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INDIVISIBILITY OF CLASS NUMBERS OF REAL QUADRATIC FIELDS

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Abstract. Let N denote the sets of positive integers and $D \in N$ be square free, and let χ_D, $h = h(D)$ denote the non-trivial Dirichlet character, the class number of the real quadratic field $K = Q(\sqrt{D})$, respectively. Let $L(s, \chi_D)$ denote the L-function attached to χ_D. In this paper, by using an upper bound different from current bounds for $L(1, \chi_D)$ and applying Dirichlet's class number formula, we will show indivisibility of $h = h(D)$ by any prime.

1. Introduction

Let Z, N, Q denote the sets of integers, positive integers and rational numbers, respectively. Throughout $D \in N$ will be assumed square free, $K = Q(\sqrt{D})$ will denote the real quadratic field, and the class number of K will be denoted by $h = h(D)$. Let Δ denote the discriminant, ε_D the fundamental unit of K.

One of the main problems deals with the structure of class group of K, and so one naturally studies the divisibility of $h(D)$ by primes. Many authors have studied such fields and considered generalizations thereof. Cohen-Lenstra are among them. They predicted that there are infinitely many real quadratic fields $K = Q(\sqrt{D})$ whose class numbers are indivisible by any prime p [1].

In this paper, we consider real quadratic fields and we will prove the following theorem.

Theorem. *Let $p > 3$ be a prime. If $p \equiv 1 \pmod 4$, then the class numbers of real quadratic fields $K = Q(\sqrt{p^2 - 4})$ and $K = Q(\sqrt{p^2 - 2})$ are indivisible by p.*

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2. Preliminaries

In order to prove above theorem we obtain an upper bound different from current bounds for the class numbers of the real quadratic fields and we need the following lemma for this.

Lemma 2.1. *i. If D is a prime with $D \equiv 1 \pmod 4$, we have*

$$\varepsilon_D > \begin{cases} \sqrt{D \mp 1}, & D = n^2 \mp 4, (n \in Z) \\ \sqrt{4D \mp 1}, & \textit{in the other cases.} \end{cases}$$

ii. If D is a prime with $D \equiv 3 \pmod 4$, we have

$$\varepsilon_D > \begin{cases} 2D \mp 1, & D = n^2 \mp 2, \\ 8D \mp 1, & \textit{in the other cases.} \end{cases}$$

Proof. i. Let

$$\varepsilon_D = \frac{t + u\sqrt{D}}{2} > 1$$

be the fundamental unit of $K = Q(\sqrt{D})$. Since ε_D is equal to the fundamental solution of the Pell's equation $x^2 - Dy^2 = \mp 4$, then we can write

$$\varepsilon_D{}^2 = (\frac{t + u\sqrt{D}}{2})^2 = \frac{1}{4}(\sqrt{Du^2 \mp 4} + u\sqrt{D})^2 > Du^2 \mp 1 \geq \begin{cases} D \mp 1, & u = 1 \\ 4D \mp 1, & u > 1. \end{cases}$$

Therefore, we have

$$\varepsilon_D > \begin{cases} \sqrt{D \mp 1}, & D = n^2 \mp 4 \\ \sqrt{4D \mp 1}, & \text{in the other cases.} \end{cases}$$

ii. If $D \equiv 3 \pmod 4$, then we have

$$\varepsilon_D{}^2 = \left(\frac{t + u\sqrt{D}}{2}\right)^2$$

from the least positive integer solution $(x, y) = (t, u)$ of the Pell's equation $x^2 - Dy^2 = \mp 2$ [5]. Similarly, we can write

$$\varepsilon_D > \begin{cases} 2D \mp 1, & D = n^2 \mp 2, \quad (n: \text{ odd integer }) \\ 8D \mp 1, & \text{in the other cases.} \end{cases}$$

□

Lemma 2.2. *Let $D \in N$ be square free, then $h(D) < \sqrt{D}$.*

In order to prove this we need the following Lemma [4].

Lemma 2.3. *Let γ be Euler's constant, then*

$$|L(1,\chi_D)| \le \begin{cases} \frac{1}{4}(\log\Delta + 2 + \gamma - \log\pi), & 2 \mid \Delta, \\ \frac{1}{2}(\log\Delta + 2 + \gamma - \log 4\pi), & \textit{otherwise.} \end{cases}$$

Proof of Lemma 2.2. By Dirichlet's class number formula, we have

$$h(D) = \frac{\sqrt{\Delta}}{2\log\varepsilon_D}|L(1,\chi_D)|$$

where Δ is a fundamental discriminant of a quadratic field define by

$$\Delta = \begin{cases} 4D, & D \equiv 2,3 \pmod 4 \\ D, & D \equiv 1 \pmod 4). \end{cases}$$

First, we consider the case $D \equiv 1 \pmod 4$ and $D = n^2 \mp 4$. Thus, by the upper bound for $L(1,\chi_D)$ in Lemma 2.3, we have that

$$h(D) < \frac{\sqrt{D}(\log D + 1,478)}{4\log\sqrt{D \mp 1}} = \frac{\sqrt{D}(\log D + 1,478)}{2\log(D \mp 1)} < \sqrt{D}, \quad (D > 5).$$

Moreover, we can write

$$h(D) \le \left[\!\!\left[\frac{\sqrt{D}(\log D + 1,478)}{2\log(D \mp 1)} \right]\!\!\right]$$

where $[\![x]\!]$ is the greatest integer less than or equal to x. It is also $h(D) < \sqrt{D}$ for $D \ne n^2 \mp 4$.

Now, we consider the case $D \equiv 3 \pmod 4$ and $D = n^2 \mp 2$ ($n \in Z$ is odd). Similarly, by applying Lemmas 2.1, 2.3 and using class number formula, we get

$$h(D) < \frac{\sqrt{D}(\log 4D + 1,478)}{2\log(2D \mp 1)} < \sqrt{D} \quad \text{and} \quad h(D) \le \left[\!\!\left[\frac{\sqrt{D}(\log 4D + 1,478)}{2\log(2D \mp 1)} \right]\!\!\right]$$

It is also true for $D \ne n^2 \mp 2$. □

3. Proof of Theorem

Specially, if we write Ds depend on prime p in the forms of $D = p^2 - 4$, $D = p^2 - 2$ we can immediately see that $h(D) < p$ for the class numbers of real quadratic fields $K = Q(\sqrt{p^2 - 4})$, $K = Q(\sqrt{p^2 - 2})$ from Lemma 2.2. Therefore we have $h(D) \not\equiv 0 \pmod p$. This prove that $h(D)$ is indivisible by prime p with $p \equiv 1 \pmod 4$ for above mentioned real quadratic fields.

Corollary 3.1. *Let $D \equiv 1,3 \pmod 4$ be a prime satisfying $D = p^2 - r$ $(r \mid 4p, r \in (-p,p])$ then the class numbers of $K = Q(\sqrt{p^2 - r})$ are indivisible by prime p with $p \equiv 1 \pmod 4$.*

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