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SPECIAL REPRESENTATIONS OF SOME SIMPLE GROUPS WITH MINIMAL DEGREES

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Abstract. If F is a subfield of C, then a square matrix over F with non-negative integral trace is called a quasi-permutation matrix over F. For a finite group G, let $q(G)$ and $c(G)$ denote the minimal degree of a faithful representation of G by quasi-permutation matrices over the rational and the complex numbers, respectively. Finally $r(G)$ denotes the minimal degree of a faithful rational valued complex character of G. In this paper $q(G), c(G)$ and $r(G)$ are calculated for Suzuki group and untwisted group of type B_2 with parameter 2^{2n+1}.

1. Introduction

In [12] Wong defined a quasi-permutation group of degree n, to be a finite group G of automorphisms of an n-dimensional complex vector space such that every element of G has non-negative integral trace. The terminology drives from the fact that if G is a finite group of permutations of a set Ω of size n, and we think of G as acting on the complex vector space with basis Ω, then the trace of an element $g \in G$ is equal to the number of points of Ω fixed by g. Wong studied the extent to which some facts about permutation groups generalize to the quasi-permutation group situation. In [2] Hartley with their colleague investigated further the analogy between permutation groups and quasi-permutation groups by studying the relation between the minimal degree of a faithful permutation representation of a given finite group G and the minimal degree of a faithful quasi-permutation representation. They also worked over the rational field and found some interesting results. We shall often prefer to work over the rational field rather than the complex field.

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By a quasi-permutation matrix we mean a square matrix over the complex field C with non-negative integral trace. Thus every permutation matrix over C is a quasi-permutation matrix. For a given finite group G, let $q(G)$ denote the minimal degree of a faithful representation of G by quasi-permutation matrices over the rational field Q, and let $c(G)$ be the minimal degree of a faithful representation of G by complex quasi-permutation matrices.

By a rational valued character we mean a character χ corresponding to a complex representation of G such that $\chi(g) \in Q$ for all $g \in G$. As the values of the character of a complex representation are algebraic numbers, a rational valued character is in fact integer valued. A quasi-permutation representation of G is then simply a complex representation of G whose character values are rational and non-negative. The module of such a representation will be called a quasi-permutation module. We will call a homomorphism from G to $GL(n, Q)$ a rational representation of G and its corresponding character will be called a rational character of G. Let $r(G)$ denote the minimal degree of a faithful rational valued character of G. It is easy to see that for a finite group G the following inequalities hold

$$r(G) < c(G) \leq q(G).$$

It is easy to see that if G is a symmetric group of degree 6, then $r(G) = 5$ and $c(G) = q(G) = 6$. If G is the quaternion group of order 8, then $r(G) = 2, c(G) = 4$ and $q(G) = 8$. Our principal aim in this paper is to investigate these quantities and inequalities further.

Finding the above quantities have been carried out in some papers, for example in [6, 5, 4] we found these for the groups $GL(2, q)$, $SU(3, q^2)$, $PSU(3, q^2)$, $SL(3, q)$ and $PSl(3, q)$.

In this paper we will apply the algorithms in [1] for the Suzuki group and untwisted group of type B_2 with parameter 2^{2n+1}.

2. Background

Let G be a finite group and χ be an irreducible complex character of G. Let $m_Q(\chi)$ denote the Schur index of χ over Q. Let $\Gamma(\chi)$ be the Galois group of $Q(\chi)$ over Q. It is known that

$$\sum_{\alpha \in \Gamma(\chi)} m_Q(\chi)\chi^{\alpha} \tag{1}$$

is a character of an irreducible QG-module ([9, Corollary 10.2 (b)]. So by knowing the character table of a group and the Schur indices of each of the irreducible characters of G, we can find the irreducible rational characters of G.

We can see all the following statements in [1].

Definition 1. Let χ be a character of G such that, for all $g \in G$, $\chi(g) \in Q$ and $\chi(g) \geq 0$. Then we say that χ is a non-negative rational valued character.

Definition 2. Let G be a finite group. Let χ be an irreducible complex character of G. Then we define

(1) $d(\chi) = |\Gamma(\chi)|\chi(1)$

(2) $m(\chi) = \begin{cases} 0 & \text{if } \chi = 1_G \\ |\min\{\sum_{\alpha\in\Gamma(\chi)}\chi^\alpha(g) : g \in G\}| & \text{otherwise} \end{cases}$

(3) $c(\chi) = \sum_{\alpha\in\Gamma(\chi)}\chi^\alpha + m(\chi)1_G$.

