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# DUAL BANACH ALGEBRAS AND CONNES-AMENABILITY

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Abstract. In this survey, we first study dual Banach algebras and completely contractive dual Banach algebras. Then, we discuss the Connes-amenability and operator Connes-amenability of these algebras and present some new advances in this area.

## 1. Introduction

Abstract harmonic analysis studies locally compact groups and Banach algebras associated with them. Amenability (which was first defined for discrete groups by von Neumann in [22]) is a distinctive property of locally compact groups ([3]). The class of amenable groups includes all finite, abelian and compact groups. On the other hand, $\mathbb{F}_2$, the free group on two generators, is not amenable. In 1972, B. Johnson defined the concept of amenability for Banach algebras ([7]) and proved that the group algebra $L^1(G)$ of a locally compact group $G$ is amenable if and only if $G$ is amenable. Since then it has been a very active area of research. Studying the amenability of a Banach algebra associated with a locally compact group gives us plenty of information about the underlying group. For example, the measure algebra $M(G)$ of a locally compact group $G$ is amenable if and only if $G$ is discrete and amenable ([1]).

The theory of abstract operator spaces, which was initiated by Ruan's representation theorem in [13], has become a very powerful tool in abstract harmonic analysis. This is because many of the classical spaces of interest, for example the Fourier algebras, have natural operator space structures on them. Taking this operator space structure into account while studying these objects yields fruitful results. In his paper ([13]), Ruan also defined the concept of operator amenability and proved that the Fourier algebra $A(G)$ is operator amenable if and only if $G$ is amenable.

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Dual Banach algebras form a very special class of Banach algebras which includes the Fourier-Stieltjes algebra and the measure algebra of a locally compact group. In [14], Runde defined the concept of Connes-amenability which is better suited for dual Banach algebras. For example, the measure algebra $M(G)$ is Connes-amenable if and only if $G$ is amenable ([16]). There is also an operator space version of Connes-amenability called operator Connes-amenability which is defined for completely contractive dual Banach algebras. In [18], Runde and Spronk proved that the reduced Fourier-Stieltjes algebra $B_r(G)$ of a locally compact group $G$ is operator Connes-amenable if and only if $G$ is amenable.

## 2. Notes From Abstract Harmonic Analysis

**Definition 2.1.** A *topological group* is a group equipped with a topology such that the group operations and the topology are compatible. That is, the maps

$$G \times G \to G,\ (g,h) \mapsto gh \text{ and } G \to G,\ g \mapsto g^{-1}$$

are continuous. If the topology on $G$ is a locally compact Hausdorff topology (that is, there is a neighborhood base for the identity element consisting of compact sets), then $G$ is called a *locally compact group.*

Clearly, every discrete group is locally compact. Also, the set of real numbers with addition and the unit circle of the complex plane with multiplication are locally compact groups. However, if $E$ is an infinite dimensional Banach space, then $(E,+)$ is not a Banach space.

Abstract harmonic analysis studies locally compact groups and Banach algebras related to them. There are many interesting Banach algebras associated with a locally compact group, for instance the Fourier algebras, the group algebra and the measure algebra. We should point out that one advantage of working with locally compact groups is the existence of (left) Haar measure (which is unique up to a positive constant multiple) on them which allows us to define $L^p$-spaces related to them.

Let $G$ be a locally compact group. A *unitary representation* of $G$ is a homomorphism $\pi$ from $G$ into the group $U(\mathcal{H}_\pi)$, unitary operators on some non-zero Hilbert space $\mathcal{H}_\pi$, that is continuous with respect to the strong operator topology. The collection of all equivalence classes of unitary representations of $G$ with respect to unitary equivalence is denoted by $\Sigma_G$. We call $f$ a *coefficient function* of a representation $\pi$ of $G$ on some Hilbert space $\mathcal{H}$ if there are $\xi, \zeta \in \mathcal{H}$ such that

$$f(x) = \langle \pi(x)\xi, \zeta \rangle \qquad (x \in G).$$

The Fourier-Stieltjes algebra of $G$ is the the algebra of coefficient functions of continuous unitary representations of G and is denoted by $B(G)$. More explicitly,

$$B(G) := \big\{ f \colon G \to \mathbb{C} :\ f(x) = \langle \pi(x)\xi, \zeta \rangle,\ \forall x \in G \big\} \tag{1}$$

where $\pi \in \Sigma_G$ and $\xi, \zeta \in \mathcal{H}_\pi$.

