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CONTRACTIVE CONDITIONS AND COMMON FIXED POINTS

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Abstract. In this paper we obtain common fixed point theorems under the Lipschitz type analogue of a strict contractive condition by using the notion of R - weak commutativity of type (A_g). In the setting of our results, we use the property (E.A) introduced by Aamri and Moutawakil [1] and compare these with the results proved by using the notion of noncompatiblity introduced by Pant [4]. Simultaneously, we provide contractive condition which ensure the existence of a common fixed point; however, the mappings are discontinuous at the common fixed point. We, thus, provide one more answer to the problem of Rhoades [10]. Our theorems extend the results of Pant and Pant (Theorem 2.1 Pant [6]), Pant, R.P. [5, Theorem 2]), Pant Vyomesh [8] and Singh and Kumar [11].

1. INTRODUCTION

In 1986, Jungck [2] generalized the notion of weak commutativity by introducing the concept of compatible maps. Two self maps f and g of a metric space (X, d) are called *compatible* if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$$

for some t in X. From the definition it follows that the maps f and g are called *noncompatible* if they are not compatible. Thus f and g will be noncompatible if there exists at least one sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$$

for some t in X but $\lim_{n\to\infty} d(fgx_n, gfx_n)$ is either non - zero or non - existent.

In the study of common fixed points of compatible mappings we often require assumptions on the completeness of the space or continuity of the mappings

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involved besides some contractive condition but the study of fixed points of noncompatible mappings can be extended to the class of nonexpansive or Lipschitz type mapping pairs even without assuming the continuity of the mappings involved or completeness of the space.

In 1994 Pant [3] defined the notion R-weakly commuting mappings. Two mappings A and S are called *R-weakly commuting at a point x* if

$$d(ASx, SAx) \leq Rd(Ax, Sx)$$

for some $R > 0$. A and S are *pointwise R-weakly commuting* if given $x \in X$, $\exists$ $R > 0$ such that $d(ASx, SAx) \leq Rd(Ax, Sx)$. In view of a paper by Pant [4], it may be observed that pointwise R - weak commutativity is

(i) equivalent to commutativity at coincident points; and
(ii) a necessary, hence minimal, condition for the existence of common fixed points of contractive type mappings.

It will be pertinent to note that the compatiblity of mappings implies pointwise R - weak commutativity, since compatible maps commute at their coincidence points. However, as shown in the examples on the following lines, pointwise R - weakly commuting maps need not be compatible.

In 1997, Pathak et. al. [9] gave an analogue of R-weak commutativity by introducing the concept of R-weak commutativity of type (A_g). Using the notion of R - weak commutativity Pant and Pant [6] (read with [7]) proved a common fixed point theorem (Theorem 2.1) under strict contractive condition. Pant ([4, Theorem 2 and Theorem 3]) used the notion of R - weak commutativity of type (A_g) and proved common fixed point theorems for a pair of maps which are discontinuous at their coincidence points.

Aamri and Moutawakil [1] introduced the property (E.A) and thus generalized the notion of noncompatible maps. Let f and g be two selfmappings of a metric space (X, d). We say that f and g satisfy the *property (E.A)* if there exists a sequence $\{x_n\}$ such that $\lim_n fx_n = \lim_n gx_n = t$ for some t in X. If two maps are noncompatible, then they satisfy the property (E.A). The converse, however, is not necessarily true. To support our assertion, we quote examples from [1].

Example 1 ([1]). Let $X = [0, +\infty)$. Define $T, S \colon X \to X$ by

$$Tx = \frac{x}{4},$$
$$Sx = \frac{3x}{4}, \quad \forall x \in X$$

Consider the sequence $\{x_n\} = \frac{1}{n}$. Clearly $\lim\limits_n Tx_n = \lim\limits_n Sx_n = 0$. Then T and S satisfy property E.A.

