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CERTAIN CLASS OF UNIFORMLY ANALYTIC FUNCTIONS

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Abstract. In this paper, we introduce a new class of functions which are analytic and univalent with negative coefficients defined by using Hadamard products. Some basic properties which include coefficient bounds, growth and distortion are given. In addition, results involving the fractional calculus are also given.

1. Introduction and Preliminaries

Denote by A the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic and univalent in the open disc $U = \{z : z \in \mathcal{C} \text{ and } |z| < 1\}$. Denote by $S^*(\alpha)$ the class of starlike functions $f \in A$ of order $\alpha(0 \leq \alpha < 1)$ satisfying

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad z \in U$$

and let $C(\alpha)$ be the class of convex functions $f \in A$ of order $\alpha(0 \leq \alpha < 1)$ such that $zf' \in S^*(\alpha)$.

A function $f \in A$ is said to be in the class of β-uniformly convex functions of order α, denoted by $\beta - UCV(\alpha)$ [8, 9] if

$$\mathrm{Re}\ \left\{1 + \frac{zf''(z)}{f'(z)} - \alpha\right\} \geq \beta \left|\frac{zf''(z)}{f'(z)} - 1\right|, \tag{1.2}$$

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and is said to be in a corresponding subclass of $\beta - UCV(\alpha)$ denote by $\beta - S_p(\alpha)$ if

$$\text{Re}\left\{\frac{zf'(z)}{f(z)} - \alpha\right\} \geq \beta\left|\frac{zf'(z)}{f(z)} - 1\right|, \tag{1.3}$$

where $-1 \leq \alpha \leq 1$ and $z \in U$.

The class of uniformly convex and uniformly starlike functions has been extensively studied by Goodman[1, 2], Ma and Minda[6]. In fact the class of uniformly β-starlike functions was introduced by Kanas and Wisniowski[4], and for which it can be generalized to $\beta - S_p(\alpha)$, the class of uniformly β-starlike functions of order α.

If f of the form (1.1) and $g(z) = z + \sum\limits_{n=2}^{\infty} b_n z^n$ are two functions in A, then the Hadamard product (or convolution) of f and g is denoted by $f * g$ and is given by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n. \tag{1.4}$$

Ruscheweyh[10] using the convolution techniques, introduced and studied an important subclass of A, the class of prestarlike functions of order α, which denoted by $\mathcal{R}(\alpha)$. Thus $f \in A$ is said to be prestarlike function of order $\alpha(0 \leq \alpha < 1)$ if $f * S_\alpha \in S^*(\alpha)$ where $S_\alpha(z) = \frac{z}{(1-z)^{2(1-\alpha)}} = z + \sum_{n=2}^{\infty} c_n(\alpha) z^n$ and $c_n(\alpha) = \frac{\Pi_{j=2}^{n}(j-2\alpha)}{(n-1)!}$ $(n \in \mathbf{N}\ \{1\} \quad \mathbf{N} := \{1, 2, 3, \ldots\})$. We note that $\mathcal{R}(0) = C(0)$ and $\mathcal{R}(\frac{1}{2}) = S^*(\frac{1}{2})$. Juneja et.al[3] define the family $\mathcal{D}(\Phi, \Psi; \alpha)$ consisting of functions $f \in A$ so that

$$\text{Re}\left(\frac{f(z) * \Phi(z)}{f(z) * \Psi(z)}\right) > \alpha, \quad z \in U$$

where $\Phi(z) = z + \sum_{n=2}^{\infty} \Upsilon_n z^n$ and $\Psi(z) = z + \sum_{n=2}^{\infty} \gamma_n z^n$ analytic in U such that $f(z) * \Psi(z) \neq 0$, $\Upsilon_n \geq 0$, $\gamma_n \geq 0$ and $\Upsilon_n > \gamma_n$ $(n \geq 2)$.

Now we define the following new class of analytic functions, and obtain some new properties.

Definition 1.1. Given η is positive real number and $\beta \geq 0$, and functions

$$\Phi(z) = z + \sum_{n=2}^{\infty} \Upsilon_n z^n, \quad \Psi(z) = z + \sum_{n=2}^{\infty} \gamma_n z^n$$

analytic in U such that $\Upsilon_n \geq 0$, $\gamma_n \geq 0$ and $\Upsilon_n > \gamma_n$ $(n \geq 2)$, we say that $f \in A$ is in $\mathcal{D}(\Phi, \Psi; \eta, \beta)$ if $f(z) * \Psi(z) \neq 0$ and

$$\text{Re}\left(1 + \frac{1}{\eta}\left(\frac{f(z) * \Phi(z)}{f(z) * \Psi(z)} - 1\right)\right) > \beta\left|\frac{1}{\eta}\left(\frac{f(z) * \Phi(z)}{f(z) * \Psi(z)} - 1\right)\right| \tag{1.5}$$

for all $z \in U$.

