Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
25 (2009), 65–84
www.emis.de/journals
ISSN 1786-0091

RECENT RESULTS ON THE STABILITY OF THE PARAMETRIC FUNDAMENTAL EQUATION OF INFORMATION

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Abstract. The purpose of this paper is to summarize the recent results on the stability of the parametric fundamental equation of information. Furthermore, by the help of a modification of a method we used in [9] we shall give a unified proof for the Hyers–Ulam stability of the equation in question, assuming that the parameter does not equal to 1. As a corollary of the main result, a system of equations, that defines the recursive and semi–symmetric information measures is also discussed.

1. Introduction and preliminaries

The study of stability problems for functional equations originates from a famous question of Ulam. In his talk he asked whether it is true that the solution of an equation differing slightly from a given one, must of necessity be close to the solution of this equation (see Ulam [21] page 63). Concerning the additive Cauchy equation, Hyers gave an affirmative answer to Ulam's question in 1941 (see Hyers [11]). Since then, this result has been extended and generalized in several ways (see e.g. Forti [6], Ger [7], Hyers–Isac–Rassias [12] and Moszner [17]), and the stability theory has become a dynamically developing field of research.

In this paper the previous problem is investigated concerning the parametric fundamental equation of information, i.e., equation

$$f(x) + (1-x)^{\alpha} f\left(\frac{y}{1-x}\right) = f(y) + (1-y)^{\alpha} f\left(\frac{x}{1-y}\right). \tag{1.1}$$

If $\alpha = 1$, then equation (1.1) is called the fundamental equation of information (see Aczél–Daróczy [3]). However, before this, we shall fix the notation and

2000 *Mathematics Subject Classification.* 39B82, 94A17.

Key words and phrases. Stability, superstability, parametric fundamental equation of information, entropy of degree alpha.

This research has been supported by the Hungarian Scientific Research Fund (OTKA) Grant NK 68040.

the terminology that will be used throughout this paper. As usual, $\mathbb{R}$ denotes the set of the real numbers and on $\mathbb{R}_+$ and on $\mathbb{R}_{++}$ we understand the set of the nonnegative and the positive real numbers, respectively. Furthermore, let n be a fixed positive integer and define the following sets

$$\Gamma_n = \left\{(p_1, \ldots, p_n) \in \mathbb{R}^n | p_i \geq 0, i = 1, \ldots, n, \sum_{i=1}^{n} p_i = 1\right\},$$

$$\Gamma_n^{\circ} = \left\{(p_1, \ldots, p_n) \in \mathbb{R}^n | p_i > 0, i = 1, \ldots, n, \sum_{i=1}^{n} p_i = 1\right\},$$

$$D = \left\{(x, y) \in \mathbb{R}^2 | x, y \in [0, 1[, x + y \leq 1\right\}$$

and

$$D^{\circ} = \left\{(x, y) \in \mathbb{R}^2 | x, y, x + y \in]0, 1[\right\}.$$

To make our result comprehensible, first we list some basic facts from the theory of functional equations. These can be found e.g. in Kuczma [13] and in Radó–Baker [18].

Definition 1.1 ([13, 18])**.** Let $I \subset \mathbb{R}_+$ and

$$\mathcal{A} = \left\{(x, y) \in \mathbb{R}_+^2 | x, y, x + y \in I\right\}.$$

A function $a\colon I \to \mathbb{R}$ is called *additive on* $\mathcal{A}$ if

$$a(x + y) = a(x) + a(y) \tag{1.2}$$

holds for all pairs $(x, y) \in \mathcal{A}$.
Consider the set

$$\mathcal{I} = \left\{(x, y) \in \mathbb{R}_+^2 | x, y, xy \in I\right\}.$$

We say that $\mu\colon I \to \mathbb{R}$ is *multiplicative on* $\mathcal{I}$ if the functional equation

$$\mu(xy) = \mu(x)\mu(y) \tag{1.3}$$

is fulfilled for all $(x, y) \in \mathcal{I}$.
If

$$\mathcal{L} = \left\{(x, y) \in \mathbb{R}_{++}^2 | x, y, xy \in I\right\}$$

then a function $l\colon I \to \mathbb{R}$ is called *logarithmic on* $\mathcal{L}$ if it satisfies the functional equation

$$l(xy) = l(x) + l(y) \tag{1.4}$$

for all $(x, y) \in \mathcal{L}$.

The parametric fundamental equation of information arises in a natural way in the characterization problem of information measures. A sequence (I_n) of real-valued functions on Γ_n° or on Γ_n is called an *information measure* on the open or on the closed domain, respectively. The usual information-theoretical interpretation is that $I_n(p_1, \ldots, p_n)$ is a measure of uncertainty as to the outcome of an experiment having n possible outcomes with probabilities

$p_1, \ldots, p_n$, or, in other words, it is the amount of information received from the knowledge of which of the possible outcomes occurred.

Some desiderata for information measures can be found in Aczél–Daróczy [3] as well as in Ebanks–Sahoo–Sander [5]. Nevertheless, in this paper we will use only the following properties. The reader should consult Aczél [1], [2], Daróczy [4], Havrda–Charvát [10] and Tsallis [20], as well.

Definition 1.2. The sequence of functions $I_n\colon \Gamma_n^\circ \to \mathbb{R}$ $(n = 2, 3, \ldots)$ is

(i) *α–recursive*, if

$$I_n(p_1, \ldots, p_n) = I_{n-1}(p_1 + p_2, p_3, \ldots, p_n) + (p_1 + p_2)^\alpha I_2\left(\frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2}\right)$$

holds for all $n = 3, 4, \ldots$ and $(p_1, \ldots, p_n) \in \Gamma_n^\circ$, with some $\alpha \in \mathbb{R}$.

(ii) *3–semi–symmetric*, if

$$I_3(p_1, p_2, p_3) = I_3(p_1, p_3, p_2)$$

holds for all $(p_1, p_2, p_3) \in \Gamma_3^\circ$.

Measures depending on one probability distribution are generally referred as *entropies*. Probably the most well-known of all is the *Shannon-entropy*

$$H_n^1(p_1, \ldots, p_n) = -\sum_{i=1}^{n} p_i \log_2(p_i), \quad ((p_1, \ldots, p_n) \in \Gamma_n^\circ)$$

and *the entropy of degree α (or the Havrda-Charvát-entropy that recently has also been called Tsallis-entropy)*

$$H_n^\alpha(p_1, \ldots, p_n) = \left(2^{1-\alpha} - 1\right)^{-1} \left(\sum_{i=1}^{n} p_i^\alpha - 1\right). \quad (\alpha \neq 1, (p_1, \ldots, p_n) \in \Gamma_n^\circ)$$

It is easy to see that, for all $(p_1, \ldots, p_n) \in \Gamma_n^\circ$,

$$\lim_{\alpha \to 1} H_n^\alpha(p_1, \ldots, p_n) = H_n^1(p_1, \ldots, p_n)$$

holds and this shows that (H_n^1) can be continuously embedded to the family of (H_n^α).

The following theorem enables us to transform the characterization of information measures into solving functional equations (see Aczél–Daróczy [3] and Ebanks–Sahoo–Sander [5]).

Theorem 1.1. *If the sequence of functions $I_n\colon \Gamma_n^\circ \to \mathbb{R}$, $(n = 2, 3, \ldots)$ is α–recursive and 3–semi–symmetric, then the function $f\colon]0,1[\to \mathbb{R}$ defined by*

$$f(x) = I_2(1 - x, x) \quad (x \in]0,1[)$$

satisfies functional equation (1.1) for all $(x, y) \in D^\circ$.

