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ON S-QUASINORMAL SUBGROUPS OF FINITE GROUP

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Abstract. A subgroup H of a group G is called S-quasinormal in G if it permutes with every Sylow subgroup of G. In this paper, we extend the study on the structure of a finite group under the assumption that some subgroups of $F^*(G)$ are S-quasinormal in G.

1. Introduction

Throughout this paper, all groups are finite. Recall that two subgroups A and B of a group G are said to permute if $AB = BA$. It is easily seen that A and B permute iff the set AB is a subgroup of G. A subgroup A of the group G is called quasinormal [19] or permutable [26, 8] in G if it permutes with all subgroups of G. The permutable subgroups have many interesting properties especially in the case when G is a finite group. It was observed by Ore [19] that every permutable subgroup H of a finite group G is subnormal. By extending this result, Ito and Szép have proved in [11] that for every permutable subgroup A of a finite group G, A/A_G is nilpotent. Here A_G is the kernel of A, that is the largest normal subgroup of G contained in A. Another important result related to Ore's result was obtained by Stonehewer in [26] in which he has proved that every permutable subgroup of every finitely generated group G is subnormal in G.

Some later, Maier and Schmid proved in [18] that for every permutable subgroup A of G it is true that $A^G/A_G \subseteq Z_\infty(G/A_G)$. Here A^G is the normal closure of A in G that is the intersection of all such normal subgroups of G which contain A. This result shows that "difference" between normality and permutability in general is small and several authors have investigated subgroups of finite groups

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which are permutable with all subgroups of some given system of subgroups. In this connection we first of all have to remind here about the following paper by Kegel [15] A subgroup A of a group G is called s-quasinormal if it permutes with all Sylow subgroups of G. It was discovered by Kegel [15] and Deskins [7] that subgroups of this kind have the properties similar to the properties of permutable subgroups and, in particular, they are subnormal. After these two papers several authors were studying and applying s-quasinormal subgroups. My main goal here is to discuss some new applications of such subgroups.

Several authors have investigated the structure of a group G under the assumption that the maximal or the minimal subgroups of the Sylow subgroups of some subgroups of G are well situated in G. Buckly in [6] proved that a group of odd order is supersoluble if all its minimal subgroups are normal. Later on, Srinivasan in [14] showed that a group G is supersoluble if it has a normal subgroup N with supersoluble quotient G/N such that all maximal subgroups of the Sylow subgroups of N are normal in G. Ramadan proved in [20]: *If G is a soluble group and all maximal subgroups of any Sylow subgroup of $F(G)$ are normal in G, then G is supersoluble.* Some later several authors were studying groups G in which the maximal or the minimal subgroups of the Sylow subgroups of some subgroups of G are s-quasinormal in G (see for example [24, 5]). The most general results in this trend were obtained in [16, 17] where the following two nice theorems were proved:

Theorem A. *Let $\mathcal{F}$ be a saturated formation containing all supersoluble groups and G be a group with a normal subgroup E such that $G/N \in \mathcal{F}$. If all minimal subgroups and all cyclic subgroups with order 4 of $F^*(N)$ are s-quasinormal G, then $G \in \mathcal{F}$ (see* [17, Theorem 3.1]*.)*

Theorem B. *Let $\mathcal{F}$ be a saturated formation containing all supersoluble groups and G be a group with a normal subgroup E such that $G/N \in \mathcal{F}$. If all maximal subgroups of the Sylow subgroups of $F^*(E)$ are s-quasinormal in G, then $G \in \mathcal{F}$ (see* [16, Theorem 3.1]*).*

In the connection with Theorems A, B the following natural question arises: Let $\mathcal{F}$ be a saturated formation containing all supersoluble groups and G be a group with a normal soluble subgroup E such that $G/E \in \mathcal{F}$. Is the group G in $\mathcal{F}$ if for every Sylow subgroup P of $F(G)$ at least one of the following conditions holds:

(1) The maximal subgroups of P are s-quasinormal in G;
(2) The minimal subgroups of P and all its cyclic subgroups with order 4 are s-quasinormal in G?

We prove the following theorem which gives the positive answer to this question.

