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NON-ELEMENTARY K-QUASICONFORMAL GROUPS ARE LIE GROUPS

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ABSTRACT. Suppose that Ω is a subdomain of $\overline{\mathbb{R}^n}$ and G is a non-elementary K-quasiconformal group. Then G is a Lie group acting on Ω.

Hilbert-Smith Conjecture states that every locally compact topological group acting effectively on a connected manifold must be a Lie group. Recently Martin [8] has solved the solution of the Hilbert-Smith Conjecture in the quasiconformal category (Theorem 1.2):

Theorem 1. *Let G be a locally compact group acting effectively by quasiconformal homeomorphisms on a Riemannian manifold. Then G is a Lie group.*

We will apply the Martin's theorem in this paper to show the following theorem.

Theorem 2. *Suppose that Ω is a subdomain of $\overline{\mathbb{R}^n}$ and G is a non-elementary K-quasiconformal group. Then G is a Lie group acting on Ω.*

Let Ω and Ω' be domains in $\mathbb{R}^n$, $n \geq 2$. A homeomorphism $f\colon \Omega \to \Omega'$ is called to be K-*quasiconformal* if $f \in W^{1,n}_{loc}(\Omega, \mathbb{R}^n)$, the Sobolev space of functions whose first derivatives are locally L^n integrable, and for some $K < \infty$, f satisfies the differential inequality

$$|Df(x)|^n \leq KJ(x,f) \text{ almost everywhere in } \Omega. \tag{1}$$

Here $Df(x)$ is the derivative of f, $|Df(x)|$ is operator norm and $J(x,f)$ is the Jacobian determinant. We say f is quasiconformal if f is K-quasiconformal for some finite K. Thus, quasiconformal homeomorphisms are transformations which have uniformly bounded distortion. They provide a class of mappings

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that lie between homeomorphisms and conformal mappings. A quasiconformal homeomorphism of domain Ω in $\mathbb{R}^n$ can be extended to a subdomain in the extended Euclidean space $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$, for instance, by setting $f(\infty) = \infty$ [12].

Let Γ denote the family of all quasiconformal homeomorphisms of a domain Ω onto Ω' in $\overline{\mathbb{R}^n}$, then Γ forms a group under composition [1]. Let Γ_K denote the family of all K-quasiconformal homeomorphisms of a domain Ω onto Ω' in $\overline{\mathbb{R}^n}$. By contrast, Γ_K is not a group if $K > 1$. However, when $K = 1$, the family Γ_1 of all 1- quasiconformal self homeomorphisms of Ω in $\overline{\mathbb{R}^n}$ is the conformal group of Ω. Indeed, this group Γ_1 is a subgroup of the Möbius transformation group if $n > 2$ or if $n = 2$ with $\Omega = \overline{\mathbb{R}^n}$. In the latter case when $n = 2$ with $\Omega = \overline{\mathbb{R}^n}$, Γ_1 is just the classical Möbius transformation group, that is the group of linear fractional transformations of $\overline{\mathbb{C}}$.

Let E be a non-empty subset of Ω, and define the *stabilizer* of a subset E:

$$\Gamma(E) = \{f \in \Gamma : f(E) = E\} \tag{2}$$

It is easy to see that $\Gamma(E)$ is a quasiconformal subgroup of Γ. And

$$\Gamma = \cup_{K\geq 1}\Gamma_K, \qquad \Gamma(E) = \cup_{K\geq 1}\Gamma_K(E) \tag{3}$$

where $\Gamma_K(E) = \{f \in \Gamma_K : f(E) = E\}$.

A subfamily G of Γ_K is called a *K-quasiconformal group* if it constitutes a subgroup of Γ under composition. For example, the quasiconformal conjugate

$$G = f^{-1} \circ \Gamma_1 \circ f$$

of a subgroup of Möbius transformations Γ_1 of Ω' by a K-quasiconformal map $f\colon \Omega \to \Omega'$ is a K^2-quasiconformal group acting on Ω. For subdomains of the plane Sullivan and Tukia showed in [9, 10], using a result of Maskit regarding groups of conformal transformations, that this is in fact the only construction. Namely a K-quasiconformal group of a domain $\Omega \subset \overline{\mathbb{R}^2}$ is quasiconformally conjugate to a subgroup of Möbius transformations of a domain $\Omega' \subset \overline{\mathbb{R}^2}$. The situation in higher dimensional is different, not every K-quasiconformal group is obtained in this way [7, 11].

