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SOME GENERAL OSTROWSKI-GRÜSS TYPE INEQUALITIES

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Abstract. Some new generalizations of Ostrowski-Grüss type inequalities for functions of Lipschitzian type and probability distribution inequalities are established.

1. Introduction

Definition 1.1. The function $f\colon [a,b] \to \mathbb{R}$ is said to be (l,L)-Lipschitzian on $[a,b]$ if

$$l(x_2 - x_1) \le f(x_2) - f(x_1) \le L(x_2 - x_1), \quad \text{for } a \le x_1 \le x_2 \le b,$$

where $l, L \in \mathbb{R}$ with $l < L$.

Liu [4] generalizes the results in [2] and [5] to functions which are L-Lipschitzian and (l,L)-Lipschitzian respectively as follows.

Theorem L. *Let $f\colon [a,b] \to \mathbb{R}$ be (l,L)-Lipschizian on $[a,b]$. Then we have*

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - f(x) + \left(x - \frac{a+b}{2}\right)S\right| \le \frac{b-a}{2}(S-l) \tag{1.1}$$

and

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - f(x) + \left(x - \frac{a+b}{2}\right)S\right| \le \frac{b-a}{2}(L-S) \tag{1.2}$$

for all $x \in [a,b]$, where $S = (f(b) - f(a))/(b-a)$.

Recently there are many versions of the Ostrowski-Grüss type inequalities (see [1, 2, 3, 4, 5]). In the present paper, we will give some general Ostrowski-Grüss type inequalities in view of the probability theory.

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2. Main results

The following lemma is easy from the elementary knowledge of probability theory.

Lemma 2.1. *Let X be a random variable with uniform distribution on the support interval $[a,b]$, i.e., the probability density function of X is equal to $(b-a)^{-1}$, $x \in [a,b]$ and zero elsewhere. Accordingly, let us denote $\mathbb{E}X$ the mathematical expectation of random variable X. Then for any $x \in [a,b]$, we have*

$$\mathbb{E}(X-x)1_{X\geq x} = \frac{(b-x)^2}{2(b-a)}$$

and

$$\mathbb{E}(x-X)1_{X<x} = \frac{(x-a)^2}{2(b-a)},$$

where 1_A denotes the indicator function of the set $A \subset [a,b]$.

From the above key lemma, we will give a more precise version of Theorem L.

Theorem 2.2. *Under the assumptions of Theorem L, we have*

$$\begin{aligned}(2.1)\quad (l-S)\frac{(b-x)^2}{2(b-a)} &+ (S-L)\frac{(x-a)^2}{2(b-a)} \\ &\leq \frac{1}{b-a}\int_a^b f(t)dt - f(x) + \left(x - \frac{a+b}{2}\right)S \\ &\leq (L-S)\frac{(b-x)^2}{2(b-a)} + (S-l)\frac{(x-a)^2}{2(b-a)},\end{aligned}$$

for all $x \in [a,b]$.

Proof. Let X be a random variable with uniform distribution on the support interval $[a,b]$. Then it is easy to see that

$$\frac{1}{b-a}\int_a^b f(t)dt - f(x) = \mathbb{E}(f(X)-f(x))$$

and

$$x - \frac{a+b}{2} = \mathbb{E}(x-X).$$

Thus, we have

$$\begin{aligned}&\frac{1}{b-a}\int_a^b f(t)dt - f(x) + \left(x-\frac{a+b}{2}\right)S \\ &= \mathbb{E}(f(X)-f(x)) + S\mathbb{E}(x-X) = \mathbb{E}[(f(X)-f(x)) - S(X-x)]1_{X\geq x} \\ &\qquad + \mathbb{E}[(f(X)-f(x)) - S(X-x)]1_{X<x}.\end{aligned}$$

Furthermore, by Definition 1.1 and Lemma 2.1, it follows that

$$\begin{aligned}
(l-S)\frac{(b-x)^2}{2(b-a)}+&(S-L)\frac{(x-a)^2}{2(b-a)} = \\
&=(l-S)\mathbb{E}(X-x)1_{X\geq x}+(S-L)\mathbb{E}(x-X)1_{X<x} \\
&\leq \mathbb{E}[(f(X)-f(x))-S(X-x)]1_{X\geq x}+ \\
&\qquad +\mathbb{E}[(f(X)-f(x))-S(X-x)]1_{X<x} \\
&\leq (L-S)\mathbb{E}(X-x)1_{X\geq x}+(S-l)\mathbb{E}(x-X)1_{X<x} \\
&=(L-S)\frac{(b-x)^2}{2(b-a)}+(S-l)\frac{(x-a)^2}{2(b-a)}
\end{aligned}$$

which implies the inequality (2.1). □

Corollary 2.3. *Under the assumptions of Theorem L, we have*

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - f(x) + \left(x-\frac{a+b}{2}\right)S\right| \leq M\frac{(b-a)}{2}, \tag{2.2}$$

for all $x \in [a,b]$, where $M = \max\{(L-S),(S-l)\}$.

