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ON GARCIA NUMBERS

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ABSTRACT. We study real algebraic integers larger than 1, whose norm is ± 2 and all of whose conjugates have modulus larger than 1. These numbers are characterized in some easy cases.

1. INTRODUCTION

Some important kinds of algebraic integers are usually characterized by specifying the size of their conjugates, e.g., Pisot, Salem or Perron numbers. Here we are concerned with a similarly defined class of algebraic integers. Let us fix $d, m \in \mathbb{N}, m \geq 2$. We denote by $\mathcal{G}_d(m)$ the (finite) set of all positive real algebraic integers α of degree d with $|\alpha_i| > 1$ and $\prod_{i=1}^{d} |\alpha_i| = m$, where $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_d$ are the conjugates of α. The interest in this class of algebraic numbers primarily stems from the case $m = 2$ which seems to have first appeared in a publication of D.-J. FENG [9], where the following terminology occurred.

Definition 1.1 ([9])**.** An algebraic integer is called a Garcia number if it is real and larger than 1, if all conjugates have modulus larger than 1 and if its norm has modulus 2.

We denote by $\mathcal{P}_d(m)$ the set of minimal polynomials P_α of $\alpha \in \mathcal{G}_d(m)$. In view of the main goal of this note we shortly write $\mathcal{P}_d$ instead of $\mathcal{P}_d(2)$. Let us start with some obvious examples.

Example 1.2. (i) If $\alpha \in \mathcal{G}_d(m)$ is larger than the modulus of each of its conjugates then α is a Perron number.
(ii) $\mathcal{P}_1(m) = \{X - m\}$.
(iii) Let $d \geq 2$ and assume that m is not a p-th power in $\mathbb{Z}$ for all primes p which divide d. Then $X^d - m$ is irreducible (see e.g. [11, Ch. VIII, Theorem 16]), and therefore $X^d - m \in \mathcal{P}_d(m)$. In particular, the positive root of $X^d - 2$ is a Garcia number (see [9]).

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(iv) If α is a real CNS basis (see [7] for the definition and further details) then
$$-\alpha \in \mathcal{G}_{\deg(P_\alpha)}(P_\alpha(0)).$$

We mention that the case $P(0) = 2$ which is of interest here has extensively been studied by A. Kovács [10] and by P. Burcsi and A. Kovács [8].

In this note we determine the sets $\mathcal{P}_d$ for $d \leq 4$ and characterize totally real Garcia numbers and Garcia numbers whose minimal polynomials are trinomials.

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2. Basic properties of $\mathcal{G}_d(m)$

We start with a description of $\mathcal{P}_d(m)$ which allows its calculation in simple cases. Recall that a polynomial with complex coefficients is called expansive if each of its roots has modulus larger than 1 (see e.g. [2, 8]).

Theorem 2.1. *Let $P \in \mathbb{Z}[X]$ be a monic irreducible polynomial of degree d with $|P(0)| = m$. Then the following statements are equivalent.*

(i) $P \in \mathcal{P}_d(m)$

(ii) *P is expansive and has a positive root.*

(iii) *$P = X^d + \sum_{i=0}^{d-1} a_{d-i} X^i$ has a positive root, and for $i = 1, \ldots, d$ we have* $\operatorname{sgn}(\det(M_i)) = (-1)^i$, *where M_d is the $2d \times 2d$ matrix given by*
$$m_{ij} = \begin{cases} a_{j-i}, & \text{for } 1 \leq i \leq d \text{ and } i \leq j \leq i+d, \\ a_{i-j}, & \text{for } d+1 \leq i \leq 2d \text{ and } i-d \leq j \leq i, \\ 0, & \text{otherwise.} \end{cases}$$
M_{d-1} is created from M_d by deleting the d-th and $2d$-th rows and columns and so on.

(iv) *For $i = 1, \ldots, d$ we have* $\operatorname{sgn}(\det(M_i)) = (-1)^i$, *and the inequality $v(m) < v(1)$ holds true where for $a \in \mathbb{R}$ we denote by $v(a)$ the number of changes of sign in the sequence $P(a), P'(a), R_1(a), \ldots, R_k(a)$ given by the Sturmian sequence $P, P', R_1, \ldots, R_k$ of P.*

Proof. The first equivalence follows directly from the definition of $\mathcal{P}_d(m)$. The second equivalence is an immediate consequence of [2, Theorem 3.1]. The third equivalence is derived from Sturm's theorem (see e.g., [12, Satz 18.1]). $\square$

The specialization to Garcia numbers yields the following corollary.