Lemma 1. *Let χ be a character of G. Then* $\operatorname{Ker}\chi = \operatorname{Ker}\sum_{\alpha\in\Gamma(\chi)}\chi^\alpha$*. Moreover χ is faithful if and only if $\sum_{\alpha\in\Gamma(\chi)}\chi^\alpha$ is faithful.*

Lemma 2. *Let $\chi \in \operatorname{Irr}(G)$, then $\sum_{\alpha\in\Gamma(\chi)}\chi^\alpha$ is a rational valued character of G. Moreover $c(\chi)$ is a non-negative rational valued character of G and $c(\chi)(1) = d(\chi) + m(\chi)$.*

Now according to [1, Corollary 3.11] and above statements the following Corollary is useful for calculation of $r(G)$, $c(G)$ and $q(G)$.

Corollary 1. *Let G be a finite group with a unique minimal normal subgroup. Then*

(1) $r(G) = \min\{d(\chi) : \chi$ *is a faithful irreducible complex character of* $G\}$

(2) $c(G) = \min\{c(\chi)(1) : \chi$ *is a faithful irreducible complex character of* $G\}$

(3) $q(G) = \min\{m_Q(\chi)c(\chi)(1) : \chi$ *is a faithful irreducible complex character of* $G\}$.

Lemma 3. *Let* $\chi \in \operatorname{Irr}(G)$ $\chi \neq 1_G$. *Then* $c(\chi)(1) \geq d(\chi) + 1 \geq \chi(1) + 1$.

Lemma 4. *Let $\chi \in \operatorname{Irr}(G)$. Then*

(1) $c(\chi)(1) \geq d(\chi) \geq \chi(1)$;

(2) $c(\chi)(1) \leq 2d(\chi)$. *Equality occurs if and only if $Z(\chi)/ker\chi$ is of even order.*

Lemma 5. *Let G be a finite group. If the Schur index of each non-principal irreducible character is equal to m, then $q(G) = mc(G)$.*

3. Calculation of $q(G), c(G)$ and $r(G)$ for the group $G = B_2(q)$

The group $G = B_2(q)$ is of order $\frac{q^4(q^4-1)(q^2-1)}{(2,q-1)}$ and if the characteristic of K is two, the Lie algebras of type B_n and of type C_n are isomorphic. The complex character table of $B_2(q)$ is given in [7] as in Table 1.

Table 1. Character table of $B_2(q)$

	A_1	A_2	A_{31}	A_{32}	A_{41}	A_{42}	$B_1(i,j)$	$B_2(i)$	$B_3(i,j)$
θ_1	$q(q+1)^2/2$	$q(q+1)/2$	$q(q+1)/2$	$q/2$	$q/2$	$-q/2$	2	0	0
θ_4	q^4	0	0	0	0	0	1	-1	-1
θ_5	$q(q-1)^2/2$	$-q(q-1)/2$	$-q(q-1)/2$	$q/2$	$q/2$	$-q/2$	0	0	0
$\chi_1(k,l)$	$(q+1)^2(q^2+1)$	$(q+1)^2$	$(q+1)^2$	$2q+1$	1	1	$\alpha_{ik}\alpha_{jl}+\alpha_{il}\alpha_{jk}$	0	0
$\chi_4(k,l)$	$(q-1)^2(q^2+1)$	$(q-1)^2$	$(q-1)^2$	$-(2q-1)$	1	1	0	0	0
χ_k	$(q^2-1)^2$	$-(q^2-1)$	$-(q^2-1)$	1	1	1	0	0	0

	$B_5(i)$	$C_1(i)$	$C_2(i)$	$C_3(i)$	$C_4(i)$
θ_1	-1	$q+1$	$q+1$	0	0
θ_4	1	q	q	$-q$	$-q$
θ_5	1	0	0	$q-1$	$q-1$
$\chi_1(k,l)$	0	$(q+1)(\alpha_{ik}+\alpha_{il})$	$(q+1)\alpha_{ik}\alpha_{il}$	0	0
$\chi_4(k,l)$	0	0	0	$-(q-1)(\beta_{ik}+\beta_{il})$	$-(q-1)\beta_{ik}\beta_{il}$
χ_k	$\tau^{ik}+\tau^{-ik}+\tau^{ikq}+\tau^{-ikq}$	0	0	0	0