Example 2 ([1]). Let $X = [2, +\infty)$. Define $T, S \colon X \to X$ by

$$Tx = x + 1,$$

$$Sx = 2x + 1, \quad \forall x \in X$$

Suppose that property (E.A) holds; then there exists in X a sequence $\{x_n\}$ satisfying $\lim_n Tx_n = \lim_n Sx_n = t$, for some $t \in X$. Therefore, $\lim_n x_n = t - 1$ and $\lim_n x_n = \frac{(t-1)}{2}$.

Then $t = 1$, which is a contradiction since $1 \notin X$. Hence T and S do not satisfy property E.A.

In the following lines, we prove fixed point theorems (Theorem 1 and Theorem 2) using the property (E.A). These theorems generalize several results including those of Pant [6] (read with [7]), Pant, R.P. [5] (Theorem 2 and Theorem 3), Pant, Vyomesh [8] and Singh and Kumar [11]. In Theorem 3 we use the aspect of noncompatible maps in place of the property (E.A) and show that the mappings are discontinuous at their common fixed point. Thus Theorem 3 provides one more answer to the problem regarding the existence of contractive definition which is strong enough to guarantee the existence of common fixed point but does not force the maps to become continuous ([10]).

2. Main Results

Theorem 1. *Let f and g be selfmappings of a complete metric space (X, d) such that*

(i) $\overline{fX} \subset gX$, *where* $\overline{fX}$ *denotes the closure of range of the mapping* f,
(ii)

$$d(fx, fy) < \max\{d(gx, gy), k\frac{[d(fx, gx) + d(fy, gy)]}{2}, \frac{[d(fy, gx) + d(fx, gy)]}{2}\} \quad 1 \leq k < 2,$$

whenever the right hand side is positive. If f and g be R-weakly commuting of type of type (A_g) satisfying the property (E.A), then f and g have a unique common fixed point.

Proof. Since f and g satisfy the property (E.A), there exists a sequence $\{x_n\}$ in X such that

$$\lim_n fx_n = \lim_n gx_n = t \tag{1}$$

for some t in X. Since $t \in \overline{fX} \subset gX$, there exists some point u in X such that $t = gu$ where $t = \lim_n gx_n$. If $fu \neq gu$, the inequality

$$d(fx_n, fu) < \max\{d(gx_n, gu), k\frac{[d(fx_n, gx_n) + d(fu, gu)]}{2}, \frac{[d(fu, gx_n) + d(fx_n, gu)]}{2}\},$$

On letting $n \to \infty$ yields

$$d(gu, fu) < \max\{d(gu, gu), k\frac{[d(gu, gu) + d(fu, gu)]}{2}, \frac{[d(fu, gu) + d(gu, gu)]}{2}\},$$
$$= k\frac{[d(fu, gu)]}{2} < k(fu, gu),$$

a contradiction. Hence $fu = gu$.

Since f and g are R - weak commutating of type (A_g), we get

$$d(ffu, gfu) \leq R(d(fu, gu)) = 0$$

that is, $ffu = gfu$. If $fu \neq ffu$, using (ii) again, we get

$$d(fu, ffu) < \max\{d(gu, gfu), k\frac{[d(fu, gu) + d(ffu, gfu)]}{2},$$
$$\frac{[d(ffu, gu) + d(gu, gfu)]}{2}\} = k\frac{[d(fu, gfu)]}{2} < d(fu, ffu),$$

a contradiction. Hence $fu = ffu = gfu$ and fu is a common fixed point of f and g. Uniqueness of the common fixed point follows easily. Hence the theorem. □

We now give an example to illustrate the above theorem.

Example 3. Let $X = [2, 20]$ and d be the usual metric on X. Define $f, g\colon X \to X$ as

$$f(x) = \begin{cases} 2 & \text{if } x = 2 \text{ or } > 5 \\ 6 & \text{if } 2 < x \leq 5 \end{cases}$$
$$g(x) = \begin{cases} 2 & \text{if } x = 2 \\ 12 & \text{if } 2 < x \leq 5 \\ \frac{(x+1)}{3} & \text{if } x > 5. \end{cases}$$

Then f and g satisfy all the conditions of the above theorem and have a unique common fixed point at $x = 2$. It can also be verified that f and g are R - weakly commuting of type (A_g) mappings and satisfy the property $(E.A)$.

