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ACYCLIC NUMBERS OF GRAPHS

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ABSTRACT. A subset S of vertices in a graph G is *acyclic* if the subgraph $\langle S\rangle$ induced by S contains no cycles. The *lower acyclic number*, $i_a(G)$, is the smallest number of vertices in a maximal acyclic set in G. The *upper acyclic number*, $\beta_a(G)$, is the maximum cardinality of an acyclic set in G. Let $\mu \in \{\beta_a, i_a\}$. Any maximal acyclic set S of a graph G with $|S| = \mu(G)$ is called a μ-set of G. A vertex x of a graph G is called: (i) *μ-good* if x belongs to some μ-set, (ii) *μ-fixed* if x belongs to every μ-set, (iii) *μ-free* if x belongs to some μ-set but not to all μ-sets, (iv) *μ-bad* if x belongs to no μ-set. In this paper we deal with μ-good/bad/fixed/free vertices and present results on upper and lower acyclic numbers in graphs having cut-vertices.

1. Introduction

We consider finite, simple graphs. The vertex set and the edge set of a graph G is denoted by $V(G)$ and $E(G)$, respectively. The subgraph induced by $S \subseteq V(G)$ is denoted by $\langle S, G\rangle$. For a vertex x of G, $N(x, G)$ denote the set of all neighbors of x in G and $N[x, G] = N(x, G) \cup \{x\}$.

A subset of vertices S in a graph G is said to be *acyclic* if $\langle S, G\rangle$ contains no cycles. Note that the property of being acyclic is a hereditary property, that is, any subset of an acyclic set is itself acyclic. An acyclic set $S \subseteq V(G)$ is *maximal* if for every vertex $v \in V(G) - S$, the set $S \cup \{v\}$ is not acyclic. The *lower acyclic number*, $i_a(G)$, is the smallest number of vertices in a maximal acyclic set in G. The *upper acyclic number*, $\beta_a(G)$, is the maximum cardinality of an acyclic set in G. These two numbers were defined by S.M. Hedetniemi et al. in [4]. We denote by $\mathrm{MAS}(G)$ the set of all maximal acyclic sets of a graph G. For every vertex $x \in V(G)$, let $\mathrm{MAS}(x, G) = \{A \in \mathrm{MAS}(G) : x \in A\}$.

Let $\mu(G)$ be a numerical invariant of a graph G defined in such a way that it is the minimum or maximum number of vertices of a set $S \subseteq V(G)$ with a given property P. A set with property P and with $\mu(G)$ vertices in G is called a μ-set of G. A vertex v of a graph G is defined to be

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(a) μ-*good*, if v belongs to some μ-set of G [3];
(b) μ-*bad*, if v belongs to no μ - set of G [3];
(c) μ-*fixed* if v belongs to every μ-set [5];
(d) μ-*free* if v belongs to some μ-set but not to all μ-sets [5].

For a graph G and $\mu \in \{i_a, \beta_a\}$ we define:

$$\begin{aligned}
\mathbf{G}(G,\mu) &= \{x \in V(G) : x \text{ is } \mu\text{-good}\};\\
\mathbf{B}(G,\mu) &= \{x \in V(G) : x \text{ is } \mu\text{-bad}\};\\
\mathbf{F}i(G,\mu) &= \{x \in V(G) : x \text{ is } \mu\text{-fixed}\};\\
\mathbf{F}r(G,\mu) &= \{x \in V(G) : x \text{ is } \mu\text{-free}\};\\
\mathbf{V}_0(G,\mu) &= \{x \in V(G) : \mu(G-x) = \mu(G)\};\\
\mathbf{V}_-(G,\mu) &= \{x \in V(G) : \mu(G-x) < \mu(G)\};\\
\mathbf{V}_+(G,\mu) &= \{x \in V(G) : \mu(G-x) > \mu(G)\}.
\end{aligned}$$

Clearly, $\{\mathbf{V}_-(G,\mu), \mathbf{V}_0(G,\mu), \mathbf{V}_+(G,\mu)\}$ and $\{\mathbf{G}(G,\mu), \mathbf{B}(G,\mu)\}$ are partitions of $V(G)$, and $\{\mathbf{F}i(G,\mu), \mathbf{F}r(G,\mu)\}$ is a partition of $\mathbf{G}(G)$.