Corollary 2.2. *Let $P \in \mathbb{Z}[X]$ be a monic polynomial with $|P(0)| = 2$. Then P is the minimal polynomial of a Garcia number if and only if P is expansive and has a positive root.*

Proof. Clear by Theorem 2.1 and the following Lemma 2.3. $\square$

Lemma 2.3. *Let $f \in \mathbb{Z}[X]$ be a monic expansive polynomial. If $|f(0)|$ is a prime then f is irreducible.*

Proof. Observe that the modulus of the constant term of every nonconstant factor of f would exceed 1. $\square$

Now, we combine some easy consequences of the definition of $\mathcal{G}_d(m)$ with well-known results. Lemma 2.4 deals with the moduli of the conjugates of $\alpha \in \mathcal{G}_d(m)$, and Lemma 2.6 collects some results on the coefficients of the minimal polynomial of α.

Lemma 2.4. *Let $d \geq 2$, $\alpha \in \mathcal{G}_d(m)$ with conjugates $\alpha_1 = \alpha, \alpha_2, \dots, \alpha_d$. Then we have the following inequalities.*

(i) $m^{1/d} \leq \overline{|\alpha|} < m$

(ii)

$$\sum_{i=1}^{d} |\alpha_j| < m(d-1+\frac{1}{m^{1/d}})$$

Proof. (i) Trivial.
(ii) By [14, Lemma 1] we have

$$\sum_{i=1}^{d} |\alpha_j| \leq (d-1)\overline{|\alpha|} + \frac{1}{\overline{|\alpha|}^{d-1}},$$

hence (i) yields the assertion. $\square$

The following theorem rephrases classical facts in the present terminology.

Theorem 2.5. (i) *Every totally real Garcia number is of the form $2\cos(\pi r)$ with $r \in \mathbb{Q}$.*

(ii) *Let $\alpha \in \mathcal{G}_d(m), \beta \in \mathcal{G}_k(n) \quad (n \geq 2)$ and assume $\alpha = \overline{|\alpha|}$ and $\beta = \overline{|\beta|}$. Then*

$$\mathbb{Q}(\alpha+\beta) = \mathbb{Q}(\alpha, \beta).$$

Further, if $\alpha\beta \in \mathbb{N}$ then there is some $e \in \mathbb{N}$ with $\alpha^e \in \mathbb{N}$ and $\beta^e \in \mathbb{N}$.

Proof. (i) Clear by Lemma 2.4 (i) and [13, VIII. Abschn., Nr. 21].
(ii) See [15, Corollary 2, Corollary 3]. $\square$

Lemma 2.6. *The coefficients of the polynomial $P = X^d + \sum_{j=0}^{d-1} a_j X^j \in \mathcal{P}_d(m)$ $(d \geq 2)$ satisfy the following statements.*

(i) $|a_0| = m$, *and $a_i < 0$ for some $i \in \{0, \dots, d-1\}$.*

(ii) $|a_{d-1}| \leq d+m-1$ *and*

$$|a_i| \leq \binom{d-1}{i} m + \binom{d-1}{i-1} \qquad (i = 1, \dots, d-2).$$

(iii) $\left|a_{d-1}^2 - 2a_{d-2}\right| < dm^2$

(iv) *For $i = 1, \dots, d-1$ we have*

$$|a_i| \leq 1 + m + \sum_{j=1, j\neq i}^{d-1} |a_j|.$$

(v) *If* $d \geq 3, \quad 1 + m + \sum_{i=1}^{d-1} a_i \geq 0$ *and* $a_i \geq 0$ *for all* $i \in \{0, \dots, d-1\} \setminus \{1, 2\}$ *then*

$$a_1 + 2a_2 < -d - \sum_{i=3}^{d-1} i a_i.$$

(vi) *If* $a_0 < 0$ *then*

$$\sum_{j=0}^{d-1} (-1)^j a_j \leq (-1)^{d-1} - 1 \qquad \textit{and} \qquad \sum_{j=0}^{d-1} a_j \leq -2.$$