As our next result, we generalize the above theorem and prove a common fixed point theorem for four mappings.

Theorem 2. *Let (A, S) and (B, T) be selfmaps of a metric space (X, d) satisfying the conditions*

(i) $$AX \subset TX, BX \subset SX,$$

$$\text{(ii)}\quad d(Ax,By) < \max\{d(Sx,Ty), \frac{k}{2}[d(Ax,Sx)+d(By,Ty)], \frac{1}{2}[d(Ax,Ty)+d(By,Sx)]\} \quad 1 \le k < 2,$$

Let one of the mapping pairs (A,S) or (B,T) be pointwise R - weakly commuting of type (A_g) satisfying the property (E.A). If the range of one of the mappings be a complete subspace of X then A, B, S and T have a unique common fixed point.

Proof. Let B and T satisfy the property (E.A). Then there exists a sequence $\{x_n\}$ in X such that $Bx_n \to t$ and $Tx_n \to t$ for some t in X. Since $BX \subset SX$, for each x_n there exists y_n in X such that $Bx_n = Sy_n$. Thus $Bx_n \to t$, $Tx_n \to t$ and $Sy_n \to t$. We claim that $Ay_n \to t$. If not, there exists a subsequence $\{Ay_m\}$ of $\{Ay_n\}$, a positive integer M and a number $r > 0$ such that for each $m \ge M$ we have

$$d(Ay_m,t) \ge r, \quad d(Ay_m,Bx_m) \ge r,$$

$$d(Ay_m,Bx_m) < \max\{d(Sy_m,Tx_m), \frac{k}{2}[d(Ay_m,Sy_m)+d(Bx_m,Tx_m)] \frac{1}{2}[d(Ay_m,Tx_m)+d(Bx_m,Sy_m)]\} < d(Ay_m,Sy_m) = [d(Ay_m,Sx_m)],$$

a contradiction. Hence $Ay_n \to t$.

Suppose SX is a complete subspace of X. Then, since $Sy_n \to t$, there exists a point u in X such that $t = Su$. If $Au \ne Su$, the inequality

$$d(Au,Bx_n) < \max\{d(Su,Tx_n), \frac{k}{2}[d(Au,Su)+d(Bx_n,Tx_n)] \frac{1}{2}[d(Au,Tx_n)+d(Bx_n,Su)]\}$$

on making $n \to \infty$ yields $d(Au,Su) = \frac{1}{2}[d(Au,Su)]$, a contradiction. Hence $Au = Su$. Since A and S are Pointwise R-weak commutative of type (A_g) maps, there exists $R_1 > 0$ such that $d(AAu,SAu) \le R_1 d(Au,Su) = 0$, that is $AAu = SAu$ and $AAu = ASu = SAu = SSu$. Since $AX \subset TX$, there exists a point w in X such that $Au = Tw$. We assert that $Tw = Bw$. If $Bw \ne Tw$, then by (ii), we get

$$d(Au,Bw) < \max\{d(Su,Tw), \frac{k}{2}[d(Au,Su)+d(Bw,Tw)], \frac{1}{2}[d(Au,Tw)+d(Bw,Su)]\} = \frac{1}{2}d(Bw,Au) < d(Bw,Au),$$

a contradiction. Hence $Au = Bw = Tw = Su$. Pointwise R-weak commutativity of type (A_g) of B and T implies that $BBw = TBw$ and $BBw = BTw = TBw = TTw$. Now if $Au \ne AAu$, then by (ii), we get

