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SOME UNIFYING RESULTS ON STABILITY AND STRONG CONVERGENCE FOR SOME NEW ITERATION PROCESSES

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Abstract. In this paper, we shall establish some stability results as well as strong convergence results for a pair of nonselfmappings using some newly introduced iteration processes and two general contractive conditions. Our results are improvements, generalizations and extensions of the results in some of the references listed in the reference section of this paper as well as some other analogous ones in the literature.

1. Introduction

Let (E, d) be a complete metric space and $T : E \to E$ a selfmap of E. Suppose that $F_T = \{ p \in E \mid Tp = p \}$ is the set of fixed points of T.

There are several iteration processes in the literature for which the fixed points of operators have been approximated over the years by various authors. In a complete metric space, the Picard iteration process $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = Tx_n,\ n = 0, 1, \cdots, \tag{1.1}$$

has been employed to approximate the fixed points of mappings satisfying the inequality relation

$$d(Tx, Ty) \leq \alpha d(x, y),\ \forall x, y \in E \text{ and } \alpha \in [0, 1). \tag{1.2}$$

Condition (1.2) is called the *Banach's contraction condition.* Any operator satisfying (1.2) is called *strict contraction.* Also, condition (1.2) is significant in the celebrated Banach's fixed point theorem [3].

In the Banach space setting, we shall state some of the iteration processes generalizing (1.1) as follows:

For $x_0 \in E$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n,\ n = 0, 1, \cdots, \tag{1.3}$$

where $\{\alpha_n\}_{n=0}^{\infty} \subset [0, 1]$, is called the Mann iteration process (see Mann [20]).

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For $x_0 \in E$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$\left.\begin{array}{l} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T z_n \\ z_n = (1-\beta_n)x_n + \beta_n T x_n \end{array}\right\} \; n = 0, 1, \cdots , \tag{1.4}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences in $[0, 1]$, is called the Ishikawa iteration process (see Ishikawa [14]).

The following is the iteration process introduced by Singh et al [39] to establish some stability results: Let S and T be operators on an arbitrary set Y with values in E such that $T(Y) \subseteq S(Y)$. $S(Y)$ is a complete subspace of E. Then, for $x_0 \in Y$, the sequence $\{Sx_n\}_{n=0}^{\infty}$ defined by

$$Sx_{n+1} = (1-\alpha_n)Sx_n + \alpha_n T x_n, \; n = 0, 1, \cdots , \tag{1.5}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in $[0, 1]$ is called the *Jungck-Mann* iteration process.

If $\alpha_n = 1$ and $Y = E$ in (1.5), then we obtain

$$Sx_{n+1} = Tx_n, \; n = 0, 1, 2, \cdots , \tag{1.6}$$

which is the *Jungck iteration*. See Jungck [16] for detail.

While the iteration process (1.5) extends (1.1), (1.3) and (1.6), the iteration processes (1.4) and (1.5) are independent.

Kannan [17] established an extension of the Banach's fixed point theorem by using the following contractive definition: For a selfmap T, there exists $\beta \in (0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq \beta \left[d(x, Tx) + d(y, Ty)\right], \; \forall x, y \in E. \tag{1.7}$$

Chatterjea [8] used the following contractive condition: For a selfmap T, there exists $\gamma \in (0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq \gamma \left[d(x, Ty) + d(y, Tx)\right], \; \forall x, y \in E. \tag{1.8}$$

Zamfirescu [40] established a nice generalization of the Banach's fixed point theorem by combining (1.2), (1.7) and (1.8). That is, for a mapping $T : E \to E$, there exist real numbers α, β, γ satisfying $0 \leq \alpha < 1, \; 0 \leq \beta < \frac{1}{2}, \; 0 \leq \gamma < \frac{1}{2}$ respectively such that for each $x, y \in E$, at least one of the following is true:

$$\left.\begin{array}{ll} (z_1) & d(Tx, Ty) \leq \alpha d(x, y) \\ (z_2) & d(Tx, Ty) \leq \beta \left[d(x, Tx) + d(y, Ty)\right] \\ (z_3) & d(Tx, Ty) \leq \gamma \left[d(x, Ty) + d(y, Tx)\right]. \end{array}\right\} \tag{1.9}$$