(vii) *If* $a_0 > 0$ *then*

$$\sum_{j=0}^{d-1} (-1)^j a_j \geq (-1)^{d-1} + 1 \qquad \textit{and} \qquad \sum_{j=0}^{d-1} a_j \geq 0.$$

Further, d *is odd or* P *does not have a negative root.*

Proof. (i) Trivial.
(ii) The bounds follow from [5, Lemma 3].
(iii) This is clear by [1, Proof of Theorem 1, p. 103] and Lemma 2.4 (i).
(iv) This is an easy consequence of Rouché's theorem.
(v) Observe that $X^d + \sum_{j=1}^{d-1} a_j X^j + m$ has a root in the interval $(1, m)$.
(vi) Clearly, $P(-1) < 0$ and $P(1) < 0$.
(vii) The first statement follows from the observations $P(-1) > 0$ and $P(1) > 0$ analogously as in (vi). The number n of negative roots of P is zero or odd. The sign of $P(0)$ is given by $(-1)^d(-1)^n$, hence d must be odd if $n \neq 0$. □

Lemma 2.3 and the following Proposition 2.8 are useful for constructing examples.

Example 2.7. Let $a_1, \dots, a_{d-1} \in \mathbb{Z}$ and put $s = 1 + \sum_{i=1}^{d-1} |a_i|$.

(i) The polynomial $X^d + \sum_{i=1}^{d-1} a_i X^i - m$ with $m > s$ is expansive by [3, Lemma 1]. Therefore it belongs to $\mathcal{P}_d(m)$ provided it is irreducible.
(ii) Let p be a prime and assume $p > s$. Then by (i) and Lemma 2.3 we have

$$X^d + \sum_{i=1}^{d-1} a_i X^i - p \in \mathcal{P}_d(p).$$

Proposition 2.8. *Let* $t \in \mathbb{N}_{>0}$.

(i) *If* t *divides* d *and* $f(X^t) \in \mathcal{P}_d(m)$ *then* $f(X) \in \mathcal{P}_{\frac{d}{t}}(m)$.
(ii) *Let* p *be a prime. If* $f(X) \in \mathcal{P}_d(p)$ *then* $f(X^t) \in \mathcal{P}_{td}(p)$.

Proof. (i) This is clear by Theorem 2.1 (ii).
(ii) This follows from Lemma 2.3. □

In the next section we will apply Proposition 2.8 to exhibit trinomials in $\mathcal{P}_d(p)$ for prime p.

3. Trinomial Garcia numbers

We are now aiming at the description of quadratic and trinomial Garcia numbers. Furthermore, we show that $X^d - X^q - p \in \mathcal{P}_d(p)$ for $0 < q < d$ and every odd prime p.

Lemma 3.1. *Let $0 < q < d$ and $X^d + bX^q + c \in \mathcal{P}_d(m)$.*

(i) $|b| \leq |c| + 1$.

(ii) *If $c = m$ we have $-m \leq b \leq -2$ and $d \leq -qb - 1$.*

Proof. (i) This follows from Rouché's theorem.
(ii) By (i) and Lemma 2.6 (i) we find $-(m+1) \leq b \leq -1$. As $P = X^d + bX^q + c$ cannot vanish at 1 we must have $b \neq -(m+1)$. The assumption $d + qb \geq 0$ implies that the derivative of P is nonnegative in $[1, \infty)$ which contradicts the fact that P must have a root in this interval. Therefore, $d + qb < 0$ and $b \neq -1$. □

Now, we are able to determine the set $\mathcal{P}_2(m)$.

Theorem 3.2. *For $b, c \in \mathbb{Z}$ the following statements are equivalent.*

(i) $X^2 + bX + c \in \mathcal{P}_2(m)$.

