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# UNIFORM CONVEXITY OF KÖTHE–BOCHNER FUNCTION SPACES

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Abstract. Of concern are the Köthe–Bochner function spaces $E(X)$, where $X$ is a real Banach space. Thus, of concern is the uniform convexity on the Köthe–Bochner function space $E(X)$. We show that $E(X)$ is uniformly convex if and only if both spaces $E$ and $X$ are uniformly convex. It is the uniform convexity that is the focal point.

## 1. Basic Concepts

Throughout this paper $(T, \sum, \mu)$ denote a $\delta$-finite complete measure space and $L^\circ = L^\circ(T)$ denotes the space of all (equivalence classes) of $\sum$-measurable real valued functions. For $f, g \in L^\circ, f \leq g$ means that $f(t) \leq g(t)$ $\mu$-almost every where $t \in T$.

A Banach space is said to be a Köthe space if:

(1) For any $f, g \in L^\circ$, $|f| \leq |g|$, $g \in E$ imply $f \in E$ and $\|f\|_E \leq \|g\|_E$.
(2) For each $A \in \sum$, if $\mu(A)$ is finite then $\chi_A \in E$. See [12, p. 28].

Let $E$ be a Köthe space on the measure space $(T, \sum, \mu)$ and $(X, \|\cdot\|_X)$ be a real Banach space. Then $E(X)$ is the space (of all equivalence classes of) strongly measurable functions $f : T \to X$ such that $\|f(\cdot)\|_X \in E$ equipped with the norm

$$|||f||| = \|\|f(\cdot)\|_X\|_E \, .$$

The space $(E(X), |||\cdot|||_E)$ is a Banach space called the Köthe–Bochner function space [11, p. 147]. The most important class of Köthe–Bochner function spaces $E(X)$ are the Lebesgue–Bochner spaces $L^p(X), (1 \leq p < \infty)$ and their generalization the Orlicz–Bochner spaces $L^\phi(X)$. They have been studied by many authors [1], [2], [6], [7]. The geometric properties of the Köthe–Bochner function spaces have been studied by many authors, (e.g. [1], [5], [10]).

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A Banach space is uniformly convex if and only if for every $\epsilon > 0$, there is a unique $\delta > 0$, such that for all $x, y$ in $X$, the conditions $\|x\| = \|y\| = 1$ and $\|x-y\| > \epsilon$ imply $\left\|\frac{x+y}{2}\right\| < 1-\delta$. Moreover this definition is equivalent to the following, [3, p. 127], for every pair of sequences $(x_n)$ and $(y_n)$ in $X$ with $\|x_n\| \leq 1, \|y_n\| \leq 1$ and $\|x_n + y_n\| \leq 2$, it follows $\|x_n - y_n\| \to 0$.

For a Banach space $X$, we denote by $\delta_X$ the modulus of convexity

$$\delta_X(\epsilon) = \inf\left\{1 - \frac{1}{2}\|x+y\| : x, y \in X,\ \|x\| = \|y\| = 1, \|x-y\| > \epsilon\right\}$$

for any $\epsilon \in [0,2]$. Note that $X$ is uniformly convex if and only if $\delta_X(\epsilon) > 0$ whenever $\epsilon > 0$. If $X$ is uniformly convex, we define the characteristic of convexity of by

$$\epsilon_r(X) = \sup\left\{\epsilon \in [0,2] :\ \delta_X(\epsilon) \leq r\right\}.$$

The uniform convexity of both the Lebesgue–Bochner spaces $L^p(X), (1 \leq p < \infty)$ and the Orlicz–Bochner spaces $L^\phi(X)$ have been studied by many authors (e.g. see [4], [5], [8], [9], [14]).

In [14], M. Smith and B. Truett showed that many properties akin to uniform convexity lift from $X$ the Lebesgue–Bochner spaces $L^p(X), (1 \leq p < \infty)$. A survey of rotundity notions in the Lebesgue–Bochner spaces $L^p(X)$, $(1 \leq p < \infty)$ and sequences spaces can be found in [13].

In [10], A. Kaminska and B. Truett showed that many properties akin to uniform convexity lift from $X$ to $E(X)$. The approach used in their paper is different than that we used.

This paper is devoted to the study of uniform convexity of the Köthe–Bochner function space $E(X)$, where $X$ is a real Banach space.

## 2. Main Results

Let us prove some preliminary results which will allow us in Theorem 4 to obtain a characterization of the uniform convexity of the Köthe–Bochner function space $E(X)$.