The mapping $T : E \to E$ satisfying (1.9) is called the *Zamfirescu contraction*. Any mapping satisfying condition (z_2) of (1.9) is called a *Kannan mapping*, while the mapping satisfying condition (z_3) is called *Chatterjea operator*. The contractive condition (1.9) implies

$$||Tx - Ty|| \leq 2\delta||x - Tx|| + \delta||x - y||, \; \forall x, y \in E, \tag{1.10}$$

where $\delta = \max\left\{\alpha, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\right\}, \; 0 \leq \delta < 1.$

Condition (1.9) was used by Rhoades [31, 32] to obtain some convergence results for Mann and Ishikawa iteration processes in a uniformly convex Banach space. The results of [31, 32] were recently extended by Berinde [6] to an arbitrary Banach space for the same fixed point iteration processes. Rafiq [30] proved a convergence result for the Noor iteration process in normed space using the Zamfirescu contraction. See Noor [2] for the Noor iteration process.

Singh et al [39] defined the following general iteration process:

Let $S,\ T : Y \to E$ and $T(Y) \subseteq S(Y)$. For any $x_0 \in Y$, let

$$Sx_{n+1} = f(T, x_n),\ n = 0, 1, \cdots \tag{1.11}$$

For $S = I$ (i.e. identity map on E), $f(T, x_n) = Tx_{n+1}$ and $Y = E$, then (1.11) reduces to the well-known Picard iteration process in (1.1).

If $Y = E$, and $f(T, x_n) = Tx_n,\ n = 0, 1, \cdots$, then (1.11) reduces to the Jungck iteration process of (1.6). Jungck [16] established that the maps S and T satisfying

$$d(Tx, Ty) \leq kd(Sx, Sy),\ \forall x, y \in E,\ k \in [0, 1), \tag{1.12}$$

have a unique common fixed point in complete metric space E, provided that S and T commute, $T(Y) \subseteq S(Y)$ and S is continuous. For results which are similar to Jungck [16] in uniform space, we refer to Aamril and El Moutawakil [1] as well as Olatinwo [21, 22].

The following definition of the stability of iteration process due to Singh et al [39] shall be required in the sequel.

Definition 1.1. Let $S,T : Y \to E,\ T(Y) \subseteq S(Y)$ and z a coincidence point of S and T, that is, $Sz = Tz = p$ (say). For any $x_0 \in Y$, let the sequence $\{Sx_n\}_{n=0}^{\infty}$, generated by the iteration procedure (1.11) converge to p. Let $\{Sy_n\}_{n=0}^{\infty} \subset E$ be an arbitrary sequence, and set $\epsilon_n = d(Sy_{n+1}, f(T, y_n)),\ n = 0, 1, \cdots$ Then, the iteration procedure (1.11) will be called (S,T)-stable if and only if $\lim\limits_{n\to\infty} \epsilon_n = 0$ implies that $\lim\limits_{n\to\infty} Sy_n = p$.

This definition reduces to that of the stability of iteration procedure due to Harder and Hicks [11] when $Y = E$ and $S = I$ (identity operator).

Several stability results established in metric space and normed linear space are available in the literature. Some of the various authors whose contributions are of colossal value in the study of stability of the fixed point iteration procedures are Ostrowski [29], Harder and Hicks [11], Rhoades [34, 36], Osilike [27], Osilike and Udomene [28], Jachymski [15], Berinde [5, 4] and Singh et al [39]. Harder and Hicks [11], Rhoades [34, 36], Osilike [27] and Singh et al [39] used the method of the summability theory of infinite matrices to prove various stability results for certain contractive definitions. The method has also been adopted to establish various stability results for certain contractive definitions in Olatinwo et al [24, 26]. Osilike and Udomene [28] introduced a shorter method of proof of stability results and this has also been employed by Berinde [5], Imoru and Olatinwo [12], Imoru et al [13], Olatinwo et al [25] and some others. In Harder and Hicks [11], the contractive definition stated in (1.2) was used to prove a stability result for the Kirk's iteration process. The first stability result on $T-$ stable mappings was proved by Ostrowski [29] for the Picard iteration using (1.2).