2. Known results

In this section we will shortly list the results which have been achieved in the last academic year on the stability of the parametric fundamental equation of information.

Concerning this topic, the first result was the stability of equation (1.1) on the set D, assuming that $1 \neq \alpha > 0$ (see Maksa [15]). Furthermore the stability constant, he has got in that paper is much smaller than that of our. However, the method, used in Maksa [15] does not work if $\alpha = 1$ or $\alpha \leq 0$ or if we consider the problem on the open domain.

After that, it was proved that equation (1.1) is stable in the sense of Hyers and Ulam on the set D° as well as on D, assuming that $\alpha \leq 0$ (see [9]). Recently it turned out that this method is appropriate to prove superstability in case $1 \neq \alpha > 0$. Thus we can give a unified proof for the stability problem of equation (1.1) except the case $\alpha = 1$.

3. The main result

Our main result is contained in the following theorem.

Theorem 3.1. *Let $\alpha, \varepsilon \in \mathbb{R}$ be fixed, $\alpha \neq 1, \varepsilon \geq 0$. Suppose that the function $f\colon]0,1[\to \mathbb{R}$ satisfies the inequality*

$$\left| f(x) + (1-x)^{\alpha} f\left(\frac{y}{1-x}\right) - f(y) - (1-y)^{\alpha} f\left(\frac{x}{1-y}\right) \right| \leq \varepsilon \tag{3.1}$$

for all $(x,y) \in D^{\circ}$. Then, in case $\alpha = 0$, there exists a logarithmic function $l\colon]0,1[\to \mathbb{R}$ and $c \in \mathbb{R}$ such that

$$|f(x) - [l(1-x) + c]| \leq K(\alpha)\varepsilon, \quad (x \in]0,1[) \tag{3.2}$$

furthermore, if $\alpha \notin \{0,1\}$, there exist $a, b \in \mathbb{R}$ such that

$$|f(x) - [ax^{\alpha} + b(1-x)^{\alpha} - b]| \leq K(\alpha)\varepsilon \tag{3.3}$$

holds for all $x \in]0,1[$, where

$$K(\alpha) = \begin{cases} \left|2^{1-\alpha} - 1\right|^{-1} \left(8 + 6 \cdot 2^{\alpha} + 2^{-\alpha}\right), & \text{if } \alpha < 0 \\ 63, & \text{if } \alpha = 0 \\ \left|2^{1-\alpha} - 1\right|^{-1} \left(3 + 12 \cdot 2^{\alpha} + \frac{32 \cdot 3^{\alpha+1}}{|2^{-\alpha} - 1|}\right), & \text{if } \alpha > 0. \end{cases}$$

Proof. Define the function F on $\mathbb{R}^2_{++}$ by

$$F(u,v) = (u+v)^{\alpha} f\left(\frac{v}{u+v}\right). \tag{3.4}$$

Then

$$F(tu, tv) = t^{\alpha} F(u,v) \quad (t, u, v \in \mathbb{R}_{++}) \tag{3.5}$$

and

$$f(x) = F(1-x, x), \quad (x \in]0,1[) \tag{3.6}$$

furthermore, with the substitutions

$$x = \frac{w}{u+v+w}, \quad y = \frac{v}{u+v+w} \quad (u, v, w \in \mathbb{R}_{++})$$

inequality (3.1) implies that

$$\left| f\left(\frac{w}{u+v+w}\right) + \frac{(u+v)^\alpha}{(u+v+w)^\alpha} f\left(\frac{v}{u+v}\right) \right.$$
$$\left. -f\left(\frac{v}{u+v+w}\right) - \frac{(u+w)^\alpha}{(u+v+w)^\alpha} f\left(\frac{w}{u+w}\right) \right| \leq \varepsilon$$

whence, by (3.4)

$$|F(u+v, w) + F(u, v) - F(u+w, v) - F(u, w)| \leq \varepsilon(u+v+w)^\alpha \tag{3.7}$$

follows for all $u, v, w \in \mathbb{R}_{++}$.

In the next step we define the functions g and G on $\mathbb{R}_{++}$ and on $\mathbb{R}^2_{++}$, respectively by

$$g(u) = F(u, 1) - F(1, u) \tag{3.8}$$

and

$$G(u, v) = F(u, v) + g(v). \tag{3.9}$$

We will show that

$$|G(u, v) - G(v, u)| \leq 3\varepsilon(u+v+1)^\alpha. \quad (u, v \in \mathbb{R}_{++}) \tag{3.10}$$

Indeed, with the substitution $w = 1$, inequality (3.7) implies that

$$|F(u+v, 1) + F(u, v) - F(u+1, v) - F(u, 1)| \leq \varepsilon(u+v+1)^\alpha. \tag{3.11}$$

Interchanging u and v, it follows from (3.11) that

$$|-F(u+v, 1) - F(v, u) + F(v+1, u) - F(v, 1)| \leq \varepsilon(u+v+1)^\alpha. \quad (u, v \in \mathbb{R}_{++})$$

This inequality, together with (3.11) and the triangle inequality imply that

$$\begin{aligned} |F(u, v) - F(v, u) - F(u+1, v) \\ -F(u, 1) + F(v+1, u) + F(v, 1)| \leq 2\varepsilon(u+v+1)^\alpha \end{aligned} \tag{3.12}$$

holds for all $u, v \in \mathbb{R}_{++}$. On the other hand, with $u = 1$, we get from (3.7) that

$$|F(1+v, w) + F(1, v) - F(1+w, w) - F(1, w)| \leq \varepsilon(1+v+w)^\alpha.$$

Replacing here v by u and w by v, respectively, we have that

$$|F(u+1, v) + F(1, u) - F(v+1, u) - F(1, v)| \leq \varepsilon(u+v+1)^\alpha. \quad (u, v \in \mathbb{R}_{++})$$

Again, by the triangle inequality and the definitions (3.8) and (3.9), (3.12) and the last inequality imply (3.10).

In what follows we will investigate the function g. At this point of the proof we have to distinguish three cases.

Case I. $(\alpha < 0)$

In this case we will determine the function g by proving that

$$g(u) = c(u^{\alpha} - 1) \quad (u \in \mathbb{R}_{++}) \tag{3.13}$$

with some $c \in \mathbb{R}$.