Observation 1.1. *For any nontrivial graph G the following holds:*
(1) $V(G) = \mathbf{V}_-(G,\beta_a) \cup \mathbf{V}_0(G,\beta_a)$;
(2) $\mathbf{V}_-(G,\beta_a) = \{x \in V(G) : \beta_a(G-x) = \beta_a(G) - 1\} = \mathbf{F}i(G,\beta_a)$;
(3) $\mathbf{V}_-(G,i_a) = \{x \in V(G) : i_a(G-x) = i_a(G) - 1\}$;
(4) $\mathbf{V}_+(G,i_a) \subseteq \mathbf{F}i(G,i_a)$;
(5) $\mathbf{B}(G,i_a) \subseteq \mathbf{V}_0(G,i_a)$.

Proof. (1): Let $v \in V(G)$ and M be a β_a-set of $G-v$. Then M be an acyclic set of G which implies $\beta_a(G-v) \le \beta_a(G)$.

(2): Let $v \in V(G)$ and M_1 be a β_a-set of G. First assume v be no β_a-fixed. Hence the set M_1 may be chosen so that $v \notin M_1$ and then M_1 is an acyclic set of $G-v$ implying $\beta_a(G) = |M_1| \le \beta_a(G-v)$. Now by (1) it follows $\beta_a(G) = \beta_a(G-v)$.

Let v be β_a-fixed. Then each β_a-set of $G-v$ is an acyclic set of G but is no β_a-set of G. Hence $\beta_a(G) > \beta_a(G-v)$. Since $M_1 - \{v\}$ is an acyclic set of $G-v$ then $\beta_a(G-v) \ge |M_1 - \{v\}| = \beta_a(G) - 1$.

(3), (4) and (5): Let $v \in V(G)$, M_2 be an i_a-set of G and $v \notin M_2$. Then $M_2 \in \mathrm{MAS}(G-v)$ implying $i_a(G) \ge i_a(G-v)$. Now let M_3 be an i_a-set of $G-v$. Then either M_3 or $M_3 \cup \{v\}$ is a maximal acyclic set of G. Hence $i_a(G-v) + 1 \ge i_a(G)$ and if the equality holds then v is i_a-good. $\square$

A set $D \subseteq V(G)$ is called a *decycling set* if $V(G) - D$ is acyclic. A decycling set $D \subseteq V(G)$ is a *minimal decycling set* if no proper subset $D_1 \subset D$ is a decycling set.

The minimum order of a decycling set of G is called the *decycling number* of G and is denoted by $\nabla(G)$ (see [2]). Note that the set A is in $\mathrm{MAS}(G)$ if and only if $V(G) - A$ is a minimal decycling set. Hence $\nabla(G) + \beta_a(G) = |V(G)|$. For a survey of results and open problems on $\nabla(G)$ see [1]. In [2] the decycling of combinations of two graphs were considered, namely the sum, the join and the Cartesian product. Let G_1 and G_2 be connected graphs, both of order at least two, and let they have an unique vertex in common, say x. Then a *coalescence* $G_1 \overset{x}{\circ} G_2$ is the graph $G_1 \cup G_2$. Clearly, x is a cut-vertex of $G_1 \overset{x}{\circ} G_2$. In this paper we present results on maximal acyclic sets, lower acyclic number and upper acyclic number in a coalescence of graphs.

2. Maximal acyclic sets

In this section we begin an investigation of maximal acyclic sets in graphs having cut-vertices.

Proposition 2.1. *Let $G = H_1 \overset{x}{\circ} H_2$, $M \in \mathrm{MAS}(x, G)$ and $M_j = M \cap V(H_j)$, $j = 1, 2$. Then $M_j \in \mathrm{MAS}(x, H_j)$ for $j = 1, 2$.*

Proof. Clearly M_j is an acyclic set of H_j, $j = 1, 2$. Assume $M_i \notin \mathrm{MAS}(x, H_i)$ for some $i \in \{1, 2\}$. Then there is a vertex $u \in V(H_i) - M_i$ such that $M_i \cup \{u\}$ is an acyclic set in H_i. But then $M \cup \{u\}$ is an acyclic set of G - a contradiction with the maximality of M. □

Proposition 2.2. *Let $G = H_1 \overset{x}{\circ} H_2$, $M_j \in \mathrm{MAS}(x, H_j)$ for $j = 1, 2$. Then $M = M_1 \cup M_2 \in \mathrm{MAS}(x, G)$.*

Proof. Since x is a cut-vertex then M is an acyclic set of G. If $M \notin \mathrm{MAS}(G)$ then there is $u \in V(G - M)$ such that $M \cup \{u\}$ is an acyclic set of G. Let without loss of generalities $u \in V(H_1)$. Then $M_1 \cup \{u\}$ is an acyclic set of H_1 contradicting $M_1 \in \mathrm{MAS}(H_1)$. Hence $M \in \mathrm{MAS}(G)$. □

