ for each $a \in \mathcal{A}$, then $\mathcal{A}$ is called an ideally factored algebra. Each multiple ba of a is therefore of the form $\varphi(b)a$ for some $\varphi(b) \in \mathbb{C}$. This ensures us to have a simple formula for the multiplication and helps us to consider problems concerning linear mappings such as homomorphisms and derivations involving this type of multiplication. The multiplication on an ideally factored algebra is interesting on its own right. In this paper, we use some ideas from [4] to characterize all ideally factored algebras and to investigate their general properties. We also show that these algebras are always Arens regular but not amenable. In addition, we investigate ρ-homomorphism and (ρ, τ)-derivations on ideally factored algebra.

2. Ideally Factored Algebras

A complex algebra $\mathcal{A}$ is called ideally factored if $\mathcal{I}_a = \mathbb{C}a$ is a left ideal of $\mathcal{A}$ for each $a \in \mathcal{A}$. Equivalently, $\mathcal{A}$ is an ideally factored algebra if its minimal left ideals are of the form $\mathbb{C}a$ for some $a \in \mathcal{A}$. In this section, we show that for each ideally factored algebra $\mathcal{A}$ there is a unique multiplicative linear functional φ such that $ab = \varphi(a)b$ $(a, b \in \mathcal{A})$. Clearly $\ker\varphi$ is the left annihilator of $\mathcal{A}$.

Theorem 2.1. *Let $\mathcal{A}$ be an ideally factored algebra. Then there is a multiplicative linear functional $\varphi\colon \mathcal{A} \to \mathbb{C}$ such that $ba = \varphi(b)a$ for all $b \in \mathcal{A}$.*

Proof. For each $a \in \mathcal{A}, \mathcal{I}_a = \mathbb{C}a$ is a left ideal. Thus, for each $b \in \mathcal{A}$ there is a $\varphi_a(b) \in \mathbb{C}$ such that $ba = \varphi_a(b)a$. Clearly, φ_a is linear. Since the multiplication of $\mathcal{A}$ is associative, we have

$$\varphi_a(b)\varphi_a(c)a = b(\varphi_a(c)a) = b(ca) = (bc)a = \varphi_a(bc)a.$$

Thus, φ_a is multiplicative.

Next, we show that $\varphi_a = \varphi_b$ for all $a, b \in \mathcal{A}$. If $\{a, b\}$ is a linearly independent set, then for each $c \in \mathcal{A}$ we have

$$\varphi_{a+b}(c)(a + b) = c(a + b) = ca + cb = \varphi_a(c)a + \varphi_b(c)b.$$

Thus,

$$(\varphi_{a+b} - \varphi_a)(c)a = (\varphi_b - \varphi_{a+b})(c)b$$

and since $\{a, b\}$ is linearly independent we have

$$(\varphi_{a+b} - \varphi_a)(c) = (\varphi_b - \varphi_{a+b})(c) = 0.$$

Hence

$$\varphi_a = \varphi_{a+b} = \varphi_b.$$

If $\{a, b\}$ is not linearly independent, say $b = \alpha a$ for a nonzero scalar α, then

$$\varphi_{\alpha a}(c)\alpha a = c(\alpha a) = \alpha(ca) = \alpha\varphi_a(c)a.$$

Thus,

$$\varphi_{\alpha a}(c) = \varphi_a(c)$$

and so

$$\varphi_b = \varphi_{\alpha a} = \varphi_a.$$

□

Next, assume that $\mathcal{A}$ is a complex Banach space and $\varphi\colon \mathcal{A} \to \mathbb{C}$ is a bounded linear functional. Define a multiplication on $\mathcal{A}$ by $a \cdot b = \varphi(a)b$. This multiplication evidently makes $\mathcal{A}$ into a Banach algebra denoted by $\mathcal{A}_\varphi$ which is called the ideally factored algebra associated to φ. In this section, we investigate the general properties of Banach algebra $\mathcal{A}_\varphi$.

Proposition 2.2. *The Banach algebra $\mathcal{A}_\varphi$ has the following properties:*

(i) φ is multiplicative.
(ii) $u \in \mathcal{A}_\varphi$ is a left identity if and only if $\varphi(u) = 1$.
(iii) $u \in \mathcal{A}_\varphi$ is a right identity if and only if $\mathcal{A}_\varphi$ is linearly spanned by u itself.
(iv) $(a, \mu) \in \mathcal{A}_\varphi^+$ is invertible if and only if $\mu \neq 0$ and $\varphi(a) \neq -\mu$. In this case,

$$(a, \mu)^{-1} = (\frac{-a}{\mu(\varphi(a) + \mu)}, \frac{1}{\mu}).$$

(v) $\mathrm{sp}_{\mathcal{A}_\varphi}(a) = \{0, \varphi(a)\}$.
(vi) $r_{\mathcal{A}_\varphi}(a) = |\varphi(a)|$.
(vii) φ is one to one if and only if $\mathcal{A}_\varphi$ is commutative.