In addition to (1.2), the contractive condition in (1.9) was also employed by Harder and Hicks [11] to establish some stability results for both Picard and Mann iteration processes. Rhoades [34, 36] extended the stability results of [11] to more general classes of contractive mappings. Rhoades [34] extended the results of [11] to the following independent contractive condition: there exists $c \in [0, 1)$ such that

$$d(Tx, Ty) \leq c \max \{d(x, y), d(x, Ty), d(y, Tx)\}, \ \forall x, y \in E. \tag{1.13}$$

Rhoades [36] used the following contractive definition: there exists $c \in [0, 1)$ such that

$$d(Tx, Ty) \leq c \max \left\{ d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}, d(x, Ty), d(y, Tx) \right\}, \tag{1.14}$$

for all $x, y \in E$. Moreover, Osilike [27] generalized and extended some of the results of Rhoades [36] by using a more general contractive definition than those of Rhoades [34, 36]. Indeed, he employed the following contractive definition: there exist $a \in [0, 1], \ L \geq 0$ such that

$$d(Tx, Ty) \leq Ld(x, Tx) + ad(x, y), \ \forall x, y \in E. \tag{1.15}$$

Osilike and Udomene [28] introduced a shorter method to prove stability results for the various iteration processes using the condition (1.15). Berinde [5] established the same stability results for the same iteration processes using the same set of contractive definitions as in Harder and Hicks [11] but the same method of shorter proof as in Osilike and Udomene [28].

More recently, Imoru and Olatinwo [12] established some stability results which are generalizations of some of the results of Osilike [5, 11, 27, 28, 34, 36]. In Imoru and Olatinwo [12], the following contractive definition was employed: there exist

$a \in [0, 1)$ and a monotone increasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\varphi(0) = 0$, such that

$$d(Tx, Ty) \leq \varphi(d(x, Tx)) + ad(x, y), \ \forall x, y \in E. \tag{1.16}$$

Condition (1.16) was also employed in Olatinwo et al [24] to establish some stability results in normed linear space setting with additional condition of continuity imposed on φ.

However, Singh et al [39] established some stability results for Jungck and Jungck-Mann iteration processes by employing two contractive definitions both of which generalize those of Osilike [27] but independent of that of Imoru and Olatinwo [12]. Singh et al [39] obtained stability results for Jungck and Jungck-Mann iterative procedures in metric space using both the contractive definition

(1.12) and the following: For $S,\ T : Y \to E$ and some $k \in [0,1)$, we have

$$d(Tx,Ty) \leq kd(Sx,Sy) + Ld(Sx,Tx),\ \forall x,y \in Y. \tag{1.17}$$

In the next section, we shall introduce the Jungck-Ishikawa iteration process to prove some stability and convergence results for nonselfmappings in normed linear space and arbitrary Banach space respectively. In establishing our results, more general contractive conditions than (1.9) will be considered.

2. Preliminaries

We shall consider the following iteration processes in establishing our results:

Let $(E, ||.||)$ be a Banach space and Y an arbitrary set. Let $S,\ T : Y \to E$ be two nonselfmappings such that $T(Y) \subseteq S(Y),\ S(Y)$ is a complete subspace of E and S is injective. Then, for $x_0 \in Y$, define the sequence $\{Sx_n\}_{n=0}^{\infty}$ iteratively by

$$\left.\begin{array}{l} Sx_{n+1} = (1-\alpha_n)Sx_n + \alpha_n Tz_n \\ Sz_n = (1-\beta_n)Sx_n + \beta_n Tx_n \end{array}\right\},\ n = 0, 1, \cdots, \tag{2.1}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences in $[0,1]$. The iteration process (2.1) is called the *Jungck-Ishikawa* iteration process. See Olatinwo [23] for more detail.

The iteration processes (1.1) and (1.3)–(1.6) are special cases of (2.1). For instance,

if in (2.1), S is identity operator, $Y = E,\ \beta_n = 0$ then we obtain the Mann iteration process of (1.3).

Since S is injective, if $\beta_n = 0$, then for $x_0 \in Y$, (2.1) reduces to the Jungck-Mann iteration process of (1.5).