Indeed, (3.10), (3.9) and (3.5) imply that

$$|G(tu, tv) - G(tv, tu)| \leq 3\varepsilon(tu + tv + 1)^{\alpha}, \quad (t, u, v \in \mathbb{R}_{++})$$

therefore

$$|t^{\alpha}F(u, v) + g(tv) - t^{\alpha}F(v, u) - g(tu)| \leq 3\varepsilon(tu + tv + 1)^{\alpha} \quad (t, u, v \in \mathbb{R}_{++})$$

or

$$|F(u, v) - F(v, u) - t^{\alpha}\left(g(tu) - g(tv)\right)| \leq 3\varepsilon(u + v + t^{-1})^{\alpha} \quad (t, u, v \in \mathbb{R}_{++})$$

whence

$$\lim_{t \to 0} t^{-\alpha}\left(g(tu) - g(tv)\right) = F(u, v) - F(v, u) \quad (t, u, v \in \mathbb{R}_{++})$$

follows. Particularly, with $v = 1$, by (3.8), we have that

$$g(u) = \lim_{t \to 0} t^{-\alpha}\left(g(tu) - g(t)\right). \quad (u \in \mathbb{R}_{++}) \tag{3.14}$$

Let now $u, v \in \mathbb{R}_{++}$. Then, by (3.14), we obtain that

$$\begin{aligned} g(uv) &= \lim_{t \to 0} t^{-\alpha}\left[g(tuv) - g(t)\right] \\ &= \lim_{t \to 0}\left[(tv)^{-\alpha}\left(g((tv)u) - g(tv)\right)v^{\alpha} + t^{-\alpha}(g(tv) - g(t))\right] \\ &= g(u)v^{\alpha} + g(v). \end{aligned}$$

Therefore, $g(u)v^{\alpha} + g(v) = g(v)u^{\alpha} + g(u)$, that is,

$$g(u)\left(v^{\alpha} - 1\right) = g(v)\left(u^{\alpha} - 1\right) \quad (u, v \in \mathbb{R}_{++})$$

which implies (3.13) with $c = g(2)\left(2^{\alpha} - 1\right)^{-1}$.

Thus, by (3.6), (3.13), (3.9) and (3.10), we have that

$$\begin{aligned} &|f(x) - c(1 - x)^{\alpha} - (f(1 - x) - cx^{\alpha})| \\ &= |F(1 - x, x) + cx^{\alpha} - (F(x, 1 - x) + c(1 - x)^{\alpha})| \\ &= |G(1 - x, x) - G(x, 1 - x)| \leq 3 \cdot 2^{\alpha}\varepsilon \end{aligned} \tag{3.15}$$

holds for all $x \in]0, 1[$.

In the next step we define the functions f_0 and F_0 on $]0, 1[$ and on $]0, 1[^2$ by

$$f_0(x) = f(x) - c\left[(1 - x)^{\alpha} - 1\right] \tag{3.16}$$

and

$$F_0(p, q) = f_0(p) + p^{\alpha}f_0(q) - f_0(pq) - (1 - pq)^{\alpha}f_0\left(\frac{1 - p}{1 - pq}\right), \tag{3.17}$$

respectively. Then (3.1) and (3.15) imply that

$$\left|f_0(x) + (1 - x)^{\alpha}f_0\left(\frac{y}{1 - x}\right) - f_0(y) - (1 - y)^{\alpha}f_0\left(\frac{x}{1 - y}\right)\right| \leq \varepsilon \tag{3.18}$$

for all $(x, y) \in D^{\circ}$ and

$$|f_0(x) - f_0(1-x)| \leq 3 \cdot 2^{\alpha} \varepsilon. \quad (x \in]0,1[) \tag{3.19}$$

Furthermore, with the substitutions $x = 1-p$, $y = pq$ $(p, q \in]0,1[)$, (3.18) implies that

$$\left| f_0(1-p) + p^{\alpha} f_0(q) - f_0(pq) - (1-pq)^{\alpha} f_0\left(\frac{1-p}{1-pq}\right) \right| \leq \varepsilon \tag{3.20}$$

holds for all $p, q \in]0,1[$. Therefore, due to (3.19) and the triangle inequality, (3.18) implies that

$$|F_0(p,q)| \leq (3 \cdot 2^{\alpha} + 1)\, \varepsilon. \quad (p, q \in]0,1[) \tag{3.21}$$

It can easily be checked that

$$\begin{aligned} f_0(p)\, [q^{\alpha} + (1-q)^{\alpha} - 1] - f_0(q)\, [p^{\alpha} + (1-p)^{\alpha} - 1] \\ = F_0(q,p) - F_0(p,q) \\ - (1-pq)^{\alpha} \left[F_0\left(\frac{1-q}{1-pq}, p\right) + f_0\left(1 - \frac{1-p}{1-pq}\right) - f_0\left(\frac{1-p}{1-pq}\right)\right] \end{aligned} \tag{3.22}$$

holds for all $p, q \in]0,1[$. Thus, by (3.21) and (3.19) we get that

$$\begin{aligned} &\left| f_0(p) - \frac{f_0(q)}{q^{\alpha} + (1-q)^{\alpha} - 1}\, [p^{\alpha} + (1-p)^{\alpha} - 1] \right| \\ &\qquad \leq \frac{2(1 + 3 \cdot 2^{\alpha}) + (1-pq)^{\alpha}(1 + 6 \cdot 2^{\alpha})}{q^{\alpha} + (1-q)^{\alpha} - 1} \varepsilon. \quad (p, q \in]0,1[) \end{aligned}$$

Taking into consideration (3.16), with $q = \frac{1}{2}$ with the definitions

$$a = f_0\left(\frac{1}{2}\right) \left(2^{1-\alpha} - 1\right)^{-1}, \quad b = a + c,$$

this inequality implies that

$$|f(x) - [ax^{\alpha} + b(1-x)^{\alpha} - b]| \leq \frac{8 + 6 \cdot 2^{\alpha} + 2^{-\alpha}}{2^{1-\alpha} - 1} \varepsilon. \quad (x \in]0,1[)$$

In view of the definition of $K(\alpha)$, this implies that inequality (3.3) holds for all $x \in]0,1[$.

<u>Case II.</u> $(\alpha = 0)$

In the second case we will show that there exists a logarithmic function $l \colon \mathbb{R}_{++} \to \mathbb{R}$ such that

$$|g(u) - l(u)| \leq 6\varepsilon$$

for all $u \in \mathbb{R}_{++}$. Indeed, (3.10) yields in this case that

$$|G(u,v) - G(v,u)| \leq 3\varepsilon. \quad (u, v \in \mathbb{R}_{++})$$

Due to (3.5) and (3.9) we obtain that

$$\begin{aligned} G(tu, tv) &= F(tu, tv) + g(tv) \\ &= F(u, v) + g(tv) \\ &= G(u, v) - g(v) + g(tv) \end{aligned}$$

that is,

$$G(tu, tv) - G(u, v) = g(tv) - g(v), \quad (t, u, v \in \mathbb{R}_{++})$$

therefore

$$\begin{aligned} &|g(tv) - g(v) + g(u) - g(tu)| \\ &= |G(tu, tv) - G(u, v) - G(tv, tu) + G(v, u)| \\ &\leq |G(tu, tv) - G(tv, tu)| + |G(v, u) - G(u, v)| \leq 6\varepsilon \end{aligned} \tag{3.23}$$

for all $t, u, v \in \mathbb{R}_{++}$. Now (3.23) with the substitution $u = 1$ implies that

$$|g(tv) - g(v) - g(t)| \leq 6\varepsilon$$

holds for all $t, v \in \mathbb{R}_{++}$, since obviously $g(1) = 0$. This means that the function g is approximately logarithmic on $\mathbb{R}_{++}$. Thus (see e.g. Forti [6]) there exists a logarithmic function $l \colon \mathbb{R}_{++} \to \mathbb{R}$ such that

$$|g(u) - l(u)| \leq 6\varepsilon$$

holds for all $u \in \mathbb{R}_{++}$.