As we know from Theorem 7.2 [3] that the compact-open topology of the space Γ of all quasiconformal homeomorphisms of a domain Ω onto Ω' in $\overline{\mathbb{R}^n}$ is equivalent to the topology induced from locally uniform convergence, where $\overline{\mathbb{R}^n}$ is a metric space with spherical metric. The space Γ is actually a metric space [4]. Therefore a compact subset coincides with a sequentially compact subset in Γ. And Γ possesses topological properties such as Hausdorff, normal and paracompact [3]. One of the most important aspects of quasiconformal homeomorphisms is their compactness properties. From now on every compact subset E of Ω in $\overline{\mathbb{R}^n}$ contains at least two points. We recall the following theorem from [4].

Theorem 3. *Suppose that Ω is a subdomain of $\overline{\mathbb{R}^n}$, that G is a K-quasiconformal group of Ω acting on a compact subset E of Ω, and that $G \subset \Gamma_K(E)$. Then G is a locally compact topological transformation group.*

Notice that a manifold here is an n-dimensional smooth manifold (C^∞ differentiable) and it is also second countable, thus it is paracompact [13]. A smooth manifold is called a *Riemannian manifold* if there exists a Riemannian metric on it. However, on a paracompact smooth manifold there always exists a Riemannian metric [5], and a topological manifold is paracompact. Hence:

Proposition 1. *Every smooth manifold is a Riemannian manifold. In particular, every domain Ω in $\overline{\mathbb{R}^n}$ can be regarded as a Riemannian manifold.*

Suppose that G is a topological transformation group of a topological space X. For each $x \in X$, consider the subset of G:

$$G_x = \{g \in G : g(x) = x\}. \tag{4}$$

It is a subgroup of G which is called the *isotropy* subgroup of G at the point x of X. Similarly, consider the subset of G:

$$G_X = \{g \in G : g(x) = x, \quad \text{for all } x \in X\}. \tag{5}$$

It is a normal subgroup of G, and we have

$$G_X = \cap_{x \in X} G_x. \tag{6}$$

The topological transformation group G is said to act *effectively* on a topological space X if $G_X = \{e\}$. In the case that a topological transformation group G acts effectively on a topological space X, the corresponding group action is said to be *faithful* [2], i.e., the homomorphism

$$\phi \colon G \to \mathrm{Homeo}(X), \text{ given by } g \mapsto g(x). \tag{7}$$

is faithful if ϕ is injective: $\mathrm{Ker}\,\phi = \{e\}$. A topological transformation group may not act effectively on a topological space in general. But quasiconformal homeomorphisms are different, we have

Proposition 2. *Let G be a K-quasiconformal group of a domain Ω in $\overline{\mathbb{R}^n}$. Then G is a topological transformation group acting effectively on Ω.*

Proof. Notice that G is a topological transformation group [4]. Since $G \subset \mathrm{Homeo}(\Omega)$, where $\mathrm{Homeo}(\Omega)$ is the group of all homeomorphisms of Ω, consider the inclusion ϕ of G into $\mathrm{Homeo}(\Omega)$, then ϕ is injective, i.e., $\mathrm{Ker}\,\phi = \{e\}$. It is easy to see that $G_X = \mathrm{Ker}\,\phi$. Thus $G_X = \{e\}$. □

Suppose that Ω is a subdomain of $\overline{\mathbb{R}^n}$, G is a K-quasiconformal group of Ω onto itself, and a compact subset E of Ω is invariant under G. Then the K-quasiconformal group G is a Lie group acting on Ω.