Also, with S and T as in (2.1), we define the following three-step iteration process which is an extension of (2.1):

For $x_0 \in Y$ and with S and T as above, define the sequence $\{Sx_n\}_{n=0}^{\infty}$ iteratively by

$$\left.\begin{array}{l} Sx_{n+1} = (1-\alpha_n)Sx_n + \alpha_n Tz_n \\ Sz_n = (1-\beta_n)Sx_n + \beta_n Tr_n \\ Sr_n = (1-\gamma_n)Sx_n + \gamma_n Tx_n \end{array}\right\},\ n = 0, 1, \cdots, \tag{2.2}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are sequences in $[0,1]$. The iteration process (2.2) will be called the *Jungck-Noor* iteration process.

The iteration processes (1.1) and (1.3)–(1.6) are also special cases of (2.2). In fact, the iteration process defined in (2.2) is an extension of that of Noor [2].

Definition 2.1 (Berinde [7])**.** A function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a *comparison function* if it satisfies the following conditions:

(i) ψ is monotone increasing;
(ii) $\lim\limits_{n\to\infty} \psi^n(t) = 0,\ \forall t \geq 0.$

Remark 2.2. Every comparison function satisfies $\psi(0) = 0$. See Rus [37] and Rus et al [38] for the Definition 2.1.

In addition to the iteration process (2.1), we shall employ the following contractive definitions:

Definition 2.3. For two nonselfmappings $S,\ T : Y \to E$ with $T(Y) \subseteq S(Y)$, where $S(Y)$ is a complete subspace of E, there exist:

(a) a real number $L \geq 0$, a sublinear comparison function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ and a monotone increasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\varphi(0) = 0$ and $\forall x, y \in Y$, we have

$$||Tx - Ty|| \leq \frac{\varphi(||Sx - Tx||) + \psi(||Sx - Sy||)}{1 + L||Sx - Tx||}; \tag{2.3}$$

and,

(b) real numbers $k \geq 0,\ L \geq 0,\ a \in [0, 1)$ and a monotone increasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\varphi(0) = 0$ and $\forall x, y \in Y$, we have

$$||Tx - Ty|| \leq \left(\frac{\varphi(||Sx - Tx||) + a||Sx - Sy||}{1 + L||Sx - Tx||} \right) e^{k||Sx - Tx||}. \tag{2.4}$$

In this paper, we shall consider the iteration processes defined in (2.1) and (2.2) to establish some stability results for nonselfmappings in normed linear space as well as obtain some strong convergence results for these nonselfmappings in an arbitrary Banach space by employing the contractive conditions (2.3) and (2.4). Our stability results are generalizations and extensions of those of Singh et al [39], some results of [5, 12, 13, 24, 26, 34, 36], while the convergence results extend, generalize and improve those of [6, 18, 19, 36, 31]. For more on the study of fixed point iteration processes and various contractive conditions, our interested readers can consult Berinde [4], Ciric [9, 10], Rhoades [33] and others in the reference section of this paper.

Definition 2.4. Let X and Y be two nonempty sets and $S,\ T : X \to Y$ two mappings. Then, an element $x^* \in X$ is a *coincidence point* of S and T if and only if $Sx^* = Tx^*$.

Denote the set of the coincidence points of S and F by $C(S, T)$.

There are several papers and monographs on the coincidence point theory. However, we refer our readers to Rus [37] and Rus et al [38] for the Definition 2.4 and some coincidence point results.

We shall require the following lemmas in the sequel.

Lemma 2.5 (Berinde [5, 4, 7])**.** *If* δ *is a real number such that* $0 \leq \delta < 1$, *and* $\{\epsilon_n\}_{n=0}^{\infty}$ *is a sequence of positive numbers such that* $\lim_{n \to \infty} \epsilon_n = 0$, *then for any sequence of positive numbers* $\{u_n\}_{n=0}^{\infty}$ *satisfying*

$$u_{n+1} \leq \delta u_n + \epsilon_n,\ n = 0, 1, \cdots,$$

we have $\lim_{n \to \infty} u_n = 0$.