Furthermore,

$$\begin{aligned} &|f(x) - l(1-x) - (f(1-x) - l(x))| \\ &= |F(1-x, x) - l(1-x) - F(x, 1-x) + l(x)| \\ &= |F(1-x, x) + g(x) - g(x) - l(1-x) \\ &\quad - F(x, 1-x) + g(1-x) - g(1-x) + l(x)\ | \\ &\leq |F(1-x, x) + g(x) - (F(x, 1-x) + g(1-x))| \\ &\quad + |g(1-x) - l(1-x)| + |l(x) - g(x)| \\ &= |G(1-x, x) - G(x, 1-x)| \\ &\quad + |g(1-x) - l(1-x)| + |l(x) - g(x)| \\ &\leq 3\varepsilon + 6\varepsilon + 6\varepsilon = 15\varepsilon \end{aligned} \tag{3.24}$$

As in the first part of the proof, define the functions f_0 and F_0 on $]0, 1[$ and on $]0, 1[^2$, respectively, by

$$f_0(x) = f(x) - l(1-x)$$

and

$$F_0(p, q) = f_0(p) + f_0(q) - f_0(pq) - f_0\left(\frac{1-p}{1-pq}\right)$$

Due to (3.24)

$$|f_0(x) - f_0(1-x)| \leq 15\varepsilon \tag{3.25}$$

holds for all $x \in]0,1[$. Furthermore, with the substitutions $x = 1-p$, $y = pq$ ($p, q \in]0,1[$) inequality (3.1) implies, that

$$\left| f_0(1-p) + f_0(q) - f_0(pq) - f_0\left(\frac{1-p}{1-pq}\right)\right| \leq \varepsilon \tag{3.26}$$

is fulfilled for all $p, q \in]0,1[$. Inequalities (3.25) and (3.26) and the triangle inequality imply that

$$|F_0(p,q)| \leq 16\varepsilon \tag{3.27}$$

for all $p, q \in]0,1[$. An easy calculation shows that

$$\begin{aligned} &f_0(p) - f_0(q) \\ &= F_0(q,p) - F_0(p,q) + F_0\left(\frac{1-p}{1-pq}, p\right) - f_0\left(1 - \frac{1-p}{1-pq}\right) + f_0\left(\frac{1-p}{1-pq}\right) \end{aligned}$$

therefore,

$$\begin{aligned} |f_0(p) - f_0(q)| &\leq |F_0(q,p)| + |F_0(p,q)| + \left|F_0\left(\frac{1-p}{1-pq}, p\right)\right| \\ &\quad + \left| f_0\left(1 - \frac{1-p}{1-pq}\right) - f_0\left(\frac{1-p}{1-pq}\right)\right| \\ &\leq 3 \cdot 16\varepsilon + 15\varepsilon = 63\varepsilon \end{aligned} \tag{3.28}$$

holds for all $p, q \in]0,1[$. With the substitution $q = \frac{1}{2}$ inequality (3.28) implies that

$$\left| f_0(p) - f_0\left(\frac{1}{2}\right)\right| \leq 63\varepsilon. \quad (p \in]0,1[)$$

Using the definition of the function f_0, we obtain that inequality

$$|f(x) - l(1-x) - c| \leq 63\varepsilon$$

is satisfied for all $x \in]0,1[$, where $c = f_0\left(\frac{1}{2}\right)$. Hence inequality (3.2) holds, indeed.

<u>Case III.</u> ($1 \neq \alpha > 0$)

Finally, in the last case, we will prove that there exists $c \in \mathbb{R}$ such that

$$|g(x) - c(x^\alpha - 1)| \leq \frac{4 \cdot 3^{\alpha+1}\varepsilon}{|2^{-\alpha} - 1|}$$

holds for all $x \in]0,1[$.

Due to inequalities (3.4) and (3.8),

$$\begin{aligned} G(tu, tv) &= F(tu, tv) + g(tv) \\ &= t^\alpha F(u,v) + g(tv) \\ &= t^\alpha G(u,v) - t^\alpha g(v) + g(tv), \end{aligned}$$

that is,

$$G(tu, tv) - t^\alpha G(u,v) = g(tv) - t^\alpha g(v)$$

holds for all $t, v \in \mathbb{R}_{++}$. Therefore,

$$\begin{aligned} &|g(tv) - t^{\alpha}g(v) + t^{\alpha}g(u) - g(tu)| \\ &= |G(tu, tv) - G(u, v) - G(tv, tu) + G(v, u)| \\ &\leq |G(tu, tv) - G(tv, tu)| + |G(u, v) - G(v, u)| \\ &\leq 3\varepsilon(t(u + v) + 1)^{\alpha} + 3\varepsilon(u + v + 1)^{\alpha} \end{aligned} \tag{3.29}$$

holds for all $t, u, v \in \mathbb{R}_{++}$, where we used (3.10). With the substitution $u = 1$, (3.29) implies that

$$\begin{aligned} &|g(tv) - t^{\alpha}g(v) - g(t)| \\ &\qquad \leq 3\varepsilon(t(v + 1) + 1)^{\alpha} + 3\varepsilon(v + 2)^{\alpha} \quad (t, v \in \mathbb{R}_{++}) \end{aligned} \tag{3.30}$$

Interchanging t and v in (3.30), we obtain that

$$\begin{aligned} &|g(tv) - v^{\alpha}g(t) - g(v)| \\ &\qquad \leq 3\varepsilon(v(t + 1) + 1)^{\alpha} + 3\varepsilon(t + 2)^{\alpha} \quad (t, v \in \mathbb{R}_{++}) \end{aligned} \tag{3.31}$$

Inequalities (3.30), (3.31) and the triangle inequality imply that

$$|t^{\alpha}g(v) + g(t) - v^{\alpha}g(t) - g(v)| \leq B(t, v) \tag{3.32}$$

is fulfilled for all $t, v \in \mathbb{R}_{++}$, where

$$\begin{aligned} B(t, v) = &\, 3\varepsilon(t(v + 1) + 1)^{\alpha} + 3\varepsilon(v + 2)^{\alpha} \\ &+ 3\varepsilon(v(t + 1) + 1)^{\alpha} + 3\varepsilon(t + 2)^{\alpha}. \end{aligned}$$

With the substitution $t = \frac{1}{2}$ and with the definition $c = \frac{g\left(\frac{1}{2}\right)}{2^{-\alpha}-1}$, we obtain

$$|g(v) - c(v^{\alpha} - 1)| \leq \frac{B\left(\frac{1}{2}, v\right)}{|2^{-\alpha} - 1|} \tag{3.33}$$

for all $v \in \mathbb{R}_{++}$.

Let us observe that

$$|B(t, v)| \leq 4 \cdot 3^{\alpha+1}\varepsilon$$

holds, if $t, v \in]0, 1[$. Thus

$$|g(v) - c(v^{\alpha} - 1)| \leq \frac{B\left(\frac{1}{2}, v\right)}{|2^{-\alpha} - 1|} \leq \frac{4 \cdot 3^{\alpha+1}\varepsilon}{|2^{-\alpha} - 1|} \tag{3.34}$$

for all $v \in]0,1[$. Therefore (3.6), (3.9), (3.10), (3.34) and the triangle inequality imply that

$$\begin{aligned}
&|f(x) - c(1-x)^\alpha + c - (f(1-x) - cx^\alpha + c)| \\
&= |F(1-x,x) - c(1-x)^\alpha + c - (F(x,1-x) - cx^\alpha + c)| \\
&\le |F(1-x,x) + g(x) - F(x,1-x) - g(1-x)| \\
&\quad + |g(x) - c(x^\alpha - 1)| + |g(1-x) - c((1-x)^\alpha - 1)| \\
&= |G(1-x,x) - G(x,1-x)| \\
&\quad + |g(x) - c(x^\alpha - 1)| + |g(1-x) - c((1-x)^\alpha - 1)| \\
&\le 3 \cdot 2^\alpha \varepsilon + \frac{8 \cdot 3^{\alpha+1}\varepsilon}{|2^{-\alpha} - 1|}
\end{aligned} \tag{3.35}$$

holds for all $x \in]0,1[$.