Theorem C. *Let $\mathcal{F}$ be a saturated formation containing all supersoluble groups and G be a group with a normal subgroup E such that $G/E \in \mathcal{F}$. Suppose* *that every non-cyclic Sylow subgroup P of $F^*(N)$ has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with order $|H| = |D|$ and with order $2|D|$ (if P is a non-abelian 2-group) are s-quasinormal in G. Then $G \in \mathcal{F}$.*

Finally, note that some results of the papers [6, 7, 8, 11, 15, 18, 19, 26, 24] and, in particular, the mentioned above main results in [16, 17] may be obtained as special cases of this theorem (see Section 4).

2. Preliminaries

Recall that a formation is a hypomorph $\mathcal{F}$ of groups such that each group G has a smallest normal subgroup (denoted by $G^{\mathcal{F}}$) whose quotient is still in $\mathcal{F}$. A formation $\mathcal{F}$ is said to be saturated if it contains each group G with $G/\Phi(G) \in \mathcal{F}$. In this paper we use $\mathcal{U}$ to denote the class of the supersoluble groups; $Z_\infty^{\mathcal{U}}(G)$ denotes the $\mathcal{U}$-hypercenter of a group G that is the product of all such normal subgroups H of G whose G-chief factors have prime order.

Lemma 2.1 ([22, Theorem 9.15])**.** *$G/C_G(Z_\infty^{\mathcal{U}}(G)) \in \mathcal{U}$.*

Lemma 2.2 ([13, Lemma 2.2])**.** *Let G be a group and $P = P_1 \times \cdots \times P_t$ be a p-subgroup of G where $t > 1$ and $P_1, \dots, P_t$ are minimal normal subgroups of G. Assume that P has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with $|H| = |D|$ is normal in G. Then the order of every subgroup P_i is prime.*

Lemma 2.3 ([13, Lemma 2.4])**.** *Let p be odd prime and P be a normal p-subgroup of a group G. Assume that every minimal subgroup of P is normal in G. Then every minimal subgroup of $G/\Omega_i(G)$ is normal in $G/\Omega_i(G)$ for all $i = 1, 2, \dots$ In particular, $P \leq Z_\infty^{\mathcal{U}}(G)$.*

We shall need in our proofs the following facts about s-quasinormal subgroups.

Lemma 2.4 ([15])**.** *[Let G be a group and $H \leq K \leq G, T \leq G$. Then*

1. *If H is s-quasinormal in G, then H is s-quasinormal in K.*
2. *Suppose that H is normal in G. Then K/H is s-quasinormalin G if and only if K is s-quasinormal in G.*
3. *If H s-quasinormal is in G, then H is subnormal in G.*
4. *If H and T are s-quasinormal in G, then $< H, T >$ does.*

The following observation is well known (see, for example, [21, Lemma A]).

Lemma 2.5. *If H is a s-quasinormal subgroup of the group G and H is a p-group for some prime p, then $O^p(G) \leq N_G(H)$.*

Lemma 2.6. *Let N be an elementary abelian normal p-subgroup of a group G. Assume that N has a subgroup D such that $1 < |D| < |N|$ and every subgroup H of N satisfying $H| = |D|$ is s-quasinormal in G. Then some maximal subgroup of N is normal in G.*

Proof. Assume that this lemma is false and G is a counterexample of minimal order. Let M be a maximal subgroup of N. Then $N = N_G(M) \neq G$ and by Lemma 2.5, M is s-quasinormal in G, as M is the product of some s-quasinormal in G subgroups. By Lemma 2.5, $O^p(G) \leq N_G(M)$ and so $|G : N_G(M)| = p^n$ for some natural $n > 0$. Thus for the set Σ of all maximal subgroups of N we have $p||\Sigma|$, which contradicts [9, Lemma 8.5(d)]. □

Lemma 2.7. *Let $\mathcal{F}$ be a saturated formation containing all nilpotent groups and let G be a group with the soluble $\mathcal{F}$-residual $P = G^{\mathcal{F}}$. Suppose that every maximal subgroup of G not containing P belongs to $\mathcal{F}$. Then $P = G^{\mathcal{F}}$ is a p-group for some prime p and if every cyclic subgroup of P with prime order and order 4 (in the case when $p = 2$ and P is non-abelian) is s-quasinormal in G, then $|P/\Phi(P)| = p$.*

Proof. By [22, Theorem 24.2], $P = G^{\mathcal{F}}$ is a p-group for some prime p and the following hold:

(1) $P/\Phi(P)$ is a G-chief factor of P;
(2) P is a group of exponent p or exponent 4 (if $p = 2$ and P is non-abelian).