Lemma 2.6 (Imoru et al [13]). *Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a comparison function and $\{v_n\}_{n=0}^{\infty}$ a sequence of positive numbers such that $\lim_{n\to\infty} v_n = 0$. Then, we have*

$$\lim_{n\to\infty} \sum_{k=0}^{n} \psi^{n-k}(v_k) = 0, \text{ for each } k.$$

Proof. Since ψ is monotone increasing, there exists a convergent series $\sum a_{nk}$ of positive numbers $a_{nk},\ k = 0, 1, 2, \cdots, n$ such that

$$\psi^{n-k}(v_k) = a_{nk} v_k.$$

Therefore,

$$\sum_{k=0}^{n} \psi^{n-k}(v_k) = \sum_{k=0}^{n} a_{nk} v_k.$$

Let A be the lower triangular matrix with entries $a_{nk}, k = 0, 1, \cdots, n$. Clearly, $\lim_{n\to\infty} a_{nk} = 0$, for each k. Since $\sum a_{nk}$ is convergent, let $\lim_{n\to\infty} \sum a_{nk} = s < \infty$. Therefore, A is multiplicative (See Rhoades [36]). Then,

$$\begin{aligned}\lim_{n\to\infty} \sum_{k=0}^{n} \psi^{n-k}(v_k) &= \lim_{n\to\infty} \sum_{k=0}^{n} a_{nk} v_k = \lim_{n\to\infty} \sum_{k=0}^{n} a_{nk} . \lim_{n\to\infty} v_n \\ &= \lim_{n\to\infty} \sum a_{nk} \lim_{n\to\infty} v_n = 0.\end{aligned}$$

□

Lemma 2.7 (Imoru et al [13]). *Let $\{\psi^k(t)\}_{k=0}^{n}$ be a sequence of comparison functions. Then, any linear combination $\sum_{j=0}^{n} c_j \psi^j(t)$ of the comparison functions is also a comparison function, where $\sum_{j=0}^{n} c_j = 1$ and $c_o, c_1, \cdots, c_n$ are positive constants.*

Proof. Let $\bar{\psi}(t) = c_0\psi^0(t) + c_1\psi(t) + c_2\psi^2(t) + \cdots + c_n\psi^n(t)$. Since each $\psi^k(t)$, $k = 0, 1, , \cdots, n$ is a comparison function, then each $\psi^k(t),\ k = 0, 1, \cdots, n$ is monotone increasing. Also, since each $c_k > 0,\ k = 0, 1, \cdots, n$, then each $c_k\psi^k(t)$ is monotone increasing, from which it follows that $\bar{\psi}(t)$ is monotone increasing. Moreover, since $\psi^k(t) \to 0,\ \forall t \geq 0,\ k = 0, 1, \cdots, n$ then $c_k\psi^k(t) \to 0,\ \forall k = 0, 1, \cdots, n$. Therefore, $\bar{\psi}(t) \to 0, \forall\, t \geq 0$. Hence, $\bar{\psi}(t)$ is a comparison function. □

Lemma 2.8 (Imoru et al [13]). *If $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a subadditive comparison function and $\{\epsilon_n\}_{n=0}^{\infty}$ is a sequence of positive numbers such that $\lim_{n\to\infty} \epsilon_n = 0$, then for any sequence of positive numbers $\{u_n\}_{n=0}^{\infty}$ satisfying*

$$u_{n+1} \leq \sum_{k=0}^{m} \delta_k \psi^k(u_n) + \epsilon_n,\ n = 0, 1, \cdots, \tag{2.5}$$

where $\delta_0, \delta_1, \cdots, \delta_m \in [0, 1]$ with $0 \leq \sum_{k=0}^{m} \delta_k \leq 1$, and we have $\lim_{n\to\infty} u_n = 0$.