$$d(Au, AAu) = d(AAu, Bw) < \max\{d(SAu, Tw),$$
$$\frac{k}{2}[d(AAu, SAu)+d(Bw, Tw)], \frac{1}{2}[d(AAu, Tw)+d(Bw, SAu)]\} = d(AAu, Au),$$

a contradiction. Thus $Au = AAu = SAu$ and Au is a common fixed point of A and S. Similarly $Au = Bw$ is a common fixed point of B and T. The proof is similar when TX is assumed to be a complete subspace of X. The cases that AX or BX be complete subspace of X are similar to the cases that TX or SX respectively be complete since $AX \subset TX$ and $BX \subset SX$. Uniqueness of the common fixed point follows easily. Hence the theorem. □

We now give an example to illustrate the above theorem.

Example 4. Let $X = [2, 20]$ with the usual metric d. Define A, B, S and $T\colon X \to X$, as follows,

$$Ax = 2 \text{ for all } x,$$
$$Bx = \begin{cases} 2 & \text{if, } x = 2 \text{ or } > 5 \\ 8 & \text{if, } 2 < x \leq 5, \end{cases}$$
$$Sx = \begin{cases} x & \text{if, } x \leq 8 \\ 8 & \text{if, } x > 8, \end{cases}$$
$$T2 = 2,$$
$$Tx = \begin{cases} \text{12+x} & \text{if, } 2 < x \leq 5 \\ \text{x-3} & \text{if, } x > 5. \end{cases}$$

Then A, B, S and T satisfy all the conditions of the Theorem 2.2 and have a unique common fixed point at $x = 2$.

It can be verified in the above example that B and T are R-weakly commuting type (A_g) maps and satisfy the property (E.A).

Remark 1. Singh and Kumar [11] have assumed that the mapping pairs commute at their coincidence point. In view of the discussion in the introductory section, the condition of Singh and Kumar [11] is equivalent to the condition that the mappings are assumed R-weekly commuting. Our theorems, therefore, improve the results of Singh and Kumar [11].

Above two theorems have been proved by using the (E.A) property. The (E.A) property was introduced by Aamri and Moutawakil [1] by generalizing the notion of noncompatible maps introduced by Pant [5]. It is, however, pertinent to mention here that if we replace the notion of noncompatibility by the (E.A) property, we get a contractive condition which ensures the existence of common fixed point for mappings which are discontinuous at the common fixed point. Thus we provide one more answer to the problem of Rhoades [10]. We show this in the following theorem.

Theorem 3. *Let (A, S) and (B, T) be pointwise R-weakly commuting selfmaps of type (A_g) of a metric space (X, d) satisfying the conditions*

(i) $$AX \subset TX, BX \subset SX,$$

(ii) $$d(Ax, By) < \max\{d(Sx, Ty), \frac{k}{2}[d(Ax, Sx) + d(By, Ty)] \\ \frac{1}{2}[d(Ax, Ty) + d(By, Sx)]\} \quad 1 \le k < 2$$

If mappings in one of the pairs (A, S) or (B, T) be noncompatible and the range of one of the mappings be a complete subspace of X then A, B, S and T have a unique common fixed point and the fixed point is a point of discontinuity.

Proof. First suppose that B and T be noncompatible maps. Then there exists a sequence $\{x_n\}$ in X such that

(2) $$\lim_n Bx_n = t \text{ and } \lim_n Tx_n = t$$

for some $t \in X$ but $\lim_n d(BTx_n, TBx_n)$ is either nonzero or non existent. Since $BX \subset SX$, for each x_n there exists $y_n \in X$ such that $Bx_n = Sy_n$. Thus $Bx_n \to t, Tx_n \to t$ and $Sy_n \to t$. We claim that $Ay_n \to t$. If not, there exists a subsequence $\{Ay_m\}$ of $\{Ay_n\}$, a positive integer M and a number $r > 0$ such that for each $m \ge M$ we have

$$d(Ay_m, t) \ge r, d(Ay_m, Bx_m) \ge r$$

and

$$d(Ay_m, Bx_m) < \max\{d(Sy_m, Tx_m), \frac{k}{2}[d(Ay_m, Sy_m) + d(Bx_m, Tx_m)], \\ \frac{1}{2}[d(Ay_m, Tx_m) + d(Bx_m, Sy_m)]\} < d(Ay_m, Sy_m) = [d(Ay_m, Bx_m)],$$

a contradiction. Hence $Ay_n \to t$.