For suitable choices of Φ, Ψ, and having $\eta = 1-\alpha$, we easily obtain the various subclasses of A. For example

$$\mathcal{D}(\frac{z}{(1-z)^2}, \frac{z}{1-z}; 1-\alpha, 0) = S^*(\alpha),$$

$$\mathcal{D}(\frac{z+z^2}{(1-z)^3}, \frac{z}{(1-z)^2}; 1-\alpha, 0) = C(\alpha),$$

$$\mathcal{D}(\frac{z+(1-2\alpha)z^2}{(1-z)^{3-2\alpha}}, \frac{z}{(1-z)^{2-2\alpha}}; 1-\alpha, 0) = \mathcal{R}(\alpha),$$

$$\mathcal{D}(\frac{z}{(1-z)^2}, \frac{z}{1-z}; 1-\alpha, \beta) = \beta - S_p(\alpha),$$

and

$$\mathcal{D}(\frac{z+z^2}{(1-z)^3}, \frac{z}{(1-z)^2}; 1-\alpha, \beta) = \beta - UCV(\alpha).$$

Also denote by T [11] the subclass of A consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z_n. \tag{1.6}$$

Let $T^*(\alpha)$ and $C_T(\alpha)$ denote the subfamilies of T that are starlike of order α and convex of order α. Silverman [11] studied $T^*(\alpha)$ and $C_T(\alpha)$ and Silverman and Silvia [12] studied $R_T(\alpha) = T \cap R_\alpha$ and obtained many interesting results. Now let us write

$$\mathcal{D}_T(\Phi, \Psi; \alpha) = \mathcal{D}(\Phi, \Psi; \alpha) \cap T. \tag{1.7}$$

$$\mathcal{D}_T(\Phi, \Psi; \eta, \beta) = \mathcal{D}(\Phi, \Psi; \eta, \beta) \cap T. \tag{1.8}$$

Note also that the class $\mathcal{D}_T(\Phi, \Psi; \alpha)$ has been extensively studied by Juneja et. al. [3].

In this paper we shall investigate various properties for the class $\mathcal{D}_T(\Phi, \Psi; \eta, \beta)$. It would be assumed throughout that $\Phi(z)$ and $\Psi(z)$ satisfy the conditions stated in Definition 1.1 and that $f(z) * \Psi(z) \neq 0$ for $z \in U$.

2. Characterization property

In this section we give a necessary and sufficient condition for a function to be in $\mathcal{D}_T(\Phi, \Psi; \eta, \beta)$.

Theorem 2.1. *(Coefficient Bounds.) Let a function f given by (1.1) be in A. If η is positive real number and $\beta \geq 0$,*

$$\sum_{n=2}^{\infty} \frac{[(1+\beta)\Upsilon_n - (1+\beta-\eta)\gamma_n]}{\eta} |a_n| \leq 1 \tag{2.1}$$

then $f \in \mathcal{D}(\Phi, \Psi; \eta, \beta)$.

Proof. Let the condition (2.1) holds. It is sufficient to show that

$$\beta\left|\frac{1}{\eta}\left(\frac{f(z)*\Phi(z)}{f(z)*\Psi(z)}-1\right)\right| \leq \mathrm{Re}\ \left\{1+\frac{1}{\eta}\left(\frac{f(z)*\Phi(z)}{f(z)*\Psi(z)}-1\right)\right\},$$

and we have

$$\beta\left|\frac{1}{\eta}\left(\frac{f(z)*\Phi(z)}{f(z)*\Psi(z)}-1\right)\right| \leq \mathrm{Re}\ \left\{\left\{1+\frac{1}{\eta}\left(\frac{f(z)*\Phi(z)}{f(z)*\Psi(z)}-1\right)\right\}-1\right\}+1.$$

That is

$$\begin{aligned}&\beta\left|\frac{1}{\eta}\left(\frac{f(z)*\Phi(z)}{f(z)*\Psi(z)}-1\right)\right| - \mathrm{Re}\ \left\{\frac{1}{\eta}\left(\frac{f(z)*\Phi(z)}{f(z)*\Psi(z)}-1\right)\right\}\\ &\leq (\beta+1)\left|\frac{1}{\eta}\left(\frac{f(z)*\Phi(z)}{f(z)*\Psi(z)}-1\right)\right|\\ &\leq \frac{\sum_{n=2}^{\infty}(1+\beta)(\Upsilon_n-\gamma_n)|a_n|}{\eta-\sum_{n=2}^{\infty}\eta\gamma_n|a_n|}.\end{aligned}$$