Proposition 2.3. *Let $G = H_1 \overset{x}{\circ} H_2$, $M \in \mathrm{MAS}(G)$, $x \notin M$ and $M_j = M \cap V(H_j)$, $j = 1, 2$. Then one of the following holds:*

1. *$M_j \in MAS(H_j)$ for $j = 1, 2$;*
2. *there are l and m such that $\{l, m\} = \{1, 2\}$, $M_l \in \mathrm{MAS}(H_l)$, $M_m \in \mathrm{MAS}(H_m - x)$ and $M_m \cup \{x\} \in \mathrm{MAS}(H_m)$.*

Proof. Clearly M_i is an acyclic set of H_i, $i = 1, 2$. Assume there be $j \in \{1, 2\}$ such that $M_j \notin \mathrm{MAS}(H_j)$, say $j = 1$. If $M_1 \notin \mathrm{MAS}(H_1 - x)$ then there is $v \in V(H_1 - x), v \notin M_1$ such that $M_1 \cup \{v\}$ is an acyclic set of $H_1 - x$ and since $x \notin M$ then $M \cup \{v\}$ is an acyclic set of G - a contradiction. So, $M_1 \in \mathrm{MAS}(H_1 - x)$. Since $M_1 \notin \mathrm{MAS}(H_1)$ then there is $u \in V(H_1 - M_1)$ such that $M_1 \cup \{u\}$ is an acyclic set of H_1. Since $M_1 \in \mathrm{MAS}(H_1 - x)$ then $u = x$. Hence $M_1 \cup \{x\} \in \mathrm{MAS}(H_1)$. Suppose $M_2 \notin \mathrm{MAS}(H_2)$. Then $M_2 \cup \{x\} \in \mathrm{MAS}(H_2)$ and by Proposition 2.2, $M \cup \{x\} \in \mathrm{MAS}(G)$ contradicting $M \in \mathrm{MAS}(G)$. □

Proposition 2.4. *Let $G = H_1 \overset{x}{\circ} H_2$, $M_j \in \mathrm{MAS}(H_j)$ for $j = 1, 2$ and $x \notin M = M_1 \cup M_2$. Then $M \in \mathrm{MAS}(H)$.*

Proof. The proof is analogous to the proof of Proposition 2.2. □

Proposition 2.5. *Let $G = H_1 \overset{x}{\circ} H_2$, $M_1 \in \mathrm{MAS}(x, H_1)$, $M_2 \in \mathrm{MAS}(H_2)$ and $x \notin M_2$. Then $M = M_1 \cup M_2$ is no acyclic set of G and there is a set M_3 such that $M_1 - \{x\} \subseteq M_3 \in \mathrm{MAS}(H_1 - x)$ and $M_3 \cup M_2 \in \mathrm{MAS}(G)$.*

Proof. Since $M_1-\{x\}$ is an acyclic set of H_1-x then there is $M_3 \in \mathrm{MAS}(H_1-x)$ with $M_1 - \{x\} \subseteq M_3$. Hence $U = M_3 \cup M_2$ is an acyclic set of G. Assume $U \notin \mathrm{MAS}(G)$. Then there is $v \in V(G) - U$ such that $U \cup \{v\}$ is an acyclic set of G. Now either $M_3 \cup \{u\}$ is an acyclic set of $H_1 - x$ or $M_2 \cup \{u\}$ is an acyclic set of H_2 depending on whether $u \in V(H_1 - x)$ or $u \in V(H_2)$. In both cases we have a contradiction. □

3. β_a-sets and i_a-sets

In this section we present some results concerning the lower acyclic number and the upper acyclic number of graphs having cut-vertices.

Theorem 3.1. *Let $G = H_1 \overset{x}{\circ} H_2$. Then $\beta_a(H_1) + \beta_a(H_2) - 1 \leq \beta_a(G) \leq \beta_a(H_1) + \beta_a(H_2)$. Moreover, $\beta_a(G) = \beta_a(H_1) + \beta_a(H_2)$ if and only if x is no β_a-fixed vertex of H_i, $i = 1, 2$.*

Proof. We need the following claims:

Claim 1. If x is a β_a-fixed vertex of G then $\beta_a(G) \leq \beta_a(H_1) + \beta_a(H_2) - 1$.