(ii) *$b^2 - 4c$ is not a square, and either*

$$c = -m, \qquad -m + 2 \leq b \leq 0$$

or

$$c = m \geq 5, \qquad -m \leq b \leq -\sqrt{4m+1}.$$

Proof. Let $P = X^2 + bX + c \in \mathcal{P}_2(m)$. Then clearly $|c| = m$, $b^2 - 4c$ is not a square, $b^2 > 4c$ and $|b| < |c + 1|$ by Theorem 2.1. In the case of $c = -m$, this yields $|b| \leq m - 2$ and $b \leq 0$ because otherwise the roots of P would not exceed 1. If $c = m$ Lemma 3.1 yields $b \leq -3$, and we conclude $4m + 1 \leq b^2 \leq m^2$.

Finally, if (ii) is satisfied then one easily checks that P is irreducible and that its roots fulfill the required conditions. □

In particular, we have shown that there is only a single quadratic Garcia number.

Corollary 3.3. $\mathcal{P}_2 = \{X^2 - 2\}$.

Proposition 3.4. *Let $0 < q < d$ and p be a prime. In case $p = 2$ we require that $d/(d, q)$ be odd or $q/(d, q)$ be even. Then $X^d - X^q - p \in \mathcal{P}_d(p)$.*

Proof. Obviously, $P = X^d - X^q - p$ has a real root > 1 because $P(1) < 0$. Therefore, using Lemma 2.3 it suffices to show that all roots of P lie outside the closed unit circle. In view of Proposition 2.8 we may assume that $(d, q) = 1$. By [4] P does not vanish inside the unit circle, hence we are left with the boundary of the unit circle. Assume on the contrary that there is some $\zeta \in \mathbb{C}$ with $|\zeta| = 1$ and $P(\zeta) = 0$. Then $w = \zeta^q$ lies on the boundaries of the unit circle and of the circle of radius 1 centered at $-p$. This is only possible for

$p = 2, w = -1$, thus $\zeta^d = 1$. Pick $a, b \in \mathbb{Z}$ with $ad + bq = 1$. Then $\zeta^d = (-1)^b$ and $(-1)^{bd} - (-1)^{bq} = 2$ which contradicts our assumptions on d and q. □

Now, we are in a position to complete the characterization of those Garcia numbers whose minimal polynomials are trinomials. Our result slightly extends an example given in [9].

Theorem 3.5. *Let $d \geq 3$. The trinomials inside $\mathcal{P}_d$ are given by*

$$\{X^d - 2\} \cup \{X^d - X^q - 2 \,:\, 0 < q < d,\ d/(d,q) \text{ odd or } q/(d,q) \text{ even }\} \cup \mathcal{P}_d^\star,$$

where

$$\mathcal{P}_d^\star = \{X^d - 2X^{d/3} - 2\}$$

if $d \equiv 0 \pmod 3$, and $\mathcal{P}_d^\star = \emptyset$ otherwise.

Proof. We easily check $X^3 - 2X - 2 \in \mathcal{P}_3$, hence by Example 1.2, Proposition 3.4 and Proposition 2.8 all trinomials mentioned above belong to $\mathcal{P}_d$.

Let $P = X^d + bX^q + c \in \mathcal{P}_d$, $t = (d,q), n = d/t$, and $k = q/t$. Then $f = X^n + bX^k + c \in \mathcal{P}_n$ by Proposition 2.8. By Lemma 3.1 we have $c = -2$ and $|b| \leq 3$. Clearly, $b = 1$ is impossible, and if $b = -1$ then d must be odd or q must be even, because otherwise $f(-1) = 0$. Using Bohl's theorem [4] it can be checked that f has a root inside the unit circle for $|b| \in \{2, 3\}$. □

Remark 3.6. If α is a Garcia number then the minimal polynomial of $-\alpha$ is not necessarily a CNS polynomial. For instance, let α be a root of $X^5 - X^3 - 2$. Then the minimal polynomial of $-\alpha$ is $X^5 - X^3 + 2$ which is not a CNS polynomial by [6, Theorem 3]. Thus, Theorem 3.5 shows that the converse of Example 1.2 (iii) does not hold.

4. Cubic and quartic Garcia numbers

In this section we determine the minimal polynomials of Garcia numbers of the third and fourth degrees.