Assume that every cyclic subgroup of P with prime order and order 4 (if $p = 2$ and P is non-abelian) is s-quasinormal in G. Let $\Phi = \Phi(P)$, X/Φ is a subgroup of P/Φ with prime order, $x \in X \setminus \Phi$ and $L = \langle x \rangle$. Then $|L| = p$ or $|L| = 4$ and so it is s-quasinormal in G.

Then by Lemma 2.4, $L\Phi(P)/\Phi(P) = X/\Phi(P)$ is s-quasinormalin $G/\Phi(P)$. Now by Lemma 2.6 we have to conclude that $|P/\Phi(P)| = p$. □

Lemma 2.8 ([22, Lemma 7.9])**.** *Let P be a nilpotent normal subgroup of a group G. If $P \cap \Phi(G) = 1$, then P is the direct product of some minimal normal subgroup of G.*

Lemma 2.9 ([9, Theorem 3.5])**.** *Let A, B be normal subgroups of a group G and $A \leq \Phi(G)$. Suppose that $A \leq B$ and B/A is nilpotent. Then B is nilpotent.*

Let p be a prime. A group G is said to be p-closed if a Sylow p-subgroup of G is normal.

Lemma 2.10 ([22, p. 34])**.** *Let p be a prime. Then the class of all p-closed groups is a saturated formation.*

Lemma 2.11 ([13, Lemma 2.10])**.** *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G be a group with a normal subgroup E such that $G/E \in \mathcal{F}$. If E is cyclic, then $G \in \mathcal{F}$.*

Lemma 2.12 ([23, Theorem 1], [10, Theorem 1])**.** *Let A be a p'-group of automorphisms of the p-group P of odd order. Assume that every subgroup of P with prime order is A-invariant. Then A is cyclic.*

Lemma 2.13 ([13, 21, Lemma 2.12]). *Let $G = AB$ where A, B are normal subgroups of G. Suppose that A is a p-group for some prime p and $O_{p'}(G) = 1$. Let $\mathcal{F}$ be a saturated formation and $B \in \mathcal{F}$. Then $G \in \mathcal{F}$.*

The generalized Fitting subgroup $F^*(G)$ of a group G is the product of all normal quasinilpotent subgroups of G. We shall need in our proofs the following well known facts about this subgroup (see Chapter X in [16]).

Lemma 2.14. *Let G be a group. Then*

(1) *If N is a normal subgroup of G, then $F^*(N) \leq F^*(G)$.*
(2) *If N is a normal subgroup of G and $N \leq F^*(G)$, then $F^*(G)/N \leq F^*(G/N)$.*
(3) *$F(G) \leq F^*(G) = F^*(F^*(G))$. If $F^*(G)$ is soluble, then $F^*(G) = F(G)$.*
(4) *$F^*(G) = F(G)E(G)$ and $F(G) \cap E(G) = Z(E(G))$ where $E(G)$ is the layer G (see p. 128 in [13]).*
(5) *$C_G(F^*(G)) \leq F(G)$.*

Lemma 2.15 ([14, Lemma 2.3(6)]). . *Let P be a normal subgroup of a group G. Then*

$$F^*(G/\Phi(P)) = F^*(G)/\Phi(P).$$

Lemma 2.16 ([14, Lemma 2.3(7)]). *Let P be a normal p-subgroup of a group G contained in $Z(G)$. Then $F^*(G/P) = F^*(G)/P$.*

Lemma 2.17 ([15, Theorem 1], [10, Theorem 1]). *Let A be a p'-group of automorphisms of the p-group P of odd order. Assume that every subgroup of P with prime order is Alemma-invariant. Then A is cyclic.*

Lemma 2.18 ([10, Lemma 2.11]). *Let $G = AB$ where A, B are normal subgroups of G. Suppose that A is a p-group for some prime p and $O_{p'}(G) = 1$. Let $\mathcal{F}$ be a saturated formation and $B \in \mathcal{F}$. Then $G \in \mathcal{F}$.*

Finally, we shall need the following results which are proved in [12].