Suppose that SX is a complete subspace of X. Then, since $Sy_n \to t$, there exists a point $u \in X$ such that $t = Su$. If $Au \ne Su$, the inequality

$$d(Au, Bx_n) < \max\{d(Su, Tx_n), \frac{k}{2}[d(Au, Su) + d(Bx_n, Tx_n)], \\ \frac{1}{2}[d(Au, Tx_n) + d(Bx_n, Su)]\}$$

on making $n \to \infty$ yields $d(Au, Su) \le \frac{1}{2}[d(Au, Su)]$, a contradiction. Hence $Au = Su$. Since A and S are pointwise R-weak commutative mappings of type (A_g); there exists $R_1 > 0$ such that $d(AAu, SAu) \le R_1 d(Au, Su) = 0$, that is $AAu = SAu$ and $AAu = ASu = SAu = SSu$. Since $AX \subset TX$, there exists a point $w \in X$ such that $Au = Tw$. We assert that $Tw = Bw$. If $Bw \ne Tw$, then by (ii), we get

$$d(Au, Bw) < \max\{d(Su, Tw), \frac{k}{2}[d(Au, Su) + d(Bw, Tw)],$$
$$\frac{1}{2}[d(Au, Tw) + d(Bw, Su)]\} = d(Bw, Au)/2 < d(Bw, Au),$$

a contradiction. Hence $Au = Bw = Tw = Su$. Pointwise R-weak commutativity of type (A_g) of B and T implies that $BBw = TBw$ and

$$BBw = BTw = TBw = TTw.$$

Now if $Au \neq AAu$, then by (ii), we get

$$d(Au, AAu) = d(AAu, Bw) < \max\{d(SAu, Tw),$$
$$\frac{k}{2}[d(AAu, SAu)+d(Bw, Tw)], \frac{1}{2}[d(AAu, Tw)+d(Bw, SAu)]\} = d(AAu, Au),$$

a contradiction. Thus $Au = AAu = SAu$ and Au is a common fixed point of A and S. Similarly $Au = Bw$ is a common fixed point of B and T. The proof is similar when TX is assumed to be a complete subspace of X. The cases that AX or BX be complete subspace of X are similar to the cases that TX or SX respectively be complete since $AX \subset TX$ and $BX \subset SX$. Uniqueness of the common fixed point follows easily.

We now show that the mappings are discontinuous at the common fixed point. If possible, first suppose B is continuous at the common fixed point $t = Au = Bw$. Then considering the sequence $\{x_n\}$ as assumed in (2) we get $\lim_n BBx_n = Bt = t$ and $\lim_n BTx_n = Bt = t$. R-weak commutativity of type (A_g) now implies that $d(BBx_n, TBx_n) \leq Rd(Bx_n, Tx_n)$. On letting $n \to \infty$ this yields $\lim_n TBx_n = Bt = t$. This, in turn, yields $\lim_n d(BTx_n, TBx_n) = d(Bt, Bt) = 0$. This contradicts the fact that $\lim_n d(BTx_n, TBx_n)$ is either nonzero or non existent for the sequence $\{x_n\}$ of (2). Hence B is discontinuous at the fixed point. Next, suppose that T is continuous. Then for the aforesaid sequence $\{x_n\}$ new get $\lim_n TBx_n = Tt = t$ and $\lim_n TTx_n = Tt = t$. In view of these limits, the inequality,