The *left regular representation* of $G$ is defined by $\lambda\colon G \to \mathcal{B}(L^2(G))$

$$(\lambda(g)\xi)(h) := \xi(g^{-1}h) \qquad (g, h \in G,\ \xi \in L^2(G)).$$

We define the *Fourier algebra*, $A(G)$ of $G$ to be the space of all coefficient functions of $\lambda$, that is:

$$A(G) := \big\{ f\colon G \to \mathbb{C} :\ \exists\, \xi, \zeta \in L^2(G) \text{ such that } \ f(x) = \langle \lambda(x)\xi, \zeta \rangle,\ \forall x \in G \big\}.$$

Fourier and Fourier-Stieltjes algebras were first defined in 1964 by Eymard ([5]) for arbitrary locally compact groups. Fourier-Stieltjes algebra becomes a Banach space with the norm defined by

$$\|f\|_{B(G)} := \inf \big\{ \|\xi\| \|\zeta\| : f \text{ is represented as in (1) } \big\}.$$

Indeed in [5], Eymard proved more:

**Theorem 2.2.** *Let $G$ be a locally compact group. Then (with pointwise multiplication) $B(G)$ is a commutative unital Banach algebra that contains $A(G)$ as a norm-closed ideal.*

For more information on Fourier algebras, we refer the reader to [5].

## 3. Amenability

A bounded linear functional, $m\colon L^\infty(G) \to \mathbb{C}$, is called a *mean* if

$$\|m\| = \langle 1, m \rangle = 1,$$

where $L^\infty(G)$ is the space of all measurable essentially bounded functions on $G$. For a function $f\colon G \to \mathbb{C}$, we define its left translate $L_g f$ by $g \in G$ through

$$(L_g f)(h) := f(gh) \qquad (h \in G).$$

A mean $m$ on $L^\infty(G)$ is called *left invariant* if

$$\langle L_g f, m \rangle = \langle f, m \rangle \qquad (g \in G, f \in L^\infty(G)).$$

The existence of left invariant means on discrete groups (which was first investigated by J. von Neumann ([22])) has strong ties with the well-known Banach-Tarski paradox. We have the following definition due to M. Day ([3]).

**Definition 3.1.** A locally compact group $G$ is *amenable* if there is a left invariant mean on $L^\infty(G)$.

All finite, abelian, and compact groups are amenable; however, the free group on two generators is not ([22]).

The concept of amenability also carries over to Banach algebras.

**Definition 3.2.** Let $\mathfrak{A}$ be a Banach algebra and $X$ be a Banach $\mathfrak{A}$-bimodule. A bounded linear map $D\colon \mathfrak{A} \to X$ is called a *derivation* if

$$D(ab) = a.D(b) + D(a).b \qquad (a, b \in \mathfrak{A}).$$

Each $x \in X$ defines a map

$$\mathrm{ad}_x \colon \mathfrak{A} \to X,\ a \mapsto a.x - x.a.$$

It is easy to verify that $ad_x$ is a derivation. Derivations of this type are called *inner derivations.*

Amenability for Banach algebras was defined first by Johnson in [7] in 1972.

**Definition 3.3.** A Banach algebra $\mathfrak{A}$ is said to be *amenable* if every derivation from $\mathfrak{A}$ into $X^*$ is inner for each Banach $\mathfrak{A}$-bimodule $X$.

