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RANGE-COMMUTATIVITY MAPS ON *-ALGEBRAS

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ABSTRACT. Let $\mathcal{A}$ and $\mathcal{B}$ be $*$-algebras, it is shown that if φ be a linear map from $\mathcal{A}$ into $\mathcal{B}$ such that $\phi(x^p)$ commutes with $\phi(x^*)$ for all x in $\mathcal{A}$ then the range of φ is commutative. The paper also studies other conditions for commutativity range of φ on $*$-algebra $\mathcal{A}$.

1. Introduction

A map φ from a Banach algebra $\mathcal{A}$ into a Banach algebra $\mathcal{B}$ is called preserving commutativity if

$$\varphi(x)\varphi(y) = \varphi(y)\varphi(x), \text{ whenever } xy = yx, \tag{1}$$

and it is called Range-commutativity if

$$\varphi(x)\varphi(y) = \varphi(y)\varphi(x), \quad \forall x, y \in \mathcal{A}. \tag{2}$$

The problem of describing commutativity preserving linear maps is one of the most studied linear preserver problems. The usual solution is that they are standard. The problem was first considered in the case where $\mathcal{A} = M_n(F)$ is the algebra of $n \times n$ matrices over F [7], and has been afterwards studied in various much more general algebras (see [3, 6] for references). A rather general theorem, and also the first ring-theoretic result in this area, was obtained in [3] by Bresar. This theorem treats the case where $\mathcal{A}$ is a prime algebra satisfying certain technical conditions, and instead of the commutativity preserving assumption it only requires a milder assumption that

$$\varphi(x^2)\varphi(x) = \varphi(x)\varphi(x^2), \quad \forall x \in \mathcal{A}. \tag{3}$$

The paper [3], particularly the idea to consider the condition (3), initiated a systematic study of different topics in ring theory and functional analysis; the reader is referred to [1, 4, 6] for historical accounts and further references. In the present paper we shall use this idea in a somewhat different way.

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Bresar and Semrl [4] proved the following theorem:

Theorem 1.1 ([5]). *Let F be a Field with char$(F) = 0$, and let $\mathcal{A}$ be a finite dimensional central simple algebra over F with dim$\mathcal{A} \neq 4$. If a linear map $f\colon \mathcal{A} \longrightarrow \mathcal{A}$ satisfies (3), then f is either a standard commutativity preserving map or its range is commutative.*

Can the assumption that $\mathcal{A}$ is finite dimensional be omitted? This is the question that initiated the present paper. The paper consider the case where $\mathcal{A}$ is a $*$-algebra.

We will, first consider a more general problem concerning certain bilinear maps and then is devoted to the proof of the main result.

Let $\mathcal{A}$ be a $*$-algebra, X be a vector space and $f\colon \mathcal{A}\times\mathcal{A} \longrightarrow X$. Our question is: does the condition

$$f(x,y) = 0 \quad \forall x,y \in \mathcal{A} \tag{4}$$

follow from the condition

$$f(x,x) = f(x^2,x^*) = 0, \quad \forall x \in \mathcal{A}. \tag{5}$$

The connection with the aforementioned question is simple. Namely, if ϕ is a linear map $\varphi\colon \mathcal{A} \longrightarrow \mathcal{B}$ such that $\phi(x^p)$ commutes with $\phi(x^*)$ for all x in $\mathcal{A}$, then a map f defined by $f(x,y) = \varphi(x)\varphi(y) - \varphi(y)\varphi(x)$ satisfies (5). Therefore, the affirmative answer to the latter question implies the affirmative answer to the former question.

This paper considers range-commutativity mappings on $*$-algebras. We show that if φ be a linear or conjugate linear map from $\mathcal{A}$ into $\mathcal{B}$ satisfies

$$\varphi(x^2)\varphi(x^*) = \varphi(x^*)\varphi(x^2), \quad \forall x \in \mathcal{A}, \tag{6}$$

then the range of φ is commutative. The paper also studies other conditions for commutativity range of φ on $*$-algebra $\mathcal{A}$.