$$d(At, BTx_n) < \max\{d(St, TTx_n), \frac{k}{2}[d(At, St) + d(BTx_n, TTx_n)],$$
$$\frac{1}{2}[d(At, TTx_n) + d(BTx_n, St)]\}$$

yields a contradiction unless $\lim_n BTx_n = TTx_n = Tt = t$. But

$$\lim_n BTx_n = Tt = t$$

and $\lim_n TBx_n = Tt = t$ contradicts the fact that $\lim_n d(BTx_n, TBx_n)$ is either nonzero or non existent. Thus both B and T are discontinuous at their common fixed point. Similarly it can be shown that A and S are also discontinuous at the common fixed point. Thus all the A, B, S and T are discontinuous at the common fixed point. This establishes the theorem. □

We now illustrate the above theorem by way of the following example.

Example 5. Let $X = [2, 20]$ and d be the usual metric on X. Define $A, B, S, T : X \to X$ by

$$Ax = \begin{cases} 2 & \text{if, } x = 2 \\ 3 & \text{if, } x > 2, \end{cases}$$

$$Bx = \begin{cases} 2 & \text{if, } x = 2 \text{ or } \geq 5 \\ 6 & \text{if, } 2 < x < 5, \end{cases}$$

$$x = \begin{cases} 2 & \text{if, } x = 2 \\ 6 & \text{if, } x > 2, \end{cases}$$

$$Tx = \begin{cases} 2 & \text{if, } x = 2 \\ 7 + x & \text{if, } 2 < x < 5 \\ \frac{1+x}{2} & \text{if, } x \geq 5. \end{cases}$$

Then A, B, S and T satisfy all the conditions of above theorem and have a unique common fixed point $x = 2$. It can be verified in this example that A, B, S and T satisfy contractive condition of the above theorem. It can also be seen that A and S satisfy the property (E.A) and all the mappings A, B, S and T are discontinuous at the common fixed point.

References

[1] M. Aamri and D. El Moutawakil. Some new common fixed point theorems under strict contractive conditions. *J. Math. Anal. Appl.*, 270(1):181–188, 2002.

[2] G. Jungck. Compatible mappings and common fixed points. II. *Internat. J. Math. Math. Sci.*, 11(2):285–288, 1988.

[3] R. P. Pant. Common fixed points of noncommuting mappings. *J. Math. Anal. Appl.*, 188(2):436–440, 1994.

[4] R. P. Pant. Common fixed point theorems for contractive maps. *J. Math. Anal. Appl.*, 226(1):251–258, 1998.

[5] R. P. Pant. Discontinuity and fixed points. *J. Math. Anal. Appl.*, 240(1):284–289, 1999.

[6] R. P. Pant and V. Pant. Common fixed points under strict contractive conditions. *J. Math. Anal. Appl.*, 248(1):327–332, 2000.

[7] R. P. Pant, V. Pant, and K. Jha. Note on: "Common fixed points under strict contractive conditions" [J. Math. Anal. Appl. **248** (2000), no. 1, 327–332; MR1757544 (2001b:34053)] by Pant and Pant. *J. Math. Anal. Appl.*, 274(2):879–880, 2002.

[8] V. Pant. Common fixed points of R-weakly commuting maps of type (A_g). *J. Indian Math. Soc. (N.S.)*, 73(1-2):65–70, 2006.

[9] H. K. Pathak, Y. J. Cho, and S. M. Kang. Remarks on R-weakly commuting mappings and common fixed point theorems. *Bull. Korean Math. Soc.*, 34(2):247–257, 1997.

[10] B. E. Rhoades. Contractive definitions and continuity. In *Fixed point theory and its applications (Berkeley, CA, 1986)*, volume 72 of *Contemp. Math.*, pages 233–245. Amer. Math. Soc., Providence, RI, 1988.

[11] S. L. Singh and A. Kumar. Common fixed point theorems for contractive maps. *Mat. Vesnik*, 58(3-4):85–90, 2006.

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