	$B_4(i,j)$	$D_1(i)$	$D_2(i)$	$D_3(i)$	$D_4(i)$
θ_1	0	1	1	0	0
θ_4	1	0	0	-1	-1
θ_5	-2	0	0	-1	-1
$\chi_1(k,l)$	0	$\alpha_{ik}+\alpha_{il}$	$\alpha_{ik}\alpha_{il}$	0	0
$\chi_4(k,l)$	$\beta_{ik}\beta_{jl}+\beta_{il}\beta_{jk}$	0	0	$\beta_{ik}+\beta_{il}$	$\beta_{ik}\beta_{il}$
χ_k	0	0	0	0	0

TABLE 2

χ	$d(\chi)$	$c(\chi)(1)$
θ_1	$\frac{q(q+1)^2}{2}$	$\frac{q(q^2+2q+2)}{2}$
θ_4	q^4	$q(q^3+1)$
θ_5	$\frac{q(q-1)^2}{2}$	$\frac{q^2(q-1)}{2}$
$\chi_1(k,l)$	$\geq (q+1)^2(q^2+1)$	$\geq (q+1)^2(q^2+1)+1$
$\chi_4(k,l)$	$\geq (q-1)^2(q^2+1)$	$\geq q^2(q^2-2q+2)$
$\chi_5(k)$	$\geq (q^2-1)^2$	$\geq q^2(q^2-1)$

Theorem 1. *Let $G = B_2(2)$, then*

$$r(B_2(2)) = 5,\ c(B_2(2)) = 6.$$

Proof. We know that $B_2(q) \cong S_6$, and by the Atlas of finite groups [6],it is easy to see that

$$r(B_2(2)) = 5, c(B_2(2)) = q(B_2(2)) = 6.$$

□

Theorem 2. *Let $G = B_2(q), q \neq 2$, then*

(1) $r(G) = \frac{q(q-1)^2}{2}$
(2) $c(G) = \frac{q^2(q-1)}{2}$

Proof. The group $B_2(q)$, $q \neq 2$ is simple so their non-trivial irreducible characters are faithful and therefore we need to look at each faithful irreducible character χ say and calculate $d(\chi), c(\chi)(1)$.

By the Table 1, we know that $\theta_1, \theta_4, \theta_5$ are rational valued characters, so by Definition 2.2 and Lemma 2.4 we have $d(\theta_1) = |\Gamma(\theta_1)|\theta_1(1) = \frac{q(q+1)^2}{2}$ and $m(\theta_1) = -\frac{q}{2}$ and so $c(\theta_1(1)) = \frac{q(q^2+2q+2)}{2}$.

$d(\theta_4) = |\Gamma(\theta_4)|\theta_4(1) = q^4$ and $m(\theta_4) = -q$ and then $c(\theta_4)(1) = q(q^3+1)$.

$d(\theta_5) = |\Gamma(\theta_5)|\theta_5(1) = \frac{q(q-1)^2}{2}$ and $m(\theta_5) = -\frac{q(q-1)}{2}$ and therefore

$$c(\theta_5)(1) = \frac{q^2(q-1)}{2}.$$

For other characters by Lemmas 2.6, 2.7 we have

$$d(\chi_1(k,l)) = |\Gamma(\chi_1(k,l))|\chi_1(k,l)(1) \geq (q+1)^2(q^2+1)$$

and $m(\chi_1(k,l)) \geq 1$ and so $c(\chi_1(k,l))(1) \geq (q+1)^2(q^2+1)+1$.

$d(\chi_4(k,l)) \geq (q-1)^2(q^2+1)$ and $m(\chi_4(k,l)) \geq 2q-1$ and so

$$c(\chi_4(k,l))(1) \geq q^2(q^2-2q+2).$$

$d(\chi_5(k)) \geq (q^2-1)^2$ and $m(\chi_5(k)) \geq q^2-1$ and so $c(\chi_5(k))(1) \geq q^2(q^2-1)$.