Proof. Let $g(x) = A(b-x)^2+B(x-a)^2$, where A, B are two positive constants. Then it is easy to check that the maximum point of $g(x)$ is at $x = a$ or $x = b$. From this fact and Theorem 2.2, the desired inequality (2.2) is obtained. □

Corollary 2.4. *Under the assumptions of Theorem L, we have*

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)}{8}(L-l). \tag{2.3}$$

Corollary 2.5. *Under the assumptions of Theorem L, we have*

$$-(S-l)\frac{(b-a)}{2} \leq \frac{1}{b-a}\int_a^b f(t)dt - \frac{f(a)+f(b)}{2} \leq (L-S)\frac{(b-a)}{2} \tag{2.4}$$

and

$$-(L-S)\frac{(b-a)}{2} \leq \frac{1}{b-a}\int_a^b f(t)dt - \frac{f(a)+f(b)}{2} \leq (S-l)\frac{(b-a)}{2}. \tag{2.5}$$

3. Inequalities in Probability distribution

In this section we will give some results for probability distribution.

Lemma 3.1. *Let X be a random variable on $[a,b]$ and $\mathbb{E}(X)$ is the expectation of X. Then we have the following equality,*

$$\begin{aligned}
&\mathbb{P}(X\le x)-\frac{1}{b-a}\left(x-\frac{a+b}{2}\right)-\frac{b-\mathbb{E}(X)}{b-a}=\\
&=\mathbb{P}(X\le x)+\frac{\mathbb{E}X-x}{b-a}-\frac{1}{2} \qquad (3.1)\\
&=\frac{\int_a^x\mathbb{P}(t\le X\le x)dt-\int_x^b\mathbb{P}(x<X<t)dt}{b-a}+\frac{a-x}{b-a}+\frac{1}{2}.
\end{aligned}$$

Proof. Noting the following facts,

$$\begin{aligned}
&\mathbb{P}(X\le x)+\frac{\mathbb{E}X}{b-a}=\\
&=\frac{\int_a^b\mathbb{P}(X\le x)dt+\mathbb{E}(\int_a^X dt)+a}{b-a}\\
&=\frac{\int_a^b\mathbb{P}(X\le x)dt+\int_a^b\mathbb{P}(X\ge t)dt+a}{b-a}\\
&=\frac{\int_a^x[\mathbb{P}(X\le x)+\mathbb{P}(X\ge t)]dt+\int_x^b[\mathbb{P}(X\le x)+\mathbb{P}(X\ge t)]dt+a}{b-a}\\
&=\frac{x-a+\int_a^x\mathbb{P}(t\le X\le x)dt+b-x-\int_x^b\mathbb{P}(x<X<t)dt+a}{b-a}\\
&=\frac{b-a+\int_a^x\mathbb{P}(t\le X\le x)dt-\int_x^b\mathbb{P}(x<X<t)dt+a}{b-a}
\end{aligned}$$

and

$$\mathbb{P}(X\le x)-\frac{1}{b-a}\left(x-\frac{a+b}{2}\right)-\frac{b-\mathbb{E}X}{b-a}=\mathbb{P}(X\le x)+\frac{\mathbb{E}X-x}{b-a}-\frac{1}{2},$$

we have

$$\begin{aligned}
&\mathbb{P}(X\le x)+\frac{\mathbb{E}X-x}{b-a}-\frac{1}{2}\\
&\qquad=\frac{\int_a^x\mathbb{P}(t\le X\le x)dt-\int_x^b\mathbb{P}(x<X<t)dt}{b-a}+\frac{a-x}{b-a}+\frac{1}{2}.
\end{aligned}$$

$\square$

Corollary 3.2. *Under the assumptions of Lemma 3.1, we have the following inequalities*

$$
\begin{aligned}
&-\frac{(b-x)\mathbb{P}(X>x)}{b-a}+\frac{a-x}{b-a}+\frac{1}{2}\leq \\
&\leq\frac{\int_a^x \mathbb{P}(t\leq X\leq x)dt-\int_x^b \mathbb{P}(x<X<t)dt}{b-a}+\frac{a-x}{b-a}+\frac{1}{2} \\
&\leq\frac{(x-a)\mathbb{P}(X\leq x)}{b-a}+\frac{a-x}{b-a}+\frac{1}{2}.
\end{aligned}
\tag{3.2}
$$

Furthermore

$$\left|\frac{\int_a^x \mathbb{P}(t\leq X\leq x)dt-\int_x^b \mathbb{P}(x<X<t)dt}{b-a}+\frac{a-x}{b-a}+\frac{1}{2}\right|\leq\frac{1}{2}.$$

Corollary 3.3. *Under the assumptions of Lemma 3.1, we have*

$$\left|\mathbb{E}(X)-\frac{a+b}{2}\right|\leq\frac{b-a}{2}. \tag{3.3}$$

Proof. We set $x=a$ or $x=b$ in (3.1) and by Corollary 3.2 to get the desired result. □

Corollary 3.4. *Under the assumptions of Lemma 3.1, we have*

$$-\frac{\mathbb{P}\left(X>\frac{a+b}{2}\right)}{2}\leq\mathbb{P}\left(X\leq\frac{a+b}{2}\right)-\frac{b-\mathbb{E}(X)}{b-a}\leq\frac{\mathbb{P}\left(X\leq\frac{a+b}{2}\right)}{2}. \tag{3.4}$$

In particular, assume that m is the median of the random variable X, i.e., $\mathbb{P}(X\leq m)\geq 1/2$ and $\mathbb{P}(X\geq m)\geq 1/2$, then we have

$$\left|\mathbb{P}(X\leq m)-\frac{1}{b-a}\left(m-\frac{a+b}{2}\right)-\frac{b-\mathbb{E}(X)}{b-a}\right|\leq\frac{1}{4}.$$

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