As in the previous cases, we define the functions f_0 and F_0 on $]0,1[$ and on $]0,1[^2$ by

$$f_0(x) = f(x) - c(1-x)^\alpha \tag{3.36}$$

and

$$F_0(p,q) = f_0(p) + p^\alpha f_0(q) - f_0(pq) - (1-pq)^\alpha f_0\left(\frac{1-p}{1-pq}\right), \tag{3.37}$$

respectively. Then (3.1), (3.35) and (3.36) imply that

$$\left| f_0(x) + (1-x)^\alpha f_0\left(\frac{y}{1-x}\right) - f_0(y) - (1-y)^\alpha f_0\left(\frac{x}{1-y}\right) \right| \le \varepsilon \tag{3.38}$$

for all $(x,y) \in D^\circ$ and

$$|f_0(x) - f_0(1-x)| \le 3 \cdot 2^\alpha \varepsilon + \frac{8 \cdot 3^{\alpha+1}\varepsilon}{|2^{-\alpha} - 1|}. \quad (x \in]0,1[) \tag{3.39}$$

Furthermore, with the substitutions $x = 1-p$, $y = pq$ ($p,q \in]0,1[$), (3.38) implies that

$$\left| f_0(1-p) + p^\alpha f_0(q) - f_0(pq) - (1-pq)^\alpha f_0\left(\frac{1-p}{1-pq}\right) \right| \le \varepsilon \tag{3.40}$$

holds for all $p,q \in]0,1[$. Thus (3.39) and (3.40) and the triangle inequality imply that

$$|F_0(p,q)| \le \varepsilon + 3 \cdot 2^\alpha \varepsilon + \frac{8 \cdot 3^{\alpha+1}\varepsilon}{|2^{-\alpha} - 1|}. \quad (x \in]0,1[)$$

As in the previous cases, it is easy to see that the identity (3.22) is satisfied for all $p, q \in]0,1[$. Therefore

$$\left| f_0(p) - \frac{f_0(q)}{q^\alpha + (1-q)^\alpha - 1} \left[p^\alpha + (1-p)^\alpha - 1\right] \right|$$
$$\leq |q^\alpha + (1-q)^\alpha - 1|^{-1} \left(3\left(\varepsilon + 3\cdot 2^\alpha \varepsilon + \frac{8\cdot 3^{\alpha+1}\varepsilon}{|2^{-\alpha}-1|}\right) + 3\cdot 2^\alpha\varepsilon + \frac{8\cdot 3^{\alpha+1}\varepsilon}{|2^{-\alpha}-1|}\right)$$

for all $p, q \in]0,1[$. In view of (3.36), with $q = \frac{1}{2}$ with the definitions

$$a = f_0\left(\frac{1}{2}\right)\left(2^{1-\alpha}-1\right)^{-1} \quad \text{and} \quad b = a + c,$$

this inequality implies that

$$|f(p) - [ap^\alpha + b(1-p)^\alpha - b]| \leq K(\alpha)\varepsilon \tag{3.41}$$

holds for all $p \in]0,1[$, where

$$K(\alpha) = \left|2^{1-\alpha}-1\right|^{-1}\left(3 + 12\cdot 2^\alpha + \frac{32\cdot 3^{\alpha+1}}{|2^{-\alpha}-1|}\right),$$

which had to be proved. □

4. Corollaries and remarks

First we explain, why our method does not work, in case $\alpha = 1$.

Remark 1. Since

$$\lim_{\alpha\to 1} K(\alpha) = +\infty,$$

our method is inappropriate if $\alpha = 1$. Hence we cannot prove stability concerning the fundamental equation of information on the set D°.

An easy calculation shows that

$$\sup_{\alpha<0} K(\alpha) = \sup_{\alpha<0} \frac{8 + 6\cdot 2^\alpha + 2^{-\alpha}}{2^{1-\alpha}-1} = 15,$$

therefore, in case $\alpha < 0$, 15 can also be considered as a stability constant.

Using Theorem 3.1., with the choice $\varepsilon = 0$, we get the general solution of equation (1.1) (see Ebanks–Sahoo–Sander [5] or Maksa [14]).

Corollary 4.1. *Let $\alpha \neq 1$ be arbitrary but fixed real number and assume that the function $f\colon]0,1[\to \mathbb{R}$ satisfies the functional equation*

$$f(x) + (1-x)^\alpha f\left(\frac{y}{1-x}\right) = f(y) + (1-y)^\alpha f\left(\frac{x}{1-y}\right)$$

for all pairs $(x, y) \in D^\circ$. Then, and only then, in case $\alpha = 0$, there exists a logarithmic function $l\colon]0,1[\to \mathbb{R}$ and $c \in \mathbb{R}$ such that

$$f(x) = l(1-x) + c, \quad (x \in]0,1[)$$

furthermore, in case $\alpha \notin \{0,1\}$, *there exist* $a, b \in \mathbb{R}$ *such that*

$$f(x) = ax^{\alpha} + b(1-x)^{\alpha} - b$$

holds for all $x \in]0,1[$.

Remark 2. In view of Corollary 4.1., Theorem 3.1. says that the parametric fundamental equation of information is stable on the open domain in the sense of Hyers and Ulam, provided that the parameter does not equal to one.

Remark 3. Let us observe that the solutions of (1.1) are bounded on D°, assuming that $1 \neq \alpha > 0$. Therefore Theorem 3.1. means that the parametric fundamental equation of information is not only stable but also superstable in this case(as to the superstability, the reader can consult Ger [7] and Moszner [17]).

In the following theorem we shall prove that equation (1.1) is stable not only on D° but also on D. During the proof of this theorem the following function will be needed. For all $1 \neq \alpha > 0$ we define the function $T(\alpha)$ by

$$T(\alpha) = 3 \cdot 2^{\alpha} + \frac{8 \cdot 3^{\alpha+1}}{|2^{-\alpha} - 1|},$$

that is, $T(\alpha)$ is that function which appears in inequality (3.39). Furthermore, the following relationship is fulfilled between $K(\alpha)$ and $T(\alpha)$

$$K(\alpha) = \frac{4T(\alpha) + 3}{|2^{1-\alpha} - 1|}$$

for all $1 \neq \alpha > 0$.