Theorem 4.1. *The minimal polynomials of the cubic and quartic Garcia numbers are given by*

$$\begin{aligned}\mathcal{P}_3 = \{&X^3 - 2, X^3 - X^2 - 2, X^3 - X - 2, X^3 - 2X - 2,\\ &X^3 - X^2 + X - 2, X^3 + X^2 - X - 2, X^3 - 2X^2 + 2X - 2\}.\end{aligned}$$

and

$$\begin{aligned}\mathcal{P}_4 = \{&X^4 - 2, X^4 - X^2 - X - 2, X^4 - X^2 + X - 2,\\ &X^4 - 2X^3 + 2X - 2, X^4 + 2X^3 - 2X - 2,\\ &X^4 - X^3 + X - 2, X^4 + X^3 - X - 2\}.\end{aligned}$$

Proof. (i) Theorem 3.5 shows that $X^3-2, X^3-X^2-2, X^3-X-2, X^3-2X-2 \in \mathcal{P}_3$ and $X^4 - 2 \in \mathcal{P}_4$. By checking the roots and using Lemma 2.3 one verifies that the other polynomials actually belong to $\mathcal{P}_3$ and $\mathcal{P}_4$, respectively.

(ii) Let $X^3+aX^2+bX+c \in \mathcal{P}_3$. By Theorem 2.1 and [2, Lemma 3.1] we have

$$|b-ac| \le 2 \tag{4.0.1}$$

and

$$|b+1| < |a+c|\,, \tag{4.0.2}$$

and Lemma 2.6 yields $|a| \le 4$. If $ab=0$ we are done by Theorem 3.5. Therefore we assume $ab \ne 0$ for the remainder of the proof for the case $d=3$.

The assumption $c=2$ yields $b<0$ by (4.0.2) and Lemma 2.6 (i). Now, $a \ge 0$ is impossible because then $-(b-2a) \le 2$ which implies $-b \le 0$ which is absurd. In view of (4.0.2) we are left with $a \in \{-1,-3,-4\}$. The polynomial X^3-X^2-X+2 does not have a positive root, $a=-3$ yields $b=-1$ by (4.0.2) contradicting (4.0.1), and analogously $a=-4$ yields $b \in \{-1,-2\}$ by (4.0.2) contradicting (4.0.1).

Now, let $c=-2$. If $b \ge 0$ then $a \in \{-4,\dots,-1\}$ by (4.0.2), and we exclude $a=-4,-3$ similarly as above. Finally, let $b<0$, hence $a \in \{0,1,3,4\}$ by (4.0.1). In view of (4.0.1) and (4.0.2) the cases $a=3,4$ are impossible.

(iii) Let $P=X^4+aX^3+bX^2+cX+d \in \mathcal{P}_4$. Lemma 2.6 (ii) yields

$$|a| \le 5, \qquad |b| \le 9, \qquad |c| \le 7.$$

If two of the coefficients a,b,c vanish then P is a trinomial, and we are done by Theorem 3.5. Therefore in the sequel we may assume that at most one of the quantities a,b,c is zero.

In case $d=2$ Theorem 2.1 and [2, Lemma 3.2] yield

$$|c-2a| \le 2, \qquad |a+c| \le 2+b, \qquad c^2+2a^2-3ac \le 2-b.$$

Similarly as above these inequalities and elementary analysis show that none of the cases $a \in \{-5,\dots,5\}$ can occur.

Finally, in case $d=-2$ [2, Lemma 3.2] yields

$$|c+2a| \le 2, \qquad |a+c| \le -b, \qquad c^2-2a^2+ac \ge -8-9b,$$

hence $b \le 0$. A long, but straightforward calculation shows that these inequalities yield the polynomials mentioned above. This completes the proof. $\square$

Remark 4.2. Note that a Garcia number need not be maximal among the moduli of its conjugates. For instance, the modulus of the complex roots of the polynomial X^3+X^2-X-2 is larger than its positive root, and the modulus of the negative root of the polynomial X^4-X^2+X-2 exceeds its positive root. Further, if a Garcia number α is the maximal real conjugate with $\alpha \ne |\alpha_i|$ for all nonreal conjugates α_i then α need not be a Perron number. E.g., take α the (unique) real root of the polynomial X^3+X^2-X-2. Then α is smaller than the moduli of its nonreal conjugates.

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