Proof. Let $\bar{\psi}(u_n) = \sum_{k=0}^{m} \delta_k \psi^k(u_n)$. Then, inequality (2.5) becomes

$$u_{n+1} \leq \bar{\psi}(u_n) + \epsilon_n,\ n = 0, 1, 2, \cdots, \tag{2.6}$$

Using Lemma 2.7, then we have that $\bar{\psi}(u_n)$ is a comparison function. Also, using (2.6) yields

$$\begin{aligned}
&u_1 \leq \bar{\psi}(u_0) + \epsilon_0\\
&u_2 \leq \bar{\psi}(u_1) + \epsilon_1 \leq \bar{\psi}(\bar{\psi}(u_0) + \epsilon_0) + \epsilon_1 = \bar{\psi}^2(u_0) + \bar{\psi}(\epsilon_0) + \epsilon_1\\
&u_3 \leq \bar{\psi}(u_2) + \epsilon_2 \leq \bar{\psi}^3(u_0) + \bar{\psi}^2(\epsilon_0) + \bar{\psi}(\epsilon_1) + \epsilon_2,
\end{aligned}$$

and in general, we have

$$u_n \leq \bar{\psi}^n(u_0) + \bar{\psi}^{n-1}(\epsilon_0) + \bar{\psi}^{n-2}(\epsilon_1) + \cdots + \bar{\psi}(\epsilon_{n-2}) + \epsilon_{n-1} \tag{2.7}$$

Replacing n by $(n+1)$ in (2.7) yields

$$\begin{aligned}
u_{n+1} &\leq \bar{\psi}^{n+1}(u_0) + \bar{\psi}^n(\epsilon_0) + \bar{\psi}^{n-1}(\epsilon_1) + \cdots + \bar{\psi}(\epsilon_{n-1}) + \epsilon_n\\
&= \bar{\psi}^{n+1}(u_0) + \sum_{k=0}^{n} \bar{\psi}^{n-k}(\epsilon_k). \qquad (2.8)
\end{aligned}$$

Since $\bar{\psi}$ is a comparison function, then $\bar{\psi}^{n+1}(u_0) \to 0$ as $n \to \infty$. Using Lemma 2.6, then we have that

$$\lim_{n\to\infty} \sum_{k=0}^{n} \bar{\psi}^{n-k}(\epsilon_k) = 0 \text{ for each } k.$$

Thus, inequality (2.8) yields $\lim_{n\to\infty} u_n = 0$. □

We establish our main results in the next two sections. Our stability results are established by using the method of Berinde [5] and Osilike and Udomene [28]. Section 3 deals with some stability results in normed linear space, while a strong convergence result is proved in section 4.

3. Some Stability Results in Normed Linear Space

Theorem 3.1. *Let $(E, ||.||)$ be a normed space and Y an arbitrary set. Suppose that $S,\ T : Y \to E$ are nonselfoperators such that $T(Y) \subseteq S(Y)$, $S(Y)$ a complete subspace of E, and S is an injective operator. Let z be a coincidence point of S and T (that is, $Sz = Tz = p$). Suppose that S and T satisfy condition (2.3). Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous sublinear comparison function and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ a monotone increasing function such that $\varphi(0) = 0$. For $x_0 \in Y$, let $\{Sx_n\}_{n=0}^{\infty}$ be the Jungck-Ishikawa iteration process defined by (2.1) converging to p, where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences in $[0, 1]$. Then, the Jungck-Ishikawa iteration process is (S, T)-stable.*

Proof. Suppose that $\{Sy_n\}_{n=0}^{\infty} \subset E$, $\epsilon_n = ||Sy_{n+1} - (1 - \alpha_n)Sy_n - \alpha_n Tb_n||$, $n = 0, 1, \ldots$, where $Sb_n = (1 - \beta_n)Sy_n + \beta_n Ty_n$ and let $\lim_{n\to\infty} \epsilon_n = 0$. Then, we shall establish that $\lim_{n\to\infty} Sy_n = p$, using the contractive condition and the triangle inequality:

$$\begin{aligned}
||Sy_{n+1} - p|| &\leq \epsilon_n + ||(1-\alpha_n)(Sy_n - p) + \alpha_n(Tb_n - p)|| \\
&\leq \epsilon_n + (1-\alpha_n)||Sy_n - p|| + \alpha_n||Tz - Tb_n|| \\
&\leq \epsilon_n + (1-\alpha_n)||Sy_n - p|| \\
&\qquad + \alpha_n\left[\frac{\varphi(||Sz - Tz||) + \psi(||Sz - Sb_n||)}{1 + M||Sz - Tz||}\right] \\
&= (1-\alpha_n)||Sy_n - p|| + \alpha_n\psi(||p - Sb_n||) + \epsilon_n \\
&\leq (1-\alpha_n)||Sy_n - p|| + \alpha_n[(1-\beta_n)\psi(||p - Sy_n||) \\
&\qquad + \beta_n\psi(||Tz - Ty_n||)] + \epsilon_n \\
&\leq (1-\alpha_n)||Sy_n - p|| + \alpha_n(1-\beta_n)\psi(||Sy_n - p||) \qquad (3.1) \\
&\qquad + \alpha_n\beta_n\psi^2(||p - Sy_n||) + \epsilon_n.
\end{aligned}$$