Johnson in [7] also gave the first connection between group amenability and Banach algebra amenability:

**Theorem 3.4.** *Let $G$ be a locally compact group. Then $G$ is amenable if and only if $L^1(G)$ is an amenable Banach algebra.*

For more information on amenability, we refer the reader to the monographs [8] and [15].

## 4. Operator Spaces

If $E$ is a linear space, then for each $m, n \in \mathbb{N}$, $M_{m,n}(E)$ will denote the space of all $m \times n$ matrices with entries in $E$. If $m = n$, then $M_{m,n}(E)$ will be denoted by $M_n(E)$ and in particular, $M_n = M_n(\mathbb{C})$ will denote the space of all scalar $n \times n$ matrices.

**Definition 4.1.** Let $E$ be a linear space with a norm $\|\cdot\|_n$ on $M_n(E)$ for each $n \in \mathbb{N}$ such that

$$\left\| \begin{array}{c|c} x & 0 \\ \hline 0 & y \end{array} \right\|_{n+m} = \max\{\|x\|_n, \|y\|_m\} \qquad (n, m \in \mathbb{N},\ x \in M_n(E),\ y \in M_m(E)) \tag{2}$$

and

$$\|\alpha x \beta\|_n \leq \|\alpha\| \|x\|_n \|\beta\| \qquad (n \in \mathbb{N},\ x \in M_n(E),\ \alpha, \beta \in M_n). \tag{3}$$

Then $(\|\cdot\|_n)_{n\in\mathbb{N}}$ is called a matricial norm for $E$. Moreover, if each $\|\cdot\|_n$ is complete, then $E$ is called an *abstract operator space.*

A *concrete operator space* is a closed subspace of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.

**Definition 4.2.** Let $E$ and $F$ be two abstract operator spaces, and let $T \in \mathcal{B}(E, F)$. Then say that

(1) $T$ is *completely bounded* if

$$\|T\|_{\mathrm{cb}} := \sup\left\{ \left\|T^{(n)}\right\|_{\mathcal{B}(M_n(E), M_n(F))} :\ n \in \mathbb{N}\right\} < \infty.$$

(2) $T$ is a *complete contraction* if $\|T\|_{\mathrm{cb}} \leq 1$.

(3) $T$ is a *complete isometry* if $T^{(n)}$ is an isometry for each $n \in \mathbb{N}$.

Not every linear bounded operator between operator spaces is completely bounded:

*Example* 4.3. Let $\mathcal{H} = l_2$. Then the (Banach space) adjoint operator

$$\mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}), \quad T \mapsto T^*$$

is an isometry but is not completely bounded ([4]).

The set of completely bounded operators from $E$ to $F$ is denoted by $\mathcal{CB}(E, F)$.

If $E$ and $F$ are two operator spaces, then $\mathcal{CB}(E, F)$ turns into an abstract operator space with the identification

$$M_n(\mathcal{CB}(E, F)) = \mathcal{CB}(E, M_n(F)), \qquad (n \in \mathbb{N}).$$

The following theorem is known as Ruan's representation theorem that initiated the theory of abstract operator spaces ([13]).

**Theorem 4.4.** *Let $X$ be an abstract operator space. Then there is a complete isometry from $X$ into $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.*

Thanks to Ruan's representation theorem, we do not have to distinguish abstract and concrete operator spaces. Operator spaces arise naturally in functional analysis. Every infinite dimensional Banach space $X$ can be turned into an operator space in at least two different ways, namely $minX$ and $maxX$.