**Lemma 1.** *If $(f_n)$ and $(g_n)$ are sequences in the Köthe–Bochner function space $E(X)$ with $|||f_n||| = |||g_n||| = 1$ and $|||f_n + g_n||| \to 2$ then*

$$\|\|f_n(\cdot)\|_X + \|g_n(\cdot)\|_X\|_E \to 2.$$

*Proof.* Using the following inequalities we get the required result

$$\begin{aligned} |||f_n + g_n||| &= \|\|f_n(\cdot) + g_n(\cdot)\|_X\|_E \\ &\leq \|\|f_n(\cdot)\|_X + \|g_n(\cdot)\|_X\|_E \\ &\leq |||f_n||| + |||g_n||| = 2. \end{aligned}$$

□

**Lemma 2.** *Let $X$ be a real Banach space and $E$ be a Köthe space. If $(E, \|\cdot\|_E)$ is uniformly convex Köthe space and $(f_n)$, $(g_n)$ are sequences in the Köthe–Bochner function space $E(X)$ with $|||f_n||| = |||g_n||| = 1$ and $|||f_n + g_n||| \to 2$ then,*

(i) $\|\|f_n(\cdot)\|_X - \|g_n(\cdot)\|_X\|_E \to 0.$
(ii) $\|\|f_n(\cdot)\|_X + \|g_n(\cdot)\|_X - \|f_n(\cdot) + g_n(\cdot)\|_X\|_E \to 0.$

*Proof.* Note that $(\|f_n(\cdot)\|_X)$ and $(\|g_n(\cdot)\|_X)$ are sequences in $E$ with

$$\|\|f_n(\cdot)\|_X\|_E = \|\|g_n(\cdot)\|_X\|_E = 1.$$

Using Lemma 1 and the fact that $E$ is uniformly convex Köthe space, it is straightforward to get (i). To prove (ii) we note first the inequalities

$$\begin{aligned} 2\,|||f_n + g_n||| &= 2\,\|\|f_n(\cdot) + g_n(\cdot)\|_X\|_E \\ &\le \|\|f_n(\cdot)\|_X + \|g_n(\cdot)\|_X + \|f_n(\cdot) + g_n(\cdot)\|_X\|_E \\ &\le 2\,\|\|f_n(\cdot)\|_X + \|g_n(\cdot)\|_X\|_E \\ &\le 2(|||f_n||| + |||g_n|||) = 4. \end{aligned}$$

In this line, we get

$$\left\| \left\| \frac{f_n(\cdot) + g_n(\cdot)}{2} \right\|_X + \frac{\|f_n(\cdot)\|_X}{2} + \frac{\|g_n(\cdot)\|_X}{2} \right\|_E \to 2.$$

Due to the uniform convexity of $E$ and the facts that $\left\|\left\|\frac{f_n(\cdot)+g_n(\cdot)}{2}\right\|_X\right\|_E \le 1$ and $\left\|\frac{\|f_n(\cdot)\|_X}{2} + \frac{\|g_n(\cdot)\|_X}{2}\right\|_E \le 1$, result (ii) is completely proved. □

**Lemma 3.** *Let $X$ be a uniformly convex Banach space and $E(X)$ be a Köthe–Bochner function space. If $f, g \in E(X)$ then*

$$|||f - g||| = \epsilon_r(X)(|||f||| + |||g|||) + \frac{2}{\epsilon}(\|\|f(\cdot)\|_X - \|g(\cdot)\|_X\|_E + \\ \|\|f(\cdot)\|_X + \|g(\cdot)\|_X - \|f(\cdot) + g(\cdot)\|_X\|_E\,.$$

*Proof.* Holding $t \in T$ fixed and letting $\epsilon > 0$, we have two inequalities

(i) $|\|f(t)\|_X - \|g(t)\|_X| \le \epsilon \max\{\|f(t)\|_X, \|g(t)\|_X\}$.
(ii) $\|f(t)\|_X + \|g(t)\|_X - \|f(t) + g(t)\|_X \le \epsilon \max\{\|f(t)\|_X, \|g(t)\|_X\}$.

We have three cases to be considered

*Case 1.* (i) and (ii) are true. Assuming that $\|f(t)\|_X \ge \|g(t)\|_X$ and letting $\widetilde{f}(t) = \frac{f(t)}{\|f(t)\|_X}$, $\widetilde{g}(t) = \frac{g(t)}{\|f(t)\|_X}$, we find that $\|\widetilde{g}(t)\|_X \le \left\|\widetilde{f}(t)\right\|_X = 1$. Furthermore, it is straightforward to verify that

$$\begin{aligned} \left\|\widetilde{g}(t) + \widetilde{f}(t)\right\|_X &\ge \frac{\|f(t)\|_X + \|g(t)\|_X - \epsilon\,\|f(t)\|_X}{\|f(t)\|_X} \\ &= 1 + \frac{\|g(t)\|_X}{\|f(t)\|_X} - \epsilon \end{aligned}$$