Theorem 4. *Suppose that Ω is a subdomain of $\overline{\mathbb{R}^n}$ and $G \subset \Gamma_K(E)$ is a K-quasiconformal group. Then G is a Lie group acting on Ω.*

Proof. Apply Theorem 3, Proposition 1 and Proposition 2 to Theorem 1, we have the result. □

A quasiconformal group G of self homeomorphisms of a domain Ω in $\overline{\mathbb{R}^n}$ is said to be *discontinuous* at a point $x \in \Omega$ if there exists a neighborhood U of x such that $g(U) \cap U = \varnothing$ for all but finite many $g \in G$. The *ordinary set* of G, denoted $O(G)$, is the set of all $x \in \Omega$ at which G is discontinuous. We say that G is a discontinuous group if $O(G) \neq \varnothing$. In other words, there exists one point of Ω which has a neighborhood that is carried outside of itself by all but finitely many elements of G. The complement of $O(G)$ is called the *limit set* of G and is denoted by $L(G)$: $L(G) = \Omega \setminus O(G)$. We say that G is an *elementary group* if the limit set $L(G)$ contains at most two points. Otherwise we say that G is *non-elementary.* Now it is ready for the main theorem mentioned at beginning.

Theorem 2. *Suppose that Ω is a subdomain of $\overline{\mathbb{R}^n}$ and G is a non-elementary K-quasiconformal group. Then G is a Lie group acting on Ω.*

Proof. Clearly, the ordinary set $O(G)$ is an open set in Ω hence in $\overline{\mathbb{R}^n}$. It follows that the limit set $L(G)$ is a closed set in Ω and $\overline{\mathbb{R}^n}$, thus $L(G)$ is a compact. Since the limit set $L(G)$ is invariant under G (Page 511, [6]), apply for $E = L(G)$ in Theorem 4, we immediately have the result. □

This result leads to a natural question. Is the hypothesis of Theorem 2 that the group is non-elementary?

Indeed, Theorem 2 is held for an elementary K-quasiconformal group if its limit set $L(G)$ contains two points, because the subset E in Theorem 3 contains at least two points. Also, we belive that Theorem 2 will be true for an elementary K-quasiconformal group if its limit set $L(G)$ contains at most one point.

References

[1] L. V. Ahlfors. *Lectures on quasiconformal mappings.* Manuscript prepared with the assistance of Clifford J. Earle, Jr. Van Nostrand Mathematical Studies, No. 10. D. Van Nostrand Co., Inc., Toronto, Ont.-New York-London, 1966.

[2] M. Berger. *Geometry. I.* Universitext. Springer-Verlag, Berlin, 1987. Translated from the French by M. Cole and S. Levy.

[3] J. Dugundji. *Topology.* Allyn and Bacon Inc., Boston, Mass., 1966.

[4] J. Gong and G. J. Martin. Compactness of uniformly quasiconformal groups. In *Geometric Groups on the Gulf Coast.* Pensacola, USA, 2008.

[5] K. Itō, editor. *Encyclopedic dictionary of mathematics.* MIT Press, Cambridge, MA, second edition, 1987. Translated from the Japanese.

[6] T. Iwaniec and G. Martin. *Geometric function theory and non-linear analysis.* Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 2001.

[7] G. J. Martin. Discrete quasiconformal groups that are not the quasiconformal conjugates of Möbius groups. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 11(2):179–202, 1986.

[8] G. J. Martin. The Hilbert-Smith conjecture for quasiconformal actions. *Electron. Res. Announc. Amer. Math. Soc.*, 5:66–70 (electronic), 1999.

[9] D. Sullivan. On the ergodic theory at infinity of an arbitrary discrete group of hyperbolic motions. In *Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978)*, volume 97 of *Ann. of Math. Stud.*, pages 465–496. Princeton Univ. Press, Princeton, N.J., 1981.
[10] P. Tukia. On two-dimensional quasiconformal groups. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 5(1):73–78, 1980.
[11] P. Tukia. A quasiconformal group not isomorphic to a Möbius group. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 6(1):149–160, 1981.
[12] J. Väisälä. *Lectures on n-dimensional quasiconformal mappings*. Lecture Notes in Mathematics, Vol. 229. Springer-Verlag, Berlin, 1971.
[13] F. W. Warner. *Foundations of differentiable manifolds and Lie groups*. Scott, Foresman and Co., Glenview, Ill.-London, 1971.

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