An overall picture is provided by the Table 2

Now by Corollary 2.5 and above table we obtain

$$\min\{d(\chi) : \chi \text{ is a faithful irreducible complex character of } G\} = \frac{q(q-1)^2}{2}$$

and

$$\min\{c(\chi)(1) : \chi \text{ is a faithful irreducible complex character of G}\} = \frac{q^2(q-1)}{2}.$$

□

4. Quasi-permutation representations of the group $Sz(q)$

A group G is called a (ZT)-group if :

(1) G is a doubly transitive group on $1+N$ symbols,
(2) the identity is the only element which leaves three distinct symbols invariant,
(3) G contains no normal subgroup of order $1+N$, and
(4) N is even.

There is a unique (ZT)-group of order $q^2(q-1)(q^2+1)$ for any odd power q of 2 (see [11, Theorem 8]). This group will be denoted here as $Sz(q)$ and called a Suzuki group. The Suzuki groups are simple for all $q > 2$.

By [10] the Suzuki group $G(q)$ is isomorphic to a subgroup of $SP_4(F_q)$ consisting of points left fixed by an involutive mapping of $SP_4(F_q)$ onto itself.

Now we shall identify $SP_4(K)^\sigma$ with the Suzuki group $G(q)$, where $SP_4(K)^\sigma$ is the set composed of all $x \in SP_4(K)$ such that $x^\sigma = x$.

Let $K = F_q, q = 2^{2n+1} (n \geq 1)$ and let θ be an automorphism of K defined by $\alpha \to \alpha^{2^n}, \alpha \in K$. It is easy to see that θ generates the Galois group of K over the prime field. Our purpose is to define an involutive mapping σ (which will not be an automorphism) of $SP_4(K)$ onto itself by making use of φ and θ so that the Suzuki group $G(q)$ is isomorphic to the subgroup $SP_4(K)^\sigma$ of $SP_4(K)$ consisting of matrices left fixed by σ.

Using Suzuki's notation, $G(q)$ is generated by $S(\alpha, \beta), M(\xi)$ and T:

$$S(\alpha,\beta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \alpha^\theta & 1 & 0 & 0 \\ \beta & \alpha & 1 & 0 \\ q(\alpha,\beta) & p(\alpha,\beta) & \alpha^\theta & 1 \end{pmatrix},$$

$M(\xi) = \mathrm{diag}(\xi^\theta, \xi^{1-\theta}, \xi^{\theta-1}, \xi^{-\theta})$,

$$T = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Define a matrix P by setting:

$$P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Then, one can easily verify that

$$PS(\alpha, \beta)P^{-1} = R(\alpha, \beta)^{-1},\ PM(\xi)P^{-1} = h(\xi^{\theta}),\ PTP^{-1} = J.$$

Thus $x \to PxP^{-1}$ gives an isomorphism $G(q) \cong SP_4(K)^{\sigma}$. So Suzuki group is a simple group of order $q^2(q-1)(q^2+1)$.

Remark 1. The involution $\sigma\colon SP_4(K) \to SP_4(K)$can not be an automorphism. For, if σ is so, then σ can be expressed as

$$x^{\sigma} = Ax^{\omega}A^{-1},$$

with $A \in GL_4(K)$ and an automorphism ω of K. Put $x = x_a(t) = I + tX_a$. Then $x^{\sigma} = x_b(t^{2\theta}) = I + t^{2\theta}X_b = I + t^{\omega}AX_aA^{-1}$. If we take $t = 1$, then $X_b = AX_aA^{-1}$. But this is absurd since $X_a = E_{12} - E_{43}$ is of rank 2 and $X_b = E_{24}$ is of rank 1.

The character table of $Sz(q)$ is computed in [11], is as follows:

TABLE 3. Character table of $Sz(q)$

	1	σ_0	ρ_0	ρ_0^{-1}	π_0^l	π_1^l	π_2^l
1	1	1	1	1	1	1	1
χ	q^2	0	0	0	1	-1	-1
ζ	$\theta(q-1)$	$-\theta$	$\theta\sqrt{-1}$	$-\theta\sqrt{-1}$	0	1	-1
$\overline{\zeta}$	$\theta(q-1)$	$-\theta$	$-\theta\sqrt{-1}$	$\theta\sqrt{-1}$	0	1	-1
ψ_i	q^2+1	1	1	1	$\varepsilon_0^i(\pi_0^l)$	0	0
μ_j	$(q-2\theta+1)(q-1)$	$2\theta-1$	-1	-1	0	$-\varepsilon_1^j(\pi_1^l)$	0
φ_k	$(q+2\theta+1)(q-1)$	$-2\theta-1$	-1	-1	0	0	$-\varepsilon_2^k(\pi_2^l)$

Where ε_0, $varepsilon_1$, ε_2 are primitive $q-1$, $q+2\theta+1$, $q-2\theta+1$-th root of 1, respectively.