$$\geq 1 + \frac{\|f(t)\|_X - \epsilon \|f(t)\|_X}{\|f(t)\|_X} - \epsilon$$
$$= 2 - 2\epsilon$$

and, since $X$ is uniformly convex and $\|f(t)\|_X, \|g(t)\|_X$ are elements of $X$ for a fixed $t \in T$, find that

$$\left\| \widetilde{f}(t) - \widetilde{g}(t) \right\|_X \leq \epsilon_r(X).$$

Therefore we deduce that

$$\|f(t) - g(t)\|_X \leq \epsilon_r(X) \max\{\|f(t)\|_X, \|g(t)\|_X\}. \tag{1}$$

*Case 2.* (i) is not true. We imply that if

$$|\|f(t)\|_X - \|g(t)\|_X| > \max\{\|f(t)\|_X, \|g(t)\|_X\}.$$

And so

$$\begin{aligned} \|f(t) - g(t)\|_X &\leq 2 \max\{\|f(t)\|_X, \|g(t)\|_X\} \\ &< \frac{2}{\epsilon} |\|f(t)\|_X - \|g(t)\|_X|. \end{aligned} \tag{2}$$

*Case 3.* (ii) is not true. We can get that

$$\begin{aligned} \|f(t) - g(t)\|_X &\leq 2 \max\{\|f(t)\|_X, \|g(t)\|_X\} \\ &< \frac{2}{\epsilon}(\|f(t)\|_X + \|g(t)\|_X - \|f(t) + g(t)\|_X). \end{aligned} \tag{3}$$

Inequalities (1), (2), and (3) give the inequality

$$\begin{aligned} \|f(t) - g(t)\|_X \leq \epsilon_r(X)(\|f(t)\|_X + \|g(t)\|_X) + \frac{2}{\epsilon}(\|f(t)\|_X + \\ \|g(t)\|_X + \|f(t)\|_X + \|g(t)\|_X - \|f(t) + g(t)\|_X) \end{aligned}$$

and (ii) is completely proved. □

In this line, we are able to introduce the following main theorem about the uniform convexity of the Köthe–Bochner function space $E(X)$.

**Theorem 4.** *Let $X$ be a real Banach space and $E$ be a Köthe space. Then $E(X)$ is a uniformly convex Köthe–Bochner space if and only if both $X$ and $E$ are uniformly convex.*

*Proof.* Suppose $E(X)$ is uniformly convex Köthe–Bochner function space. Since both spaces $X$ and $E$ are embedded isometrically into $E(X)$, and due to the fact that uniform convexity inherited by subspaces, we deduce that both spaces $X$ and $E$ are uniformly convex.

Conversely, suppose $X$ and $E$ are uniformly convex spaces. Let $(f_n)$ and $(g_n)$ be sequences in $E(X)$ with $|||f_n||| = |||g_n||| = 1$ and $|||f_n + g_n||| \to 2$, then by Lemma 2 it follows that

$$\| |\|f_n(\cdot)\|_X - \|g_n(\cdot)\|_X| \|_E \to 0$$

and

$$\| \|f_n(\cdot)\|_X + \|g_n(\cdot)\|_X - \|f_n(\cdot) + g_n(\cdot)\|_X \|_E \to 0.$$

Take $\epsilon^* > 0$ and choose $\delta > 0$ such that $\epsilon_r(X) \le \frac{\epsilon^*}{4}$. Choose $K$ sufficiently large so that for all $n > K$,

$$\| \|f_n(\cdot)\|_X + \|g_n(\cdot)\|_X \|_E + \| \|f_n(\cdot)\|_X + \|g_n(\cdot)\|_X - \|f_n(\cdot) + g_n(\cdot)\|_X \|_E < \frac{1}{4}\epsilon\epsilon^*.$$

From Lemma 3 it follows that

$$\begin{aligned} |||f_n - g_n||| &= \epsilon_r(X)(|||f_n||| + |||g_n|||) + \frac{2}{\epsilon}(\| \|f_n(\cdot)\|_X - \|g_n(\cdot)\|_X \|_E + \\ &\qquad \| \|f_n(\cdot)\|_X + \|g_n(\cdot)\|_X - \|f_n(\cdot) + g_n(\cdot)\|_X \|_E) \\ &\le 2\epsilon_r(X) + \frac{2}{\epsilon}(\frac{1}{4}\epsilon\epsilon^*) \\ &\le \frac{1}{2}\epsilon^* + \frac{1}{2}\epsilon^* = \epsilon^*, \qquad \text{for all } n > K. \end{aligned}$$

Consequently $|||f_n - g_n||| \to 0$, which means that $E(X)$ is a uniformly convex Köthe–Bochner space, and thereby the theorem is completely proved. □

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