Using Lemma 2.8 in (3.1) yields $\lim_{n\to\infty} ||Sy_n - p|| = 0$, that is, $\lim_{n\to\infty} Sy_n = p$.

Conversely, let $\lim_{n\to\infty} Sy_n = p$. Then, by using the triangle inequality and the contractive definition, we have the following:

$$\begin{aligned}
\epsilon_n &= ||Sy_{n+1} - (1-\alpha_n)Sy_n - \alpha_n Tb_n|| \\
&\leq ||Sy_{n+1} - p|| + (1-\alpha_n)||Sy_n - p|| + \alpha_n||Tz - Tb_n|| \\
&\leq ||Sy_{n+1} - p|| + (1-\alpha_n)||Sy_n - p|| + \alpha_n(1-\beta_n)\psi(||p - Sy_n||) \\
&\qquad + \alpha_n\beta_n\psi(||Tz - Ty_n||) \\
&\leq ||Sy_{n+1} - p|| + (1-\alpha_n)||Sy_n - p|| + \alpha_n(1-\beta_n)\psi(||Sy_n - p||) \\
&\qquad + \alpha_n\beta_n\psi^2(||Sy_n - p|| \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

□

Theorem 3.2. *Let $(E, ||.||)$ be a normed space and Y an arbitrary set. Suppose that S, $T : Y \to E$ are nonselfoperators such that $T(Y) \subseteq S(Y)$, $S(Y)$ a complete subspace of E, and S is an injective operator. Let z be a coincidence point of S and T (that is, $Sz = Tz = p$). Suppose that S and T satisfy condition (2.4). Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a monotone increasing function such that $\varphi(0) = 0$. For $x_0 \in Y$, let $\{Sx_n\}_{n=0}^{\infty}$ be the Jungck-Noor iteration process defined by (2.2) converging to p, where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are sequences in $[0, 1]$ such that $0 < \alpha \leq \alpha_n$, $0 < \beta \leq \beta_n$ and $0 < \gamma \leq \gamma_n$, $(n = 0, 1, \cdots)$. Then, the Jungck-Noor iteration process is (S, T)-stable.*

Proof. Suppose that $\{Sy_n\}_{n=0}^{\infty} \subset E$, $\epsilon_n = ||Sy_{n+1} - (1-\alpha_n)Sy_n - \alpha_n Tb_n||$, $n = 0, 1, \ldots$, where $Sb_n = (1-\beta_n)Sy_n + \beta_n Tc_n$, $Sc_n = (1-\gamma_n)Sy_n + \gamma_n Ty_n$ and let $\lim_{n\to\infty} \epsilon_n = 0$. Then, we shall establish that $\lim_{n\to\infty} Sy_n = p$, using the contractive condition (2.4) and the triangle inequality:

$$
\begin{aligned}
||Sy_{n+1} - p|| &\leq (1-\alpha_n)||Sy_n - p|| + \alpha_n||Tz - Tb_n|| + \epsilon_n \\
&\leq (1-\alpha_n)||Sy_n - p|| + a\alpha_n||p - Sb_n|| + \epsilon_n \\
&\leq (1-\alpha_n)||Sy_n - p|| + a\alpha_n[(1-\beta_n)||Sy_n - p|| \\
&\qquad + \beta_n||Tz - Tc_n|| + \epsilon_n \\
&= [1-(1-a)\alpha_n - a\alpha_n\beta_n]||Sy_n - p|| + a\alpha_n\beta_n||Tz - Tc_n||] + \epsilon_n \\
&\leq [1-(1-a)\alpha_n - a\alpha_n\beta_n]||Sy_n - p|| + a^2\alpha_n\beta_n||Sz - Sc_n|| + \epsilon_n \\
&= [1-(1-a)\alpha_n - (1-a)a\alpha_n\beta_n \\
&\qquad -(1-a)a^2\alpha_n\beta_n\gamma_n]||Sy_n - p|| + \epsilon_n \\
&\leq [1-(1-a)\alpha - (1-a)a\alpha\beta - (1-a)a^2\alpha\beta\gamma]||Sy_n - p|| + \epsilon_n. \qquad (3.2)
\end{aligned}
$$