Proof. (i) $\varphi(ab) = \varphi(\varphi(a)b) = \varphi(a)\varphi(b)$.

(ii) $b = u \cdot b = \varphi(u)b$ for all $b \in \mathcal{A}_\varphi$ if and only if $\varphi(u) = 1$.

(iii) $b = b \cdot u = \varphi(b)u$ if and only if each element of $\mathcal{A}_\varphi$ is a scaler multiple of u.

(iv) We have

$$\begin{aligned}(a, \mu)(b, \lambda) &= (a \cdot b + \lambda a + \mu b, \mu\lambda)\\ &= (\varphi(a)b + \lambda a + \mu b, \mu\lambda)\\ &= ((\varphi(a) + \mu)b + \lambda a, \mu\lambda).\end{aligned}$$

Thus, $(a, \mu)(b, \lambda) = (0, 1)$ if and only if $b = \frac{-1}{\mu(\varphi(a)+\mu)}a$ and $\lambda = \frac{1}{\mu}$.

(v) By (iii), $(a, 0) - \lambda(0, 1) = (a, -\lambda)$ is invertible if and only if $\varphi(a) \neq 0$ and $\lambda \neq 0$. Hence $\mathrm{sp}_{\mathcal{A}_\varphi}(a, 0) = \{\varphi(a), 0\}$.

(vi) $r_{\mathcal{A}_\varphi}(a) = \max\{|\lambda| : \lambda \in \mathrm{sp}_{\mathcal{A}_\varphi}(a)\} = |\varphi(a)|$.
(vii) Let φ be one to one, then

$$\varphi(a \cdot b) = \varphi(\varphi(a)b) = \varphi(a)\varphi(b) = \varphi(b)\varphi(a) = \varphi(\varphi(b)a) = \varphi(b \cdot a).$$

Hence $a \cdot b = b \cdot a$. Conversely, let $\varphi(a) = \varphi(b)$ for some $a, b \in \mathcal{A}_\varphi$. Since $\mathcal{A}_\varphi$ is commutative, $a \cdot b = b \cdot a$. So $\varphi(a)b = \varphi(b)a$. Hence $a = b$. Therefore, φ is one to one. □

Now, we investigate the so-called ρ-homomorphisms and (ρ, τ)-derivations on ideally factored algebras. We refer the interested reader to papers [6, 5] for more information about some generalized notions of homomorphism and derivation.

Definition 2.3. Let $\rho \colon \mathcal{A}_\varphi \to \mathcal{A}_\varphi$ be a linear mapping. A linear mapping $\alpha \colon \mathcal{A}_\varphi \to \mathcal{A}_\varphi$ is called a ρ-homomorphism if $\alpha(ab) = \rho(a)\alpha(b)$ for all $a, b \in \mathcal{A}_\varphi$.

Theorem 2.4. *Let $\rho \colon \mathcal{A}_\varphi \to \mathcal{A}_\varphi$ be a mapping. Then a non-zero linear mapping $\alpha \colon \mathcal{A}_\varphi \to \mathcal{A}_\varphi$ is a ρ-homomorphism if and only if $\rho(a) - a \in \ker\varphi \quad (a \in \mathcal{A}_\varphi)$. If this is the case, then $\varphi \circ \rho$ is a homomorphism.*

Proof. Let $b \in \mathcal{A}_\varphi$ with $\alpha(b) \neq 0$. Then $\alpha(ab) = \rho(a)\alpha(b)$ if and only if $\alpha(\varphi(a)b) = \varphi(\rho(a))\alpha(b)$ or equivalently $\varphi(a)\alpha(b) = (\varphi \circ \rho)(a)\alpha(b)$ which is equivalent to $\varphi(\rho(a) - a) = 0$ for all $a \in \mathcal{A}_\varphi$. If α is a ρ-homomorphism then

$$\rho(a)\rho(a')\alpha(b) = \rho(a)\alpha(a'b) = \alpha(aa'b) = \rho(aa')\alpha(b) \quad (a, a' \in \mathcal{A}_\varphi).$$

Hence $\varphi(\rho(a))\varphi(\rho(a'))\alpha(b) = \varphi(\rho(aa'))\alpha(b)$ and we have the result. □

Using an analogous reasoning one can show that a linear mapping $\delta \colon \mathcal{A}_\varphi \to \mathcal{A}_\varphi$ is a derivation if and only if $\varphi \circ \delta = 0$. Now, we extend this result to (ρ, τ)-derivations.