Lemma 2.19. *Let $\mathcal{F}$ be a saturated formation containing all supersoluble groups and G be a group with a normal subgroup E such that $G/E \in \mathcal{F}$. Suppose that every non-cyclic Sylow subgroup P of E has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with order $|H| = |D|$ and with order $2|D|$ (if P is a non-abelian 2-group) are s-quasinormal in G. Then $G \in \mathcal{F}$.*

Lemma 2.20. *Let $\mathcal{F}$ be a saturated formation containing all supersoluble groups and G be a group with a soluble normal subgroup E such that $G/E \in \mathcal{F}$. Suppose that every non-cyclic Sylow subgroup P of $F(E)$ has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with order $|H| = |D|$ and with order $2|D|$ (if P is a non-abelian 2-group) are s-quasinormal in G. Then $G \in \mathcal{F}$.*

3. The proof of Theorem C.

Proof of Theorem C. Assume that this theorem is false and let G be a counterexample with minimal $|G||E|$. Let F=F(E) and let p be the largest prime divisor of $|F|$. Let P be the Sylow p-subgroup of F, $P_0 = \Omega_1(P)$ and $C = C_G(P_0)$. It is clear that C is normal in G. We divide the proof into the following steps:

(1) $F^* = F \neq E$ and $C_G(F) = C_G(F^*) \leq F$.

By Lemma 2.6(1) the hypothesis is still true for F^* (respectively F^*), and so F^* is supersoluble by Theorem 1.2. Hence $F^* = F$, by Lemma 2.14(3). Thus if $F = E$, then $G \in \mathcal{F}$, by Lemma 2-20, which contradicts the choice of G. Hence $F^* = F \neq E$. Finally, by Lemma 2.14(5), $C_G(F) = C_G(F^*) \leq F$.

(2) Every proper normal subgroup X of G containing F is supersoluble.

By Lemma 2.14(1), $F^*(X) \leq F^* = F \leq X$ and so $F^*(X) = F^*$. Thus the hypothesis is still true for X (respectively X) and so X is supersoluble, by the choice of G.

(3) If $E \neq G$, E is supersoluble (this directly follows from (2).

(4) Assume that E is soluble and let $V/P = F(E/P)$ and Q be a Sylow q-subgroup of V where q divides $|V/P|$. Then $q \neq p$ and either $Q \leq F$ or $p > q$ and $C_Q(P) = 1$.

Since V/P is nilpotent and QP/P is a Sylow q-subgroup of V/P, QP/P is characteristic in V/P and so QP is normal in E. Thus $q \neq p$. By Theorem 1.2, QP is supersoluble. Assume $q > p$. Then Q is normal in QP and so $Q \leq F = F(E)$. Next let $p > q$. Then $p > 2$ and since p is the minimal prime divisor of $|F|$, F is a q'-subgroup. Now let U be a Sylow r-subgroup of F where $r \neq p$. Then $r \neq q$ and so $[U, Q] \leq P$. Assume that for some $x \in Q$ we have $x \in C_E(P)$. Then by [18, Theorem 3.6] and since V/P is nilpotent, $[U, \langle x \rangle] = [U, \langle x \rangle, \langle x \rangle] = 1$ and so $x \in C_G(F) \leq F$, by (1). Hence $C_Q(P) = 1$.

(5) $p > 2$.

Assume that $p = 2$. First suppose that E is soluble. In this case by (4) we have $F/P = F(E/P)$. Besides, by (1) and Lemma 2.14(3), $F^*(E/P) = F(E/P) = F^*/P$. Thus by Lemma 2.6 the hypothesis is still true for G/P respectively E/P, since $G/E \simeq (G/P)/(E/P) \in \mathcal{F}$. Therefore $G/P \in \mathcal{F}$ and so $G \in \mathcal{F}$, by Theorem 1.2. This contradiction shows that E is non-soluble. In this case p is the largest prime divisor of $|F|$ and so by (1), $F^* = F$ is a 2-group. Let Q be a subgroup of E with prime order q where $q \neq 2$ and let $X = FQ$. By Theorem 1.2, X is supersoluble and so Q is normal in X. Thus $Q \leq C_E(F)$. But by (1), $C_E(F) = C_E(F^*) \leq F$, a contradiction. Hence we have (5).

(6) Every subgroup of P has no a supersoluble supplement in G.