Theorem 4.1. *Let* $\alpha, \varepsilon \in \mathbb{R}$ *be fixed,* $\alpha \neq 1$, $\varepsilon \geq 0$. *Suppose that the function* $f\colon [0,1] \to \mathbb{R}$ *satisfies inequality (3.1) for all* $(x,y) \in D$. *Then, in case* $\alpha \neq 0$ *there exist* $a, b \in \mathbb{R}$ *such that the function* h_1 *defined on* $[0,1]$ *by*

$$h_1(x) = \begin{cases} 0, & \textit{if } x = 0 \\ ax^{\alpha} + b(1-x)^{\alpha} - b, & \textit{if } x \in \ ]0,1[\\ a - b, & \textit{if } x = 1 \end{cases}$$

is a solution of (1.1) on D *and*

$$|f(x) - h_1(x)| \leq K(\alpha)\varepsilon, \quad (x \in [0,1]) \tag{4.1}$$

holds if $\alpha < 0$ *and*

$$|f(x) - h_1(x)| \leq \max\{K(\alpha), T(\alpha) + 1\}\,\varepsilon, \quad (x \in [0,1]) \tag{4.2}$$

is satisfied in case $1 \neq \alpha > 0$. *Furthermore, in case* $\alpha = 0$, *there exists* $c \in \mathbb{R}$ *such that the function* h_2 *defined on* $[0,1]$ *by*

$$h_2(x) = \begin{cases} f(0), & \textit{if } x = 0 \\ c, & \textit{if } x \in \ ]0,1[\\ f(1), & \textit{if } x = 1 \end{cases}$$

is a solution of (1.1) on D and

$$|f(x) - h_2(x)| \leq K(\alpha)\varepsilon. \quad (x \in [0, 1]) \tag{4.3}$$

Proof. An easy calculation shows that the functions h_1 and h_2 are the solutions of equation (1.1) on D in case $\alpha \neq 0$ and $\alpha = 0$, respectively.

Firstly, we investigate the case $\alpha < 0$. Theorem 3.1. implies that (4.1) holds for all $x \in]0, 1[$. Thus it is enough to prove that (4.1) holds for $x = 0$ and $x = 1$. It follows from (3.1), with the substitution $y = 0$, that

$$|(1 - x)^{\alpha} - 1| \cdot |f(0)| \leq \varepsilon. \quad (x \in]0, 1[)$$

Since $\alpha < 0$, $f(0) = 0$ follows, that is, (4.1) holds in case $x = 0$.

Let now $x \in]0, 1[$ and $y = 1 - x$ in (3.1). Then

$$|f(1 - x) - f(x) - f(1) \left((1 - x)^{\alpha} - x^{\alpha}\right)| \leq \varepsilon.$$

Applying (3.3) to $1 - x$ instead of x to get

$$|-f(1 - x) + a(1 - x)^{\alpha} + bx^{\alpha} - b| \leq K(\alpha)\varepsilon.$$

Adding this last two inequalities and inequality (3.3) up and using the triangle inequality to obtain

$$|(a - b) - f(1)| \cdot |(1 - x)^{\alpha} - x^{\alpha}| \leq (2K(\alpha) + 1)\,\varepsilon. \quad (x \in]0, 1[)$$

Since $\alpha < 0$, we get that $f(1) = a - b$ and so (4.1) holds also for $x = 1$.

Secondly, we deal with the case $\alpha > 0$. Substituting $x = 0$ into (3.1) and with $y \to 0$ we obtain that

$$|f(0)| \leq \varepsilon \leq K(\alpha)\varepsilon,$$

that is, (4.1) holds for $x = 0$. If $x \in]0, 1[$, then inequality (4.2) follows immediately from Theorem 3.1. Furthermore, with the substitution $y = 1 - x$ ($x \in]0, 1[$) inequality (3.1) implies that

$$|f(x) + (1 - x)^{\alpha} f(1) - f(1 - x) - x^{\alpha} f(1)| \leq \varepsilon. \quad (x \in]0, 1[)$$

From the proof of Theorem 3.1 (see definition (3.36)) it is known that

$$f(x) = f_0(x) + c(1 - x)^{\alpha}, \quad (x \in]0, 1[)$$

therefore the last inequality yields that

$$|f_0(x) - f_0(1 - x) + c(1 - x)^{\alpha} - cx^{\alpha} + (1 - x)^{\alpha} f(1) - x^{\alpha} f(1)| \leq \varepsilon \tag{4.4}$$

holds for all $x \in]0, 1[$. Whereas

$$|f_0(x) - f_0(1 - x)| \leq T(\alpha). \quad (x \in]0, 1[)$$

Thus after rearranging (4.4), we get that

$$|f_0(x) - f_0(1 - x) - [c + f(1)][x^{\alpha} - (1 - x)^{\alpha}]| \leq \varepsilon, \quad (x \in]0, 1[)$$

that is,

$$||f_0(x) - f_0(1 - x)| - |c + f(1)| \cdot |x^{\alpha} - (1 - x)^{\alpha}|| \leq \varepsilon$$

holds for all $x \in]0,1[$. Therefore

$$|c + f(1)| \cdot |x^{\alpha} - (1-x)^{\alpha}| \leq (T(\alpha)+1)\varepsilon$$

for all $x \in]0,1[$. Taking the limit $x \to 0+$, we obtain that

$$|c + f(1)| \leq (T(\alpha)+1)\varepsilon.$$

However, in the proof of Theorem 3.1. we used the definition $c = b - a$, thus

$$|f(1) - (a-b)| \leq (T(\alpha)+1)\varepsilon,$$

so (4.2) holds, indeed.

Finally, we investigate the case $\alpha = 0$. If $x = 0$ or $x = 1$, then (4.3) trivially holds, since

$$|f(0) - h_2(0)| = |f(0) - f(0)| = 0 \leq K(\alpha)\varepsilon$$

and

$$|f(1) - h_2(1)| = |f(1) - f(1)| = 0 \leq K(\alpha)\varepsilon.$$

Let now $x \in]0,1[$ and $y = 1 - x$ in (3.1), then we obtain that

$$|f(x) - f(1-x)| \leq \varepsilon, \quad (x \in]0,1[) \tag{4.5}$$

if fulfilled for all $x \in]0,1[$.

Due to Theorem 3.1. there exists a logarithmic function $l\colon]0,1[\to \mathbb{R}$ and $c \in \mathbb{R}$ such that

$$|f(x) - l(1-x) - c| \leq 63\varepsilon \tag{4.6}$$

holds for all $x \in]0,1[$. Hence it is enough to prove that the function l is identically zero on $]0,1[$. Indeed, due to (3.2) and (4.5)

$$\begin{aligned}
&|l(1-x) - l(x)| \\
&= |l(1-x) - f(1-x) + f(1-x) + c - l(x) + f(x) - f(x) - c| \\
&\leq |l(1-x) + c - f(x)| + |f(1-x) - l(x) - c| + |f(x) - f(1-x)| \\
&\leq 127\varepsilon
\end{aligned} \tag{4.7}$$

holds for all $x \in]0,1[$. Since the function l is uniquely extendable to $\mathbb{R}_{++}$, with the substitution $x = \frac{p}{p+q}$ $(p, q \in \mathbb{R})$, we get that

$$|l(p) - l(q)| \leq 127\varepsilon, \quad (p, q \in \mathbb{R}_{++})$$

where we used the fact that l is logarithmic, as well. This last inequality, with the substitution $q = 1$ implies that

$$|l(p)| \leq 127\varepsilon$$

holds for all $p \in \mathbb{R}_{++}$, since $l(1) = 0$. Thus l is bounded on $\mathbb{R}_{++}$. However, the only bounded, logarithmic function on $\mathbb{R}_{++}$ is the identically zero function. Therefore,

$$|f(x) - c| \leq 63\varepsilon$$

holds for all $x \in]0,1[$, i.e., (4.3) is proved. □

Applying Theorem 3.1. we can prove the stability of a system of functional equations that characterizes the α-recursive, 3-semi-symmetric information measures.