In this table $q=2\theta^2$ and the ε_j^i are defined as follows:

$$\varepsilon_0^i(\xi_0^j)=\varepsilon_0^{ij}+\varepsilon_0^{-ij} \text{ for } i=1,2,\ldots,\frac{q}{2}-1$$

where ξ_0 is a generator of cyclic group of order $q-1$.

$$\varepsilon_1^i(\xi_1^k)=\varepsilon_1^{ik}+\varepsilon_1^{ikq}+\varepsilon_1^{-ik}+\varepsilon_1^{-ikq} \text{ for } i=1,2,\ldots,q+2\theta$$

where ξ_1 is a generator of cyclic group of order $q+2\theta+1$.

$$\varepsilon_2^i(\xi_2^k)=\varepsilon_2^{ik}+\varepsilon_2^{ikq}+\varepsilon_2^{-ik}+\varepsilon_2^{-ikq} \text{ for } i=1,2,\ldots,q+2\theta$$

where ξ_2 is a generator of cyclic group of order $q-2\theta+1$.

Lemma 6. *Let $G=Sz(q)$, $q=2^{2n+1}$, then all characters of G have Schur index 1.*

Proof. See [8, Theorem 9]. □

Theorem 3. *Let $G=Sz(q)$, $q=2^{2n+1}$, then $r(G)=2\theta(q-1)$, $c(G)=q(G)=2\theta q$, where $\theta=2^n$ and $q=2\theta^2$.*

Proof. Let $G=Sz(q)$, $q=2^{2n+1}$, by Lemma 4.1 the Schur index of every irreducible character is 1, therefor $c(G)=q(G)$. The groups $G=Sz(q)$ is simple,so their non-trivial irreducible characters are faithful and therefor we need to look at each faithful irreducible character ϑ say and calculate $d(\vartheta), c(\vartheta)(1)$.

By Table 3 we know χ is a rational valued character, so by Definition 2.2 and Lemma 2.4 we have:

$$d(\chi)=|\Gamma(\chi)|\chi(1)=q^2,$$

and $m(\chi)=1$, and so $c(\chi)(1)=q^2+1$.

For the character ζ we have $|\Gamma(\zeta)|=2$ and therefore:

$$d(\zeta)=|\Gamma(\zeta)|\zeta(1)=2\theta(q-1),$$

and $m(\zeta)=2\theta$, and so $c(\zeta)(1)=2\theta q$.

In this way, by Lemmas 2.6, 2.7 we have

$$d(\psi_i)\geq q^2+1$$

and $c(\psi_i)\geq q^2+2$,

$$d(\mu_j)\geq(q-2\theta+1)(q-1)$$

and $c(\mu_j)\geq q^2-2\theta q+2\theta$, $d(\varphi_k)\geq(q+2\theta+1)(q-1)$ and $c(\varphi_k)\geq q(q+2\theta)$.

The values are set out in the following table :

By observing the Corollary 2.5 and Table 4 we have:

$$\min\{d(\chi):\chi \text{ is a faithful irreducible complex character of } G\}=2\theta(q-1)$$

and

$$\min\{c(\chi)(1):\chi \text{ is a faithful irreducible complex character of } G\}=2\theta q.$$

TABLE 4

ϑ	$d(\vartheta)$	$c(\vartheta)(1)$
χ	q^2	q^2+1
ζ	$2\theta(q-1)$	$2\theta q$
ψ_i	$\geq q^2+1$	$> q^2+1$
μ_j	$\geq (q-2\theta+1)(q-1)$	$\geq q^2-2\theta q+2\theta$
φ_k	$(q+2\theta+1)(q-1)$	$\geq q(q+2\theta)$

Hence $r(G) = 2\theta(q-1)$, $c(G) = q(G) = 2\theta q$. □

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