Assume that for some subgroup H of P we have $G = HT$ where T is supersoluble. Then $G/P \simeq T/T \cap P$ is supersoluble and so $G \in \mathcal{F}$, by Theorem 1.2, a contradiction.

(7) Some minimal subgroup of P is not quasinormal in G.

Suppose that every minimal subgroup of P is quasinormal in G. First suppose that E is soluble. Let $V/P = F(E/P)$ and Q be a Sylow q-subgroup of V where q divides $|V/P|$. Then by (4) either $Q \le F$ or $C_Q(P) = 1$. In the second case, Q is cyclic, by (5) and Lemma 2.17. Thus by Lemma 2.6(4) the hypothesis is still true for G/P (respectively E/P) and so $G/P \in \mathcal{F}$, by the choice of G. But then $G \in \mathcal{F}$, by Theorem 1.2. This contradiction shows that E is not soluble. Note that in this case $E = G$, by (3). We show that every minimal subgroup L of P is normal in G. But first we prove that $O^p(G) = G$. Indeed, assume that $O^p(G) \ne G$. By Lemma 2.14(1), $F^*(O^p(G)) \le F^*$. Hence $F^*(O^p(G)) = F^* \cap O^p(G) = F \cap O^p(G)$ and so by (5) and Lemma 2.6 the hypothesis is still true for $O^p(G)$ (respectively $O^p(G)$). Thus $O^p(G)$ is supersoluble, by the choice of G. But then G is soluble and so E is soluble, a contradiction. Therefore we have to conclude that $O^p(G) = G$ and so by Lemma 2.7, $G = O^p(G) \le N_G(L)$, since L is quasinormal in G. Therefore every minimal subgroup of P is normal in G and hence $P_0 \le Z(F)$. Next we show that the hypothesis is still true for G/P_0 (respectively C/P_0). Indeed, by Lemma 2.1, G/C is supersoluble and hence $(G/P_0)/(C/P_0) \simeq G/C \in \mathcal{F}$. Clearly $F^* = F \le F^*(C)$ and by Lemma 2.14(1), $F^*(C) \le F^*$. Hence $F^*(C) = F^*$ and so by Lemma 2.16, $F^*(C/P_0) = F^*(C)/P_0 = F^*/P_0 = F/P_0$, since $P_0 \le Z(C)$. Now by (5) and Lemmas 2.2, 2.6 we see that the hypothesis is still true for G/P_0 and so $G/P_0 \in \mathcal{F}$, by the choice of G. But $P_0 \le Z^{\mathcal{U}}_{\infty}(G)$ and so $G \in \mathcal{F}$, by Lemma 2.13. This contradiction completes the proof of (7).

(8) P is not cyclic (this directly follows from (6) and (7)).

By (8), P is not cyclic and so by hypothesis and by (6), P has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with $|H| = |D|$ is s-quasinormal in G.

(9) $|D| > p$.

Suppose that $|D| = p$. By (7), P has a subgroup H such that $|H| = p$ and H is not quasinormal in G. By (6) and Lemma 2.6(5) the subgroup H has a normal complement T in G. Then the hypothesis is true for G (respectively $V = T \cap E$). Indeed, evidently $G/V \in \mathcal{F}$ and $F^*(V) \le F^*(E)$. On the other hand, since $|G : T| = p$, every Sylow q-subgroup of $F = F^*$ where $q \ne p$ is contained in T. Thus the hypothesis is still true for G (respectively V), by Lemma 2.6. But since T is a proper subgroup of G and $ET = G$, $|V| < |E|$, which contradicts the choice of G and the subgroup E. This contradiction completes the proof of (9).

(10) If L is a minimal normal subgroup of G and $L \le P$, then $|L| > p$.

Assume that $|L| = p$. Let $C_0 = C_E(L)$. Then the hypothesis is true for G/L (respectively C_0/L). Indeed, clearly, $G/C_0 = G/E \cap C_G(L) \in \mathcal{F}$. Besides, since $L \le Z(C_0)$ and evidently $F = F^* \le C_0$ and $L \le Z(F)$, we have $F^*(C_0/L) = F^*/L$. On the other hand, if H/L is a subgroup of G/L such that $|H| = |D|$, we have $1 < |H/L| < |P/L|$, by (9). Besides, H/L is s-quasinormal in G/L, by

Lemma 2.6(2). Now, by Lemma 2.6(4) and by (5) we see that the hypothesis is still true for G/L. Hence $G/L \in \mathcal{F}$ and so $G \in \mathcal{F}$, by Lemma 2.13, a contradiction.