2. Main results

Theorem 2.1. *Let $\mathcal{A}$ be a unital $*$-algebra with identity e and X be a vector space and $f\colon \mathcal{A} \times \mathcal{A} \longrightarrow X$ be a linear map with respect to first (or second) component and it is an additive map with respect to second (or first) component. Then each of the following two conditions implies $f(x,y) = 0$ for every $x,y \in \mathcal{A}$:*

$$f(x,x) = f(x^*,x) = 0, \quad \forall x \in \mathcal{A}, \tag{i}$$

$$f(x,x) = f(x^*x,x) = 0, \quad \forall x \in \mathcal{A} \tag{ii}$$

Proof. For $x,y \in \mathcal{A}$ the assumption $f(x,x) = 0$ shows that

$$f(x,y) + f(y,x) = 0, \tag{7}$$

and

$$f(x,y) + f(y,x) = f(x,y) + f(y,x) + f(x,x) + f(y,y) = f(x+y,x+y) = 0.$$

(i) Let $x, y \in \mathcal{A}$, by replacing x by $x+y$ in (i) we have $f((x+y)^*, (x+y)) = 0$. It implies that

$$f(x^*, y) + f(y^*, x) = 0. \tag{8}$$

Now, assume for a moment that x is self-adjoint. By using (7) and (8) we conclude

$$f(x, y) + f(y^*, x) = -f(y, x) + f(y^*, x) = f(y - y^*, x) = 0. \tag{9}$$

By replacing y by iy in (9) we have

$$f(y + y^*, x) = 0. \tag{10}$$

By combining (9) and (10) we obtain $f(x, y) = f(y, x) = 0$. Note that, the first equality is held by (7). Now if x is arbitrary element in $\mathcal{A}$ we can write $x = x_1 + ix_2$ which x_1 and x_2 are self-adjoint elements in $\mathcal{A}$. Linearity of f with respect to first component follows that $f(x, y) = 0$ for all $x, y \in \mathcal{A}$.

(ii) Let $x, y \in \mathcal{A}$, by replacing x by $x + y$ in (ii) we obtain $f((x + y)^*(x + y), x + y) = 0$. It follows that

$$f(x^*x, y) + f(y^*y, x) + f(x^*y + y^*x, x + y) = 0, \tag{11}$$

By replacing y by $-y$ in (11) we have

$$-f(x^*x, y) + f(y^*y, x) + f(x^*y + y^*x, -x + y) = 0. \tag{12}$$

Combining (11) and (12) follows that

$$f(x^*x, y) + f(x^*y + y^*x, x) = 0. \tag{13}$$

Now by using (7), applying (13) once by replacing x by e and once by replacing y by e we have $f(x, e) = 0$ and $f(x^*, x) = 0$. By (i) we yield $f(x, y) = 0$ for every $x, y \in \mathcal{A}$. $\square$

Theorem 2.2. *Let $\mathcal{A}$ be a unital $*$-algebra with identity e and X be a vector space and $f\colon \mathcal{A} \times \mathcal{A} \longrightarrow X$ be a bilinear map satisfies:*

$$f(x, x) = 0, \quad \forall x \in \mathcal{A}_+, \tag{iii}$$

$$f(x^2, x^*) = 0, \quad \forall x \in \mathcal{A}. \tag{iv}$$

Then $f(x, y) = 0$ for every $x, y \in \mathcal{A}$. Where $\mathcal{A}_+$ is the set of all positive elements in $\mathcal{A}$.

Proof. Let $x \in \mathcal{A}_+$ we have

$$f(x, e) + f(e, x) = f(x, e) + f(e, x) + f(x, x) + f(e, e) = f(x + e, x + e) = 0. \tag{14}$$

Since every self adjoint element is difference of two positive elements it is true for every self adjoint elements.

Let $x, y \in \mathcal{A}$, we can replace x once by $x + |x^*|$ and once by $x - |x^*|$ in (iv) to derive

$$f((x + |x^*|)^2, x^* + |x^*|) = 0,$$

$$f((x-|x^*|)^2, x^*-|x^*|)=0.$$

These imply that

$$f(|x^*|^2, x^*)+f(x|x^*|+|x^*|x, |x^*|)=0. \tag{15}$$

Since f is bilinear by replacing x by ix in (15) we obtain $f(|x|^2, x)=0$. Similarly to the proof of Theorem 2.1 (ii) and by applying (14) we conclude $f(x,e)=0$ and $f(x^*+x, x)=0$. Since f is a bilinear map we obtain $f(x^*,x)=f(x,x)=0$. By Theorem 2.1 (i) we yield $f(x,y)=0$, for every $x,y\in\mathcal{A}$. □