Theorem 4.2. *Let $n \geq 2$ be a fixed positive integer and (I_n) be the sequence of functions $I_n \colon \Gamma_n^{\circ} \to \mathbb{R}$ and suppose that there exist a sequence (ε_n) of non-negative real numbers and a real number $\alpha \neq 1$ such that*

$$(4.8) \quad \Bigg| I_n(p_1, \ldots, p_n) - I_{n-1}(p_1 + p_2, p_3, \ldots, p_n) - (p_1 + p_2)^{\alpha} I_2 \left(\frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2} \right) \Bigg| \leq \varepsilon_{n-1}$$

for all $n \geq 3$ and $(p_1, \ldots, p_n) \in \Gamma_n^{\circ}$, and

$$(4.9) \qquad |I_3(p_1, p_2, p_3) - I_3(p_1, p_3, p_2)| \leq \varepsilon_1$$

holds on Γ_n°. Then, in case $\alpha < 0$ there exist $c, d \in \mathbb{R}$ such that

$$(4.10) \quad |I_n(p_1, \ldots, p_n) - [cH_n^{\alpha}(p_1, \ldots, p_n) + d(p_1^{\alpha} - 1)]| \leq \sum_{k=2}^{n-1} \varepsilon_k + K(\alpha)(2\varepsilon_2 + \varepsilon_1) \left(1 + \sum_{k=2}^{n-1} \left(\sum_{i=1}^{k} p_i^{\alpha} \right) \right)$$

for all $n \geq 2$ and $(p_1, \ldots, p_n) \in \Gamma_n^{\circ}$. Furthermore, in case $\alpha = 0$ there exists a logarithmic function $l \colon]0, 1[\to \mathbb{R}$ and $c \in \mathbb{R}$ such that

$$(4.11) \quad \big| I_n(p_1, \ldots, p_n) - \big[cH_n^0(p_1, \ldots, p_n) + l(p_1) \big] \big| \leq \sum_{k=2}^{n-1} \varepsilon_k + K(\alpha)(n-1)(2\varepsilon_2 + \varepsilon_1)$$

for all $n \geq 2$ and $(p_1, \ldots, p_n) \in \Gamma_n^{\circ}$. Finally, if $\alpha > 0$ then there exist $c, d \in \mathbb{R}$ such that

$$(4.12) \quad |I_n(p_1, \ldots, p_n) - [cH_n^{\alpha}(p_1, \ldots, p_n) + d(p_1^{\alpha} - 1)]| \leq \sum_{k=2}^{n-1} \varepsilon_k + (n-1)K(\alpha)(2\varepsilon_2 + \varepsilon_1)$$

holds for all $n \geq 2$ and $(p_1, \ldots, p_n) \in \Gamma_n^{\circ}$, where the convention

$$\sum_{k=2}^{1} \varepsilon_k = \sum_{k=2}^{1} \left(\sum_{i=1}^{k} p_i^{\alpha} \right) = 0$$

is adopted.

Proof. As in [15], due to (4.8) and (4.9), it can be proved that, for the function f defined on $]0, 1[$ by $f(x) = I_2(1 - x, x)$ we get that

$$\left| f(x) + (1-x)^{\alpha} f\left(\frac{y}{1-x} \right) - f(y) - (1-y)^{\alpha} f\left(\frac{x}{1-y} \right) \right| \leq 2\varepsilon_2 + \varepsilon_1$$

for all $(x, y) \in D^\circ$, i.e., (3.1) holds with $\varepsilon = 2\varepsilon_2 + \varepsilon_1$. Therefore, applying Theorem 3.1. we obtain (3.2) and (3.3), respectively, with some $a, b, c \in \mathbb{R}$ and a logarithmic function $l\colon]0,1[\to \mathbb{R}$ and $\varepsilon = 2\varepsilon_2 + \varepsilon_1$, i.e.,

$$|I_2\left(1-x,x\right) - (ax^\alpha + b(1-x)^\alpha - b)| \leq K(\alpha)(2\varepsilon_2+\varepsilon_1), \quad (x \in]0,1[)$$

in case $\alpha \neq 0$, and

$$|I_2\left(1-x,x\right) - (l(1-x)+c)| \leq K(\alpha)(2\varepsilon_2+\varepsilon_1) \quad (x \in]0,1[)$$

in case $\alpha = 0$.

Therefore (4.10) and (4.12) holds with $c = (2^{1-\alpha}-1)a$, $d = b-a$ in case $\alpha < 0$ and in case $\alpha > 0$, furthermore, (4.11) holds in case $\alpha = 0$, respectively, for $n = 2$.

We continue the proof by induction on n. Suppose that (4.10), (4.11) and (4.12) holds, resp., and for the sake of brevity, introduce the notation

$$J_n(p_1,\ldots,p_n) = \begin{cases} cH_n^\alpha(p_1,\ldots,p_n), & \text{if } \alpha \neq 0 \\ cH_n^0(p_1,\ldots,p_n) + l(p_1), & \text{if } \alpha = 0 \end{cases}$$

for all $n \geq 2$, $(p_1,\ldots,p_n) \in \Gamma_n^\circ$. It can easily be seen that (4.10), (4.11) and (4.12) hold on Γ_n° for J_n instead of I_n $(n \geq 3)$ with $\varepsilon_n = 0$ $(n \geq 2)$. Thus, for all $(p_1,\ldots,p_{n+1}) \in \Gamma_{n+1}^\circ$, we get that

$$\begin{aligned} &I_{n+1}(p_1,\ldots,p_{n+1}) - J_{n+1}(p_1,\ldots,p_{n+1}) \\ &= I_{n+1}(p_1,\ldots,p_{n+1}) - J_n(p_1+p_2,p_3,\ldots,p_{n+1}) \\ &\quad - (p_1+p_2)^\alpha J_2\left(\frac{p_1}{p_1+p_2}, \frac{p_2}{p_1+p_2}\right) \\ &= I_{n+1}(p_1,\ldots,p_{n+1}) - I_n(p_1+p_2,p_3,\ldots,p_{n+1}) \\ &\quad - (p_1+p_2)^\alpha I_2\left(\frac{p_1}{p_1+p_2}, \frac{p_2}{p_1+p_2}\right) \\ &\quad + I_n(p_1+p_2,p_3,\ldots,p_{n+1}) - J_n(p_1+p_2,p_3,\ldots,p_{n+1}) \\ &\quad + (p_1+p_2)^\alpha \left(I_2\left(\frac{p_1}{p_1+p_2}\right) - J_2\left(\frac{p_1}{p_1+p_2}, \frac{p_2}{p_1+p_2}\right)\right). \end{aligned}$$

Therefore, if $\alpha < 0$, (4.8) (with $n+1$ instead of n), (4.10) with $n = 2$ and the induction hypothesis (applying to $(p_1+p_2,\ldots,p_{n+1})$ instead of $(p_1,\ldots,p_n)$) imply that

$$\begin{aligned} &|I_{n+1}(p_1,\ldots,p_{n+1}) - J_{n+1}(p_1,\ldots,p_{n+1})| \\ &\leq \varepsilon_n + \sum_{k=2}^{n-1}\varepsilon_k + K(\alpha)(2\varepsilon_2+\varepsilon_1)\left(1 + \sum_{k=2}^{n-1}\left(\sum_{i=1}^{k+1} p_i^\alpha\right)\right) \\ &\quad + K(\alpha)(2\varepsilon_2+\varepsilon_1)(p_1+p_2)^\alpha \\ &= \sum_{k=2}^{n}\varepsilon_k + K(\alpha)(2\varepsilon_2+\varepsilon_1)\left(1 + \sum_{k=2}^{n}\left(\sum_{i=1}^{k} p_i^\alpha\right)\right), \end{aligned}$$

that is (4.10) holds for $n+1$ instead of n.