The above expression is bounded by 1 and hence the assertion of the result. Thus $f \in \mathcal{D}(\Phi, \Psi; \eta, \beta)$. □

Theorem 2.2. *Let a function f be given by (1.6), then $f \in \mathcal{D}_T(\Phi, \Psi; \eta, \beta)$ if and only if (2.1) is satisfied.*

Proof. Let $f \in \mathcal{D}_T(\Phi, \Psi; \eta, \beta)$, then for z is real (1.5) gives

$$1+\frac{1}{\eta}\left(\frac{1-\sum_{n=2}^{\infty}\Upsilon_n a_n z^{n-1}}{1-\sum_{n=2}^{\infty}\gamma_n a_n z^{n-1}}-1\right) \geq \beta\frac{\sum_{n=2}^{\infty}(\Upsilon_n-\gamma_n)a_n z^{n-1}}{\eta-\sum_{n=2}^{\infty}\eta\gamma_n a_n z^{n-1}}.$$

Letting $z \to 1^-$ along the real axis leads to the desired inequality

$$\sum_{n=2}^{\infty}[(1+\beta)\Upsilon_n-(1+\beta-\eta)\gamma_n]a_n \leq \eta.$$

which is (2.1). That (2.1) implies $f \in \mathcal{D}_T(\Phi, \Psi; \eta, \beta)$ is an immediate consequence of Theorem 2.1. Hence the theorem.

Finally, the function f given by

$$f(z) = z - \frac{\eta}{(1+\beta)\Upsilon_n-(1+\beta-\eta)\gamma_n}z^n, \quad (n \geq 2) \tag{2.2}$$

is the extremal function for the assertion of Theorem 2.1 and Theorem 2.2.

□

Corollary 2.3. *Let the function f defined by (1.6) be in the class $\mathcal{D}_T(\Phi, \Psi; \eta, \beta)$. Then*

$$a_n \leq \frac{\eta}{[(1+\beta)\Upsilon_n - (1+\beta-\eta)\gamma_n]}, n \geq 2. \tag{2.3}$$

The equality in (2.3) is attained for the function f given by (2.2).

For $\beta = 0$ and $\eta = 1-\alpha$, we have result obtained by Juneja[3].

Corollary 2.4 ([3])**.** *A function f defined by (1.6) is in the class*

$$\mathcal{D}_T(\Phi, \Psi; 1-\alpha, 0)$$

if and only if,

$$\sum_{n=2}^{\infty} \frac{[\Upsilon_n - \alpha\gamma_n]}{1-\alpha} |a_n| \leq 1. \tag{2.4}$$

Next we consider the growth and distortion theorem for the class $\mathcal{D}_T(\Phi, \Psi; \eta, \beta)$. We shall omit the proof as the techniques are tedious and standard.

Theorem 2.5. *Let the function f defined by (1.6) be in the class $\mathcal{D}_T(\Phi, \Psi; \eta, \beta)$. Then*

$$\begin{aligned} |z| - |z|^2 &\frac{\eta}{[(1+\beta)\Upsilon_2 - (1+\beta-\eta)\gamma_2]} \\ &\leq |f(z)| \\ &\leq |z| + |z|^2 \frac{\eta}{[(1+\beta)\Upsilon_2 - (1+\beta-\eta)\gamma_2]} \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} 1 - |z| &\frac{2\eta}{[(1+\beta)\Upsilon_2 - (1+\beta-\eta)\gamma_2]} \\ &\leq |f(z)'| \\ &\leq 1 + |z| \frac{2\eta}{[(1+\beta)\Upsilon_2 - (1+\beta-\eta)\gamma_2]} \end{aligned} \tag{2.6}$$

The bounds (2.5) and (2.6) are attained for functions given by

$$f(z) = z - z^2 \frac{\eta}{[(1+\beta)\Upsilon_2 - (1+\beta-\eta)\gamma_2]}. \tag{2.7}$$

Theorem 2.6. *Let a function f be defined by (1.6) and*

$$g(z) = z - \sum_{n=2}^{\infty} b_n z^n \tag{2.8}$$

be in the class $\mathcal{D}_T(\Phi, \Psi; \eta, \beta)$. Then the function h defined by

$$h(z) = (1-\lambda)f(z) + \lambda g(z) = z - \sum_{n=2}^{\infty} q_n z^n \tag{2.9}$$

where $q_n = (1-\lambda)a_n + \lambda b_n,\ \ 0 \le \lambda \le 1$ *is also in the class* $\mathcal{D}_T(\Phi, \Psi; \eta, \beta)$.