As we mentioned before, not all bounded maps between two operator spaces are completely bounded. However, this is true for some operator spaces. The following result is known as Smith's Lemma ([19]):

**Lemma 4.5.** *Let $E$ be an abstract operator space and $\mathcal{A}$ be a commutative $C^*$-algebra. Then for every $T \in \mathcal{B}(E, \mathcal{A})$ we have*

$$\|T\|_{cb} = \|T\|.$$

In particular, if $\mathcal{A} = \mathbb{C}$, then we have:

**Corollary 4.6.** *Let $E$ be an abstract operator space. Then for every $T \in E^*$ we have $\|T\|_{cb} = \|T\|$.*

Due to the duality theorem, the dual and the predual (if it exists) of an operator space have natural operator space structures. More explicitly, if $X$ is an operator space and $\phi = (\phi_{i,j}) \in M_n(X^*)$ for some $n \in \mathbb{N}$, then

$$\|\phi\|_n = \sup\Big\{ |\langle \phi_{i,j}, x_{k,l} \rangle|_{n^2} : \ x = (x_{k,l}) \in M_n(X), \|x\|_n \leq 1 \Big\}.$$

Now by using the duality theorem, we can give more examples of (abstract) operator spaces.

*Examples.* 1. If $G$ is a locally compact group, then as a dual of a commutative $C^*$-algebra, the measure algebra $M(G)$ is an operator space.
2. If $G$ is a locally compact group, then as a predual of a von Neumann algebra, the Fourier algebra $A(G)$ is an operator space.
3. If $G$ is a locally compact group, then as a dual of a $C^*$-algebra algebra, the Fourier-Stieltjes algebra $B(G)$ is an operator space.

Investigating Fourier algebras with their operator structures yields fruitful results. For example, the long time open problem "amenability of the Fourier algebra" was solved recently only by taking the operator space structure of the Fourier algebra into account. In 2005, Forrest and Runde ([6]) proved that:

**Theorem 4.7.** *Let $G$ be a locally compact group. Then the Fourier algebra $A(G)$ is amenable if and only if $G$ has an abelian subgroup of finite index.*

**Definition 4.8.** A *completely contractive Banach algebra* is an algebra which is also an operator space such that multiplication is a completely contractive bilinear map.

*Examples.* 1. Every closed subalgebra of $\mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space, is a completely contractive Banach algebra.
2. If $\mathfrak{A}$ is a Banach algebra, then *max* $\mathfrak{A}$ is a completely contractive Banach algebra.
3. If $G$ is a locally compact group, then the Fourier-Stieltjes algebra $B(G)$ is a Hopf–von Neumann algebra ([18]) and hence, it is a completely contractive Banach algebra. Since $A(G)$ is a closed ideal of $B(G)$ ([5]), it is also a completely contractive Banach algebra.

If $\mathfrak{A}$ is a completely contractive Banach algebra, then a Banach $\mathfrak{A}$-bimodule $X$ is called an operator $\mathfrak{A}$-bimodule if $X$ is an operator space and the module actions of $\mathfrak{A}$ on $X$ are completely bounded. In [13], Ruan gave the following definition: a completely contractive Banach algebra $\mathfrak{A}$ is called *operator amenable* if every completely bounded derivation from $\mathfrak{A}$ into $X^*$ is inner, for every $\mathfrak{A}$-bimodule $X$. In [13], Ruan also proved that:

**Theorem 4.9.** *Let $G$ be a locally compact group. Then $A(G)$ is operator amenable if and only if $G$ is amenable.*

For more information on operator spaces, we refer the reader to [11] and [4].

## 5. Dual Banach Algebras and Connes-amenability

**Definition 5.1.** A Banach algebra $\mathfrak{A}$ is called a dual Banach algebra if there exists a closed submodule $\mathfrak{A}_*$ of $\mathfrak{A}^*$ such that $\mathfrak{A} = (\mathfrak{A}_*)^*$.

Note that $\mathfrak{A}_*$ does not have to be unique. In general, however, when we study a dual Banach algebra, we fix its predual. Thus, we can talk about the $w^*$-topology on $\mathfrak{A}$ without any ambiguity. It is easy to see that $\mathfrak{A}$ is a dual

Banach algebra if and only if multiplication in $\mathfrak{A}$ is separately $w^*$-continuous.