Let M be a β_a-set of G. Then

$$\beta_a(G) = |M| = |M \cap V(H_1)| + |M \cap V(H_2)| - 1 \leq \beta_a(H_1) + \beta_a(H_2) - 1.$$

Claim 2. If x is no β_a-fixed vertex of G then $\beta_a(G) \leq \beta_a(H_1) + \beta_a(H_2)$.

Let M be a β_a-set of G such that $x \notin M$. Hence

$$\beta_a(G) = |M| = |M \cap V(H_1)| + |M \cap V(H_2)| \leq \beta_a(H_1) + \beta_a(H_2).$$

Claim 3. If x is no β_a-fixed vertex of H_i, $i = 1, 2$ then $\beta_a(G) \geq \beta_a(H_1)+\beta_a(H_2)$.

Let M_i be a β_a-set of H_i and $x \notin M_i$, $i = 1, 2$. Then $M = M_1 \cup M_2$ is an acyclic set of G and $\beta_a(G) \geq |M| = |M_1| + |M_2| = \beta_a(H_1) + \beta_a(H_2)$.

Claim 4. If x is β_a-fixed vertex of H_i for some $i \in \{1, 2\}$ then

$$\beta_a(G) \geq \beta_a(H_1) + \beta_a(H_2) - 1.$$

Let without loss of generalities $i = 1$. Let M_j be a β_a-set of H_j, $j = 1, 2$. Then $M = (M_1 - \{x\}) \cup M_2$ is an acyclic set of G and

$$\beta_a(G) \geq |M| = |M_1| - 1 + |M_2| = \beta_a(H_1) + \beta_a(H_2) - 1.$$

By the above claims it immediately follows

$$\beta_a(H_1) + \beta_a(H_2) - 1 \leq \beta_a(G) \leq \beta_a(H_1) + \beta_a(H_2) \tag{1}$$

If x is no β_a-fixed vertex of H_i, $i = 1, 2$ then by (1) and Claim 3 it follows $\beta_a(G) = \beta_a(H_1) + \beta_a(H_2)$. Now, let without loss of generalities x is a β_a-fixed vertex of H_1. If x is a β_a-fixed vertex of G then by Claim 1 and (1) it follows $\beta_a(G) = \beta_a(H_1) + \beta_a(H_2) - 1$. Assume x is no β_a-fixed vertex of G. Then there is a β_a-set of G with $x \notin M$. Hence

$$\beta_a(G) = |M| = |M \cap V(H_1)| + |M \cap V(H_2)| \\ \leq \beta_a(H_1 - x) + \beta_a(H_2) = (\beta_a(H_1) - 1) + \beta_a(H_2)$$

because of Observation 1.1 (2). □

Corollary 3.2. *Let $G = H_1 \overset{x}{\circ} H_2$ and x is a β_a-fixed vertex of G. Then $\beta_a(G) = \beta_a(H_1) + \beta_a(H_2) - 1$.*

Theorem 3.3. *Let $G = H_1 \overset{x}{\circ} H_2$. Then:*

(1) $i_a(G) \geq i_a(H_1) + i_a(H_2) - 1$;

(2) *Let x be an i_a-good vertex of G, $i_a(G) = i_a(H_1) + i_a(H_2) - 1$, let M be an i_a-set of G and $x \in M$. Then $M \cap V(H_j)$ is an i_a-set of H_j, $j = 1, 2$;*

(3) *Let x be an i_a-bad vertex of the graph G, $i_a(H) = i_a(H_1)+i_a(H_2)-1$ and let M be an i_a-set of G. Then there are l, m such that $\{l, m\} = \{1, 2\}$, $M \cap V(H_l)$ is a i_a-set of H_l, $M \cap V(H_m)$ is an i_a-set of $H_m - x$, $i_a(H_m - x) = i_a(H_m) - 1$ and $(M \cap V(H_m)) \cup \{x\}$ is an i_a-set of H_m;*

(4) *Let x be an i_a-good vertex of graphs H_1 and H_2. Then*

$$i_a(G) = i_a(H_1) + i_a(H_2) - 1.$$

If M_j is an i_a-set of H_j, $j = 1, 2$ and $\{x\} = M_1 \cap M_2$ then $M_1 \cup M_2$ is an i_a-set of the graph G;

(5) *Let x be an i_a-bad vertex of graphs H_1 and H_2. Then*

$$i_a(G) = i_a(H_1) + i_a(H_2).$$

If M_j is a i_a-set of H_j, $j = 1, 2$ then $M_1 \cup M_2$ is an i_a-set of G.