Definition 2.5. Let $\mathcal{X}$ be an $\mathcal{A}_\varphi$-bimodule and $\rho, \tau \colon \mathcal{A}_\varphi \to \mathcal{A}_\varphi$ be linear mappings. A linear mapping $\delta \colon \mathcal{A}_\varphi \to \mathcal{X}$ is a (ρ, τ)-derivation if

$$\delta(ab) = \delta(a)\rho(b) + \tau(a)\delta(b), \quad a, b \in \mathcal{A}_\varphi.$$

Theorem 2.6. *A nonzero linear mapping $\delta \colon \mathcal{A}_\varphi \to \mathcal{X}$ is a (ρ, τ)-derivation for some ρ and τ if either*

(i) $\rho = 0$ (in this case δ is a τ-homomorphism);
or
(ii-1) $\varphi \circ \tau$ is a homomorphism (in this case, δ is an ordinary derivation)
or
(ii-2) $\varphi \circ (\rho + \tau)$ is a homomorphism (in this case, $\delta = \lambda\rho$ for some constant $\lambda \in \mathbb{C}$).

Proof. We have

$$\varphi(a)\delta(b) = \delta(\varphi(a)b) = \delta(ab) = \delta(a)\rho(b) + \tau(a)\delta(b) = \varphi(\delta(a))\rho(b) + \varphi(\tau(a))\delta(b).$$

Hence

$$(\varphi(a) - \varphi(\tau(a))\delta(b) = \varphi(\delta(a))\rho(b).$$

If $\rho = 0$ then δ is clearly a τ-homomorphism (case (i)).

Let $\rho(b) \neq 0$ for some b.

If $\varphi(a) = \varphi(\tau(a))$ for all $a \in \mathcal{A}_\varphi$ then $\varphi(\delta(a)) = 0$ for all $a \in \mathcal{A}_\varphi$. Thus, δ is a derivation. Note that by Theorem 2.4 we can deduce form $\varphi(a - \tau(a)) = 0$ that $\varphi \circ \tau$ is a homomorphism (case (ii-1)).

Otherwise, there exists $a_0 \in \mathcal{A}_\varphi$ such that $\varphi(a_0) \neq \varphi(\tau(a_0))$. Then

$$\delta(b) = \frac{\varphi(\delta(a_0))}{\varphi(a_0) - \varphi(\tau(a_0))}\rho(b) \qquad (b \in \mathcal{A}_\varphi)$$

Putting $\lambda = \frac{\varphi(\delta(a_0))}{\varphi(a_0)-\varphi(\tau(a_0))}$ we have $\delta(b) = \lambda\rho(b)$ for all $b \in \mathcal{A}_\varphi$. Hence

$$\begin{aligned}\lambda\rho(ab) &= \delta(ab)\\ &= \delta(a)\rho(b) + \tau(a)\delta(b)\\ &= \lambda\rho(a)\rho(b) + \tau(a)\lambda\rho(b).\end{aligned}$$

Thus,

$$\rho(ab) = \rho(a)\rho(b) + \tau(a)\rho(b)$$

and so

$$\begin{aligned}\varphi(a)\rho(b) &= \rho(\varphi(a)b)\\ &= \rho(ab)\\ &= \varphi(\rho(a))\rho(b) + \varphi(\tau(a))\rho(b).\end{aligned}$$

Therefore, we have

$$\varphi(a - \rho(a) - \tau(a))\rho(b) = 0 \quad (a, b \in \mathcal{A}_\varphi)$$

Hence

$$\varphi(a - \rho(a) - \tau(a)) = 0 \quad (a \in \mathcal{A}_\varphi).$$

Therefore, $\varphi \circ (\rho + \tau)$ is a homomorphism by Theorem 2.4 (case (ii-2)). $\square$

3. Arens Regularity and Amenability

The left action of $\mathcal{A}_\varphi$ on its dual $\mathcal{A}_\varphi^*$ is $a \diamond f = f(a)\varphi$, since

$$(a \diamond f)(b) = f(b \cdot a) = f(\varphi(b)a) = \varphi(b)f(a), \qquad b \in \mathcal{A}_\varphi, f \in \mathcal{A}_\varphi^*.$$