*Example* 5.2. 1. Every von Neumann algebra is a dual Banach algebra.
2. If $E$ is a reflexive Banach space, then $\mathcal{B}(E)$ is a dual Banach algebra with the predual $E^*\hat{\otimes}E$, where $\hat{\otimes}$ represents the projective tensor product of Banach spaces.
3. If $G$ is a locally compact group, then the measure algebra $M(G)$ and the Fourier-Stieltjes algebra $B(G)$ are dual Banach algebras with preduals $C_0(G)$ and $C^*(G)$ respectively.

For more information on dual Banach algebras, we refer the reader to [14].

**Definition 5.3.** A completely contractive dual Banach algebra is a Banach algebra which is a dual operator space such that multiplication is completely contractive and separately $w^*$-continuous.

Note that there are operator spaces for which there exist predual Banach spaces but not predual operator spaces (Lemma 2.7.15, [3]).

In 2007, the following representation theorem was proved by Daws ([2]) and Uygul ([21]) independently.

**Theorem 5.4.** *Every completely contractive dual Banach algebra is completely isometric to a $w^*$-closed subalgebra of $\mathcal{CB}(E)$ for some reflexive operator space $E$.*

The construction of such a reflexive operator space heavily relies on the theory of real and complex interpolation of operator spaces defined by Xu ([25]) and Pisier ([9] and [10]) respectively.

**Definition 5.5.** Let $\mathfrak{A}$ be a dual Banach algebra and let $X$ be a dual Banach $\mathfrak{A}$-bimodule. An element $x \in X$ is called *normal* if the maps

$$\mathfrak{A} \to X,\ x \mapsto \begin{cases} a.x, \\ x.a \end{cases}$$

are $w^*$-$w^*$-continuous.
We say that $X$ is normal if every element of $X$ is normal.

When we are studying amenability properties of dual Banach algebras, it is very natural to take the $w^*$-topology into account. This leads us to the following definition:

**Definition 5.6.** A dual Banach algebra $\mathfrak{A}$ is called *Connes-amenable* if every $w^*$-continuous derivation from $\mathfrak{A}$ into a normal, dual Banach $\mathfrak{A}$-bimodule is inner.

The Connes amenability of a dual Banach algebra (associated with a locally compact group) reflects the properties of the underlying group. For instance, as proved in [15], $M(G)$ is Connes-amenable if and only if $G$ is an amenable group.

The Connes-amenability of the Fourier-Stieltjes algebra is an open problem. In [16], Runde proved that when $G$ is the direct product of a family of finite groups or when $G$ is an amenable discrete group, the Fourier-Stieltjes algebra $B(G)$ is Connes-amenable if and only if $G$ has an abelian subgroup of finite index, and he conjectured that this is true for all locally compact groups. In 2008, Uygul in [20] proved this conjecture for discrete groups.

**Theorem 5.7.** *Let $G$ be a discrete group. $B(G)$ is Connes-amenable if and only if $G$ has an abelian subgroup of finite index.*

Let $\mathfrak{A}$ be a completely contractive dual Banach algebra. Then $\mathfrak{A}$ is said to be *operator Connes-amenable* if every $w^*$-continuous completely bounded derivation from $\mathfrak{A}$ into a normal dual operator Banach $\mathfrak{A}$-bimodule is inner. It is known ([18]) that $B(\mathbb{F}_2)$ is operator Connes-amenable. On the other hand, $B_r(G)$ (as defined in [5]) is operator Connes-amenable if and only if $G$ is an amenable group.

The *group von Neumann algebra* of $G$ is defined to be the von Neumann algebra generated by $\{\lambda(g) : \ g \in G\}$ in $\mathcal{B}(L_2(G))$. The dual space of $A(G)$ can be identified by $VN(G)$. The following theorem was proved by Uygul in 2008 [20]:

**Theorem 5.8.** *Let $G$ be a locally compact group. Then $G$ is amenable if and only if $WAP(VN(G))^*$ is operator Connes-amenable.*

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