Proof. (2): Let M be an i_a-set of G and $M_j = M \cap V(H_j)$, $j = 1, 2$. If $x \in M$ then by Proposition 2.1 it follows $M_j \in \mathrm{MAS}(x, H_j)$, $j = 1, 2$. So that

$$i_a(G) = |M| = |M_1| + |M_2| - 1 \geq i_a(H_1) + i_a(H_2) - 1.$$

Clearly the equality holds if and only if M_i is an i_a-set of H_i, $i = 1, 2$.

(3): Let M be an i_a-set of G and $M_j = M \cap V(H_j)$, $j = 1, 2$. Since x is i_a-bad, $x \notin M$. If $M_j \in \mathrm{MAS}(H_j)$, $j = 1, 2$ then

$$i_a(G) = |M| = |M_1| + |M_2| \geq i_a(H_1) + i_a(H_2).$$

If there are l and m such that $\{l, m\} = \{1, 2\}$, $M_l \in \mathrm{MAS}(H_l)$, $M_m \in \mathrm{MAS}(H_m - x)$ and $M_m \cup \{x\} \in \mathrm{MAS}(H_m)$ then

$$i_a(G) = |M| = |M_l| + |M_m| \geq i_a(H_l) + i_a(H_m) - 1$$

and the equality holds if and only if M_l is an i_a-set of H_l, M_m is an i_a-set of $H_m - x$ and $M_m \cup \{x\}$ is an i_a-set of H_m. There is no other possibilities because of Proposition 2.3.

(1): Immediately follows by the proofs of (2) and (3).

(4): Let M_j be an i_a-set of H_j, $j = 1, 2$ and $\{x\} = M_1 \cap M_2$. It follows by Proposition 2.2 that $M_1 \cup M_2 \in \text{MAS}(G)$. Hence

$$i_a(G) \leq |M_1 \cup M_2| = |M_1| + |M_2| - 1 = i_a(H_1) + i_a(H_2) - 1.$$

Now, by (1), $i_a(G) = i_a(H_1) + i_a(H_2) - 1$ and then $M_1 \cup M_2$ is an i_a-set of G.

(5): Assume $i_a(G) = i_a(H_1) + i_a(H_2) - 1$. If x is an i_a-bad vertex of G then by (3) there exists $m \in \{1, 2\}$ such that $i_a(H_m - x) = i_a(H_m) - 1$. Now, by Observation 1.1(5) x is an i_a-good vertex of H_m - a contradiction. If x is an i_a-good vertex of G, M is an i_a-set of G and $x \in M$ then by (2) we have $M \cap V(H_s)$ is an i_a-set of H_s, $s = 1, 2$. But then x is an i_a-good vertex of H_s, $s = 1, 2$ which is a contradiction. Hence, $i_a(G) \geq i_a(H_1) + i_a(H_2)$. Let M_j be an i_a-set of H_j, $j = 1, 2$. By Proposition 2.4, $M = M_1 \cup M_2 \in \text{MAS}(G)$. Hence, $i_a(H_1) + i_a(H_2) \leq i_a(G) \leq |M| = |M_1| + |M_2| = i_a(H_1) + i_a(H_2)$. $\square$

Example 3.4. Let H_1 and H_2 be the graphs defined as follows:

$V(H_1) = \{x; x_{11}, \ldots, x_{1m}; x_{21}, \ldots, x_{2m}\}$,
$E(H_1) = \cup_{i=1}^{m} \{xx_{1i}, xx_{2i}, x_{1i}x_{2i}\}$,
$V(H_2) = \{x, y, z; y_{11}, \ldots, y_{1n}; y_{21}, \ldots, y_{2n}; z_{11}, \ldots, z_{1p}; z_{21}, \ldots, z_{2p}\}$,
$E(H_2) = \{xy, yz, zx\} \cup \cup_{i=1}^{n} \{yy_{1i}, yy_{2i}, y_{1i}y_{2i}\} \cup \cup_{j=1}^{p} \{zz_{1j}, zz_{2j}, z_{1j}z_{2j}\}$,

where m, n and p be positive integers such that $m + 1 \leq n \leq p$. Now, let $G = H_1 \overset{x}{\circ} H_2$. It is easy to see that $i_a(H_1) = m + 1$, $i_a(H_2) = n + p + 2$ and $i_a(G) = 2m + n + p + 2$. Hence, $i_a(G) - i_a(H_1) - i_a(H_2) = m - 1$.

This example establish the following result.

Theorem 3.5. *For each positive integer r there exists a pair of graphs H_1 and H_2 such that they have an unique vertex in common, say x, and*

$$i_a(H_1 \overset{x}{\circ} H_2) - i_a(H_1) - i_a(H_2) > r.$$

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