Theorem 2.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be $*$-algebras and $\varphi\colon\mathcal{A}\longrightarrow\mathcal{B}$ be a linear (or conjugate linear) map, then each of the following three conditions implies the range of φ is commutative:*

v $$\varphi(x^*)\varphi(x)=\varphi(x)\varphi(x^*),\quad \forall x\in\mathcal{A},$$

vi $$\varphi(x^*x)\varphi(x)=\varphi(x)\varphi(x^*x),\quad \forall x\in\mathcal{A},$$

vii $$\varphi(x^2)\varphi(x^*)=\varphi(x^*)\varphi(x^2),\quad \forall x\in\mathcal{A}.$$

Proof. First, let $\mathcal{A}$ be a unital $*$-algebra with identity e. Define

$$f\colon\mathcal{A}\times\mathcal{A}\longrightarrow\mathcal{B}$$

with

$$f(x,y)=\varphi(x)\varphi(y)-\varphi(y)\varphi(x),\quad \forall x,y\in\mathcal{A}.$$

f is a bilinear map and $f(x,x)=f(x^*,x)=0$, for all $x\in\mathcal{A}$, if (v) is true, $f(x,x)=f(x^*x,x)=0$, for all $x\in\mathcal{A}$ if (vi) is true and $f(x,x)=f(x^2,x^*)=0$, for all $x\in\mathcal{A}$ if (vii) is true. According to Theorem 2.1 and 2.2 the range of φ is commutative.

Now, assume that $\mathcal{A}$ is without identity. We can consider two case
1)$\mathcal{B}$ is without identity
2)$\mathcal{B}$ has identity e.
In the first case we set

$$\widetilde{\mathcal{A}}=\mathcal{A}\bigoplus\mathcal{C}$$

and

$$\widetilde{\mathcal{B}}=\mathcal{B}\bigoplus\mathcal{C}$$

which are unitization of $\mathcal{A}$ and $\mathcal{B}$ respectively. We can extent φ to

$$\widetilde{\varphi}\colon\widetilde{\mathcal{A}}\longrightarrow\widetilde{\mathcal{B}}$$

as follows: $\widetilde{\varphi}(x,\lambda)=(\varphi(x),\lambda)$. In the second case we can extent φ to $\widetilde{\varphi}\colon\widetilde{\mathcal{A}}\longrightarrow\mathcal{B}$ as follows: $\widetilde{\varphi}(x,\lambda)=\varphi(x)+\lambda e$. In both case it is easy to see that $\widetilde{\varphi}$ as defined above satisfies (v), (vi) and (vii). Therefore, the range of $\widetilde{\varphi}$ is commutative. It follows immediately the range of φ is commutative.

Now, assume that φ is conjugate linear map. Define $\bar{\varphi}(x)=\varphi(x)^*$, then $\bar{\varphi}$ is a linear map which satisfies (v), (vi) and (vii). Hence the range of $\bar{\varphi}$ is commutative. Consequently, the range of φ is commutative. □

References

[1] P. Ara and M. Mathieu. *Local multipliers of C^*-algebras*. Springer Monographs in Mathematics. Springer-Verlag London Ltd., London, 2003.
[2] G. M. Bergman. Centralizers in free associative algebras. *Trans. Amer. Math. Soc.*, 137:327–344, 1969.
[3] M. Brešar. Commuting traces of biadditive mappings, commutativity-preserving mappings and Lie mappings. *Trans. Amer. Math. Soc.*, 335(2):525–546, 1993.
[4] M. Brešar. Commuting maps: a survey. *Taiwanese J. Math.*, 8(3):361–397, 2004.
[5] M. Brešar and P. Šemrl. Commutativity preserving linear maps on central simple algebras. *J. Algebra*, 284(1):102–110, 2005.
[6] M. Omladič, H. Radjavi, and P. Šemrl. Preserving commutativity. *J. Pure Appl. Algebra*, 156(2-3):309–328, 2001.
[7] W. Watkins. Linear maps that preserve commuting pairs of matrices. *Linear Algebra and Appl.*, 14(1):29–35, 1976.

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