Furthermore, if $\alpha = 0$, (4.8) (with $n+1$ instead of n), (4.11) with $n = 2$ and the induction hypothesis (applying to $(p_1+p_2, \ldots, p_{n+1})$ instead of $(p_1, \ldots, p_n)$) imply that

$$\begin{aligned}
&|I_{n+1}(p_1, \ldots, p_{n+1}) - J_{n+1}(p_1, \ldots, p_{n+1})| \\
&\leq \varepsilon_n + \sum_{k=2}^{n-1} \varepsilon_k + K(\alpha)(n-1)(2\varepsilon_2 + \varepsilon_1) + K(\alpha)(2\varepsilon_2 + \varepsilon_1) \\
&= \sum_{k=2}^{n} \varepsilon_k + K(\alpha)n(2\varepsilon_2 + \varepsilon_1).
\end{aligned}$$

This yields that (4.11) holds for $n+1$ instead of n.

Finally, if $\alpha > 0$, then (4.8) (with $n+1$ instead of n), (4.12) with $n = 2$ and the induction hypothesis (applying to $(p_1+p_2, \ldots, p_{n+1})$ instead of $(p_1, \ldots, p_n)$) imply that

$$\begin{aligned}
&|I_{n+1}(p_1, \ldots, p_{n+1}) - J_{n+1}(p_1, \ldots, p_{n+1})| \\
&\leq \varepsilon_n + \sum_{k=2}^{n-1} \varepsilon_k + K(\alpha)(n-1)(2\varepsilon_2 + \varepsilon_1) + K(\alpha)(2\varepsilon_2 + \varepsilon_1) \\
&= \sum_{k=2}^{n} \varepsilon_k + K(\alpha)n(2\varepsilon_2 + \varepsilon_1),
\end{aligned}$$

that is, (4.12) holds for $n+1$ instead of n. □

Remark 4. Applying Theorem 4.2 with the choice $\varepsilon_n = 0$ for all $n \in \mathbb{N}$, we get the α–recursive, 3–semi–symmetric information measures. Hence Theorem 4.2 says that the system of α–recursive and 3–semi–symmetric information measures is stable.

5. Open problems

In the last part of the paper we list some open problems from the investigated topic.

The stability of equation (1.1) in the exceptional case $\alpha = 1$ was raised by Székelyhidi in [19], and it is still open.

Open problem 1. Prove or disprove that the fundamental equation of information is stable on the set D° or on the set D.

We remark that concerning this problem a partial result was published in Morando [16].

In the monograph of Ebanks, Sahoo and Sander (see [5]) higher dimensional information measures and functions are considered. A stability type result was published in [8], assuming the underlying multiplicative function is bounded on its closed domain. Therefore the following problem can be formulated.

Open problem 2. What can be said about the stability of the fundamental equation of information of multiplicative type on the closed as well as on the open domain?

In the inset theory (see e.g. Aczél–Daróczy [3]), measures of information may be depend on both the probabilities and events. Thus the problem of finding all inset information measures lead to the generalized fundamental equation of information of degree alpha, that is, to the functional equation

$$f(x) + (1-x)^{\alpha} g\left(\frac{y}{1-x}\right) = h(x) + (1-y)^{\alpha} k\left(\frac{x}{1-y}\right). \quad ((x,y) \in D^{\circ})$$

This equation was solved in Maksa [14].

Open problem 3. Is it true that the generalized fundamental equation of information of degree alpha is stable on the set D°?

Acknowledgement. I am grateful to Professor Gyula Maksa for encouraging me to write this paper and I would like to express my sincere thanks for his help during the preparation of the manuscript.

References

[1] J. Aczél. Notes on generalized information functions. *Aequationes Math.*, 22(1):97–107, 1981.

[2] J. Aczél. Characterizing information measures: approaching the end of an era. In *Uncertainty in knowledge-based systems (Paris, 1986)*, volume 286 of *Lecture Notes in Comput. Sci.*, pages 359–384. Springer, Berlin, 1987.

[3] J. Aczél and Z. Daróczy. *On measures of information and their characterizations.* Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1975. Mathematics in Science and Engineering, Vol. 115.

[4] Z. Daróczy. Generalized information functions. *Information and Control*, 16:36–51, 1970.

[5] B. Ebanks, P. Sahoo, and W. Sander. *Characterizations of information measures.* World Scientific Publishing Co. Inc., River Edge, NJ, 1998.

[6] G. L. Forti. Hyers-Ulam stability of functional equations in several variables. *Aequationes Math.*, 50(1-2):143–190, 1995.

[7] R. Ger. A survey of recent results on stability of functional equations. In *Proceeding of the* 4^{th} *International Conference on Functional Equations and Inequalities*, pages 5–36, Pedagogical University in Cracow, 1994.

[8] E. Gselmann. Stability type results concerning the fundamental equation of information of multiplicative type. *Colloq. Math.*, 114(1):33–40, 2009.

[9] E. Gselmann and G. Maksa. Stability of the parametric fundamental equation of information for nonpositive parameters. *Aequationes Math.*, To appear.

[10] J. Havrda and F. Charvát. Quantification method of classification processes. Concept of structural a-entropy. *Kybernetika (Prague)*, 3:30–35, 1967.

[11] D. H. Hyers. On the stability of the linear functional equation. *Proc. Nat. Acad. Sci. U. S. A.*, 27:222–224, 1941.

[12] D. H. Hyers, G. Isac, and T. M. Rassias. *Stability of functional equations in several variables.* Progress in Nonlinear Differential Equations and their Applications, 34. Birkhäuser Boston Inc., Boston, MA, 1998.

[13] M. Kuczma. *An introduction to the theory of functional equations and inequalities*, volume 489 of *Prace Naukowe Uniwersytetu Śląskiego w Katowicach [Scientific Publications of the University of Silesia]*. Uniwersytet Śląski, Katowice, 1985. Cauchy's equation and Jensen's inequality, With a Polish summary.
[14] G. Maksa. Solution on the open triangle of the generalized fundamental equation of information with four unknown functions. *Utilitas Math.*, 21:267–282, 1982.
[15] G. Maksa. The stability of the entropy of degree alpha. *J. Math. Anal. Appl.*, 346(1):17–21, 2008.
[16] A. Morando. A stability result concerning the Shannon entropy. *Aequationes Math.*, 62(3):286–296, 2001.
[17] Z. Moszner. Sur les définitions différentes de la stabilité des équations fonctionnelles. *Aequationes Math.*, 68(3):260–274, 2004.
[18] F. Radó and J. A. Baker. Pexider's equation and aggregation of allocations. *Aequationes Math.*, 32(2-3):227–239, 1987.
[19] L. Székelyhidi. 38. problem. *Aequationes Math.*, 1991. Report of Meeting: The Twenty-Eighth International Symposium on Functional Equations, August 23 - September 1, 1990, Graz-Mariatrost, Austria.
[20] C. Tsallis. Possible generalization of Boltzmann-Gibbs statistics. *J. Statist. Phys.*, 52(1-2):479–487, 1988.
[21] S. M. Ulam. *Problems in modern mathematics.* Science Editions John Wiley & Sons, Inc., New York, 1964.

Received October 22, 2008.

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