Proof. The result follows easily from (2.1)and (2.9). □

3. Integral transform of the class $\mathcal{D}_T(\Phi, \Psi; \eta, \beta)$

For $f \in A$ we define the integral transform

$$V_\lambda(f)(z) = \int_0^1 \lambda(t) \frac{f(tz)}{t} dt,$$

where λ is real valued, non-negative weight function normalized so that

$$\int_0^1 \lambda(t) dt = 1.$$

Since special cases of $\lambda(t)$ are particularly interesting such as

$$\lambda(t) = (1+c)t^c, \quad c > -1,$$

for which V_λ is known as the Bernardi operator, and

$$\lambda(t) = \frac{(c+1)^\delta}{\lambda(\delta)} t^c \left(log \frac{1}{t} \right)^{\delta - 1}, \ c > -1,\ \delta \ge 0$$

which gives the Komatu operator. For more details see [5].

First we show that the class $\mathcal{D}_T(\Phi, \Psi; \eta, \beta))$ is closed under $V_\lambda(f)$.

Theorem 3.1. *Let* $f \in \mathcal{D}_T(\Phi, \Psi; \eta, \beta)$. *Then* $V_\lambda(f) \in \mathcal{D}_T(\Phi, \Psi; \eta, \beta)$.

Proof. By definition, we have

$$\begin{aligned} V_\lambda(f) &= \frac{(c+1)^\delta}{\lambda(\delta)} \int_0^1 (-1)^{\delta-1} t^c (logt)^{\delta-1} \left(z - \sum_{n=2}^{\infty} a_n z^n t^{n-1} \right) dt \\ &= \frac{(-1)^{\delta-1}(c+1)^\delta}{\lambda(\delta)} \lim_{r \to 0^+} \left[\int_r^1 t^c (logt)^{\delta-1} \left(z - \sum_{n=2}^{\infty} a_n z^n t^{n-1} \right) dt \right], \end{aligned}$$

and a simple calculation gives

$$V_\lambda(f)(z) = z - \sum_{n=2}^{\infty} \left(\frac{c+1}{c+n} \right)^\delta a_n z^n.$$

We need to prove that

$$\sum_{n=2}^{\infty} \frac{[(1+\beta)\Upsilon_n - (1+\beta-\eta)\gamma_n]}{\eta} \left(\frac{c+1}{c+n} \right)^\delta a_n < 1. \tag{3.1}$$

On the other hand by Theorem 2.2, $f \in \mathcal{D}_T(\Phi, \Psi; \alpha, \beta)$ if and only if

$$\sum_{n=2}^{\infty} \frac{[(1+\beta)\Upsilon_n - (1+\beta-\eta)\gamma_n]}{\eta} < 1.$$

Hence $\frac{c+1}{c+n} < 1$. Therefore (3.1) holds and the proof is complete. □

Next we provide a starlike condition for functions in $\mathcal{D}_T(\Phi,\Psi;\eta,\beta)$ and $V_\lambda(f)$.

Theorem 3.2. *Let $f\in\mathcal{D}_T(\Phi,\Psi;\eta,\beta)$. Then $V_\lambda(f)$ is starlike of order $0\le\tau<1$ in $|z|<R_1$ where*

$$R_1=\inf_n\left[\left(\frac{c+n}{c+1}\right)^{\delta}\frac{(1-\tau)[(1+\beta)\Upsilon_n-(1+\beta-\eta)\gamma_n]}{(n-\tau)\eta}\right]^{\frac{1}{n-1}}$$

Proof. It is sufficient to prove

$$\left|\frac{z(V_\lambda(f)(z))'}{V_\lambda(f)(z)}-1\right|<1-\tau. \tag{3.2}$$

For the left hand side of (4.2) we have

$$\left|\frac{z(V_\lambda(f)(z))'}{V_\lambda(f)(z)}-1\right|=\left|\frac{\sum\limits_{n=2}^{\infty}(1-n)(\frac{c+1}{c+n})^{\delta}a_nz^{n-1}}{1-\sum\limits_{n=2}^{\infty}(\frac{c+1}{c+n})^{\delta}a_nz^{n-1}}\right|$$
$$\le\frac{\sum\limits_{n=2}^{\infty}(n-1)(\frac{c+1}{c+n})^{\delta}a_n|z|^{n-1}}{1-\sum\limits_{n=2}^{\infty}(\frac{c+1}{c+n})^{\delta}a_n|z|^{n-1}}.$$