The right action of $\mathcal{A}_\varphi$ on $\mathcal{A}_\varphi^*$ is $f \circ a = \varphi(a)f$. In fact,

$$(f \circ a)(b) = f(a \cdot b) = f(\varphi(a)b) = \varphi(a)f(b) \qquad b \in \mathcal{A}_\varphi, f \in \mathcal{A}_\varphi^*.$$

Moreover,

$$(F \circ f)(a) = F(f \circ a) = F(\varphi(a)f) = \varphi(a)F(f)$$

and

$$(f \diamond F)(a) = F(a \diamond f) = F(f(a)\varphi) = f(a)F(\varphi),$$

where $F \in \mathcal{A}_\varphi^{**}, f \in \mathcal{A}_\varphi^*, a \in \mathcal{A}_\varphi$.

Hence $F \circ f = F(f)\varphi$ and $f \diamond F = F(\varphi)f$ for all $f \in \mathcal{A}_\varphi^*$ and $F \in \mathcal{A}_\varphi^{**}$. Then the first Arens product on $\mathcal{A}_\varphi^{**}$ is given by

$$(F \circ G)(f) = F(G \circ f) = F(G(f)\varphi) = F(\varphi)G(f)$$

that is $F \circ G = F(\varphi)G$, for each $F, G \in \mathcal{A}_\varphi^{**}$
Similarly, the second Arens product is given by

$$(F \diamond G)(f) = G(f \diamond F) = G(F(\varphi)f) = F(\varphi)G(f)$$

that is $F \diamond G = F(\varphi)G$, for each $F, G \in \mathcal{A}_\varphi^{**}$.

Thus, we get the following theorem.

Theorem 3.1. *$\mathcal{A}_\varphi$ is Arens regular, i.e. $F \circ G = F \diamond G$ for all $F, G \in \mathcal{A}_\varphi^*$ (c.f.* [1]*). Furthermore, every F with $F(\varphi) = 1$ is a left unit for $\mathcal{A}_\varphi^{**}$ with respect to each of the Arens products.*

Theorem 3.2. *$\mathcal{A}_\varphi$ is not amenable for any ϕ.*

Proof. $\mathbb{C}$ is a Banach $\mathcal{A}_\varphi$-bimodule under the actions $\lambda \cdot a = a \cdot \lambda = \frac{\lambda}{2}\varphi(a)$. Define $\delta\colon \mathcal{A}_\varphi \to \mathbb{C}$ by $\delta(a) = \varphi(a)$. Then

$$\delta(a \cdot b) = \delta(\varphi(a)b) = \varphi(\varphi(a)b) = \varphi(a)\varphi(b)$$

and

$$a\delta(b) + \delta(a)b = a\varphi(b) + \varphi(a)b = \frac{1}{2}\varphi(b)\varphi(a) + \frac{1}{2}\varphi(a)\varphi(b) = \varphi(a)\varphi(b).$$

Hence δ is a nonzero derivation and so is in $Z^1(\mathcal{A}_\varphi, \mathbb{C})$.

For each inner derivation $\delta_\mu\colon \mathcal{A}_\varphi \to \mathbb{C}$, where $\mu \in \mathbb{C}$, we have

$$\delta_\mu(a) = a\mu - \mu a = \frac{\mu}{2}\varphi(a) - \frac{\mu}{2}\varphi(a) = 0 \qquad a \in \mathcal{A}_\varphi.$$

Hence $B^1(\mathcal{A}_\varphi, \mathbb{C}) = \{0\}$. Since $\mathbb{C}^* = \mathbb{C}$, we have $H^1(\mathcal{A}_\varphi, \mathbb{C}) = H^1(\mathcal{A}_\varphi, \mathbb{C}^*) \neq \{0\}$. Thus, $\mathcal{A}_\varphi$ is not amenable. $\square$

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Maryam Amyari,
Department of Mathematics,
Faculty of Science,
Islamic Azad University–Mashhad Branch,
Mashhad 91735,
Iran
E-mail address: amyari@mshdiau.ac.ir and maryam_amyari@yahoo.com

Madjid Mirzavaziri,
Department of Pure Mathematics,
Ferdowsi University,
P. O. Box 1159,
Mashhad 91775,
Iran

and

Center of Excellence in Analysis on Algebraic Structures (CEAAS),
Ferdowsi University of Mashhad,
Iran
E-mail address: mirzavaziri@math.um.ac.ir and madjid@mirzavaziri.com