(11) $\Phi(G) \cap P \neq 1$ and if L is a minimal normal subgroup of G contained in $\Phi(G) \cap P$, then $F^*(E/L) \neq F^*/L$.

Suppose that $\Phi(G) \cap P = 1$. Then P is the direct product of some minimal normal subgroups of G, by Lemma 2.10. Hence by Lemma 2.8, P has a maximal subgroup M such which is normal in G. Now by [11, Theorem (9.13)] for some minimal normal subgroup L of G contained in P we have $|L| = p$, which contradicts (10). Thus $\Phi(G) \cap P \neq 1$. Let $L \leq \Phi(G) \cap P$ where L is some minimal normal subgroup of G. Assume that $F^*(E/L) = F^*/L$. We show that the hypothesis is still true for G/L (respectively E/L). By Lemma 2.11, $|L| \leq |D|$ and so the hypothesis is true for G/L in the case $|P : D| = p$. Besides, by (5) the hypothesis is true for G/L in the case $|L| < |D|$. So let $|P : D| > p$ and $|L| = |D|$. By (10), L is non-cyclic and so every subgroup of G containing L is non-cyclic. Let $L \leq K$, $M \leq K$ where $M \neq L$ and L, M are maximal subgroups of K. We have only to show that K is s-quasinormal in G. It is evident if M is quasinormal in G. Now let M is not quasinormal in G. Then by Lemma 2.6(5), G has a normal subgroup S such that $MS = G = KS$ and $|G : S| = p$. Since $L \leq \Phi(G)$, we have $L \leq S$ and so $S \cap K = L$. Therefore K is s-quasinormal in G. Thus the hypothesis is true for G/L and $G/L \in \mathcal{F}$, by the choice of G. But then $G \in \mathcal{F}$, since $L \leq \Phi(G)$ and the formation $\mathcal{F}$ is saturated, by hypothesis. This contradiction shows that $F^*(E/L) \neq F^*/L$.

(12) $E = G$ is not soluble.

Assume that E is soluble. Let L be a minimal normal subgroup of G contained in $\Phi(G) \cap P$. By Lemma 2.11, $F/L = F(E/L)$. On the other hand, $F^*(E/L) = F(E/L)$, by Lemma 2.14(3). Hence by (1), $F^*(E/L) = F(E/L) = F^*/L$, which contradicts (11). Therefore E is not soluble and so $E = G$, by (3).

(13) G has a unique maximal normal subgroup containing F, M say, M is supersoluble and G/M is non-abelian simple (this directly follows from (2) and (12)).

(14) G/F is a simple non-abelian group and if L is a minimal normal subgroup of G contained in $\Phi(G) \cap P$, then G/L is a quasinilpotent group.

Let L be a minimal normal subgroup of G contained in $\Phi(G) \cap P$. Then by (11), $F^*(E/L) \neq F^*/L$. Thus $F/L = F^*/L$ is a proper subgroup of $F^*(G/L)$, by Lemma 2.14(2). By Lemma 2.14(4), $F^*(G/L) = F(G/L)E(G/L)$ where $E(G/L)$ is the layer of G/L. By (13) every chief series of G has the only non-abelian factor. But $E(G/L)/Z(E(G/L))$ is a direct product of simple non-abelian groups and so $F^*(G/L) = G/L$ is quasinilpotent group, since by Lemma 2.11, $F(G/L) = F/L$. Since by Lemma 2.14(4), $Z(E(G/L)) = F(G/L) \cap E(G/L)$, then $G/F \simeq (G/L)/(F/L)$ is a simple non-abelian group.

(15) $F^* = P$.

Assume that $P \neq F$ and let Q be a Sylow q-subgroup of F where $q \neq p$. By (14), $Q \leq Z_\infty(G)$. Hence by Lemma 2.16, $F^*(G/Q) = F^*/Q$ and so by Lemma 2.6 the hypothesis is still true for G/Q (respectively G/Q) and so G/Q is supersoluble, by the choice of G. Hence G is soluble, which contradicts (12). Hence we have (15).

(16) $\Phi(P) = 1$.