Since

$$0 \leq 1-(1-a)\alpha - (1-a)a\alpha\beta - (1-a)a^2\alpha\beta\gamma < 1 \text{ and } \lim_{n\to\infty}\epsilon_n = 0,$$

using Lemma 2.5 in (3.2) yields $\lim_{n\to\infty}||Sx_n - p|| = 0$, that is, $\lim_{n\to\infty} Sx_n = p$.

Conversely, let $\lim_{n\to\infty} Sx_{n+1} = p$. Then, by using the triangle inequality and the contractive definition, we have the following:

$$
\begin{aligned}
\epsilon_n &\leq ||Sy_{n+1} - p|| + (1-\alpha_n)||Sy_n - p|| + \alpha_n||Tz - Tb_n|| \\
&\leq ||Sy_{n+1} - p|| + (1-\alpha_n)||Sy_n - p|| + a\alpha_n||Sz - Sb_n|| \\
&\leq ||Sy_{n+1} - p|| + [1-(1-a)\alpha_n - a\alpha_n\beta_n]||Sy_n - p|| + a\alpha_n\beta_n||Tz - Tc_n|| \\
&\leq ||Sy_{n+1} - p|| + [1-(1-a)\alpha_n - a\alpha_n\beta_n]||Sy_n - p|| + a^2\alpha_n\beta_n||Sz - Sc_n|| \\
&\leq ||Sy_{n+1} - p|| + [1-(1-a)\alpha_n - a\alpha_n\beta_n]||Sy_n - p|| \\
&\quad + a^2\alpha_n\beta_n[(1-\gamma_n)||p - Sy_n|| + \gamma_n||p - Ty_n||] \\
&\leq ||Sy_{n+1} - p|| + [1-(1-a)\alpha - (1-a)a\alpha\beta \\
&\quad - (1-a)a^2\alpha\beta\gamma]||Sy_n - p|| \to 0 \text{ as } n \to \infty.
\end{aligned}
$$

□

Remark 3.3. Both Theorem 3.1 and Theorem 3.2 are generalizations and extensions of Theorem 3.5 of Singh et al [39], Theorem 3 of Berinde [5], Theorem 2 of Osilike [27], Theorem 2 and Theorem 5 of Osilike and Udomene [28], Theorem 2 of Rhoades [34], Theorem 30 of Rhoades [35], Theorem 2 of Rhoades [36], Theorem 3 of Harder and Hicks [11] as well as some of the results of the author [12, 13, 24, 26, 25]. Our stability results also extend some similar ones in Berinde [7] and Olatinwo [23].

4. Some Convergence Results in Arbitrary Banach Space

Theorem 4.1. *Let $(E, ||.||)$ be an arbitrary Banach space and Y is an arbitrary set. Suppose that S, $T : Y \to E$ are nonselfoperators such that $T(Y) \subseteq S(Y)$, $S(Y)$ a complete subspace of E, and S is an injective operator. Let z*

be a coincidence point of S and T (that is, $Sz = Tz = p$). Suppose that S and T satisfy condition (2.4). Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a monotone increasing function such that $\varphi(0) = 0$. For $x_0 \in Y$, let $\{Sx_n\}_{n=0}^{\infty}$ be the Jungck-Noor iteration process defined by (2.2), where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are sequences in $[0, 1]$ such that $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then, $\{Sx_n\}_{n=0}^{\infty}$ converges strongly to p.

Proof. Let $C(S, T)$ be the set of the coincidence points of S and T. We shall now use condition (2.4) to establish that S and T have a unique coincidence point z (i.e. $Sz = Tz = p$ (say)): Injectivity of S is sufficient.