This last expression is less than $(1-\tau)$ since

$$|z|^{n-1}<\left(\frac{c+1}{c+n}\right)^{-\delta}\frac{(1-\tau)[(1+\beta)\Upsilon_n-(1+\beta-\eta)\gamma_n]}{(n-\tau)\eta}.$$

Therefore the proof is complete. ☐

Using the fact that f is convex if and only if zf' is starlike, we obtain the following:

Theorem 3.3. *Let $f\in\mathcal{D}_T(\Phi,\Psi;\eta,\beta)$. Then $V_\lambda(f)$ is convex of order $0\le\tau<1$ in $|z|<R_2$ where*

$$R_2=\inf_n\left[\left(\frac{c+n}{c+1}\right)^{\delta}\frac{(1-\tau)[(1+\beta)\Upsilon_n-(1+\beta-\eta)\gamma_n]}{n(n-\tau)\eta}\right]^{\frac{1}{n-1}}.$$

We omit the proof as it is easily derived.
Finally,

Theorem 3.4. *Let $f\in\mathcal{D}_T(\Phi,\Psi;\eta,\beta)$. Then $V_\lambda(f)$ is close-to-convex of order $0\le\tau<1$ in $|z|<R_3$ where*

$$R_3=\inf_n\left[\left(\frac{c+n}{c+1}\right)^{\delta}\frac{(1-\tau)[(1+\beta)\Upsilon_n-(1+\beta-\eta)\gamma_n]}{n\eta}\right]^{\frac{1}{n-1}}.$$

Again we omit the proofs.

4. An Application of Fractional Calculus

Many interesting results have been studied by various authors involving the fractional calculus. Too many to be mentioned here. However, our definitions for fractional calculus are due to Owa[7]. For other definitions, see [13], [14], [15] and [16]. For $\delta > 0$, the fractional integral of order δ is defined by

$$D_z^{-\delta} \frac{1}{\Gamma(\delta)} \int_0^z \frac{f(\zeta) d\zeta}{(z-\zeta)^{1-\delta}}$$

and f is analytic functions in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{\delta-1}$ is removed by requiring $log(z-\zeta)$ to be real when $z-\zeta > 0$. In addition for $0 \leq \delta < 1$, the fractional derivative of order δ is defined by

$$D_z^{\delta} \frac{1}{\Gamma(1-\delta)} \int_0^z \frac{f(\zeta) d\zeta}{(z-\zeta)^{\delta}}$$

and f is analytic functions in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{\delta}$ is removed by requiring $log(z-\zeta)$ to be real when $z-\zeta > 0$. Under this condition, the fractional derivative of order $n+\delta$ is defined by

$$D_z^{n+\delta} f(z) = \frac{d^n}{dz^n} D^{\delta} z f(z),$$

where $0 \leq \delta < 1$ and $n = 0, 1, \ldots$. Here we give simple results regarding the application of fractional calculus for functions in $\mathcal{D}_T(\Phi, \Psi; \eta, \beta)$, and the details of proving are omitted as the results are easily derived.

Theorem 4.1. *Let the function f defined by (1.6) be in the class $\mathcal{D}_T(\Phi, \Psi; \eta, \beta)$. Then*

$$\begin{aligned} &\frac{|z|^{1+\delta}}{\Gamma(2+\delta)} - |z|^{2+\delta} \frac{\eta}{(2+\delta)[(1+\beta)\Upsilon_2 - (1+\beta-\eta)\gamma_2]} \\ &\leq |D_z^{-\delta} f(z)| \leq \frac{|z|^{1+\delta}}{\Gamma(2+\delta)} + |z|^{2+\delta} \frac{\eta}{(2+\delta)[(1+\beta)\Upsilon_2 - (1+\beta-\eta)\gamma_2]} \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} &\frac{|z|^{1-\delta}}{\Gamma(2-\delta)} - |z|^{2-\delta} \frac{2\eta}{(2-\delta)[(1+\beta)\Upsilon_2 - (1+\beta-\eta)\gamma_2} \\ &\leq |D_z^{\delta} f(z)| \leq \frac{|z|^{1-\delta}}{\Gamma(2-\delta)} + |z|^{2-\delta} \frac{2\eta}{(2-\delta)[(1+\beta)\Upsilon_2 - (1+\beta-\eta)\gamma_2]} \end{aligned} \tag{4.2}$$

The bounds (4.1) and (4.2) are attained for functions given by (2.2).

Remark. Taking $\delta = 0$ and $\delta = 1$ respectively in (4.1) and (4.2), we have Theorem 2.5 which represent the bounds (2.5) and (2.6) respectively.

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