Assume that $\Phi(P) \neq 1$ and let L be a minimal normal subgroup of G contained in $\Phi(P)$. Then by (14), G/L is quasinilpotent and so G is quasinilpotent, by Lemma 2.11. But then $F = F^* = G$, a contradiction. So we have (16).

(17) If H is a normal subgroup of P and H is s-quasinormal in G, then H is normal in G.

Indeed, by (14) and (15), $PO^p(G) = G$ and so by Lemma 2.7, H is normal in G.

(18) $|P : D| > p$.

Assume that $|P : D| = p$. By (11), $\Phi(G) \cap P \neq 1$ and let N be a minimal normal subgroup of G contained in $\Phi(G) \cap P$. By (16) for some maximal subgroup V of P we have $P = NV$. By the hypothesis V is s-quasinormal in G. By (17) we have V is normal in G, since V is a maximal subgroup of P. But, by (10), $|N| > p$ and so $N \neq N \cap V \neq 1$, which contradicts minimality of N. Thus we have (18).

(19) $O_{p'}(G) = 1$.

Indeed, assume that $O_{p'}(G) \neq 1$. Then by (13), $G = O_{p'}(G)P = O_{p'}(G) \times P = F^* = F$, a contradiction.

(20) $O^p(G) = G$.

Assume $O^p(G) \neq G$. Then G has a normal subgroup T such that $|G : T| = p$. We show that T satisfies the hypothesis. First note that $F \cap T = F^*(T)$. Indeed, clearly, $F \cap T \leq F^*(T)$. By (14), $T/F \cap T$ is simple non-abelian. Thus if $F \cap T \neq F^*(T)$, then $F^*(T) = T$ and so $G = TF = F^* = F$ is nilpotent, a contradiction. Hence $F \cap T = F^*(T)$ and so the hypothesis is still true for T, by (18). Therefore $T \in \mathcal{F}$ and so by Lemma 2.18 and by (19), $G \in \mathcal{F}$, a contradiction.

(21) Every subgroup H of P satisfying $|H| = |D|$ is normal G (this directly follows from (20) and Lemma 2.7).

Final contradiction.

Let N be a minimal normal subgroup of G contained in P. By (16) for some maximal subgroup M of P we have $P = NM$. Let H be a subgroup of P such that $H \leq M$ and $|H| = |D|$. Then by (21), H is normal in G and evidently $N \nsubseteq H$. Hence $N \cap H = 1$ and so G has a minimal normal subgroup $L \neq N$ which contained in P. Then by (11) and (14) at least one of the subgroups N, L has prime order, which contradicts (10). This contradiction completes the proof of this theorem. □

4. Some applications

Finally, consider some applications of Theorem C.

Corollary 4.1 ([7]). *Let G be a group of odd order. If all subgroups of G of prime order are normal in G, then G is supersoluble.*

Corollary 4.2 ([24]). *Let G be a group and E a normal subgroup of G with supersoluble quotient. Suppose that all minimal subgroups of E and all its cyclic subgroups with order* 4 *are s-quasinormal in G. Then G is supersoluble.*

Corollary 4.3 ([5]). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G a group with normal subgroup E such that $G/E \in \mathcal{F}$. Assume that a Sylow* 2*-subgroup of G is abelian. If all minimal subgroups of E are permutable in G, then $G \in \mathcal{F}$.*

Corollary 4.4 ([5]). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G a group with a soluble normal subgroup E such that $G/E \in \mathcal{F}$. If all minimal subgroups and all cyclic subgroups with order 4 of E are in G, then $G \in \mathcal{F}$.*

Corollary 4.5 ([20]). *Let G be a soluble group. If all maximal subgroups of the Sylow subgroups of $F(E)$ are normal in G, then G is supersoluble.*

Corollary 4.6 ([4]). *Let G be a group and E a soluble normal subgroup of G with supersoluble quotient G/E. Suppose that all maximal subgroups of any Sylow subgroup of $F(E)$ are s-quasinormal in G. Then G is supersoluble.*

Corollary 4.7 ([3]). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G be a group with a soluble normal subgroup E such that $G/E \in \mathcal{F}$. If all minimal subgroups and all cyclic subgroups with order 4 of $F(E)$ are s-quasinormal in G, then $G \in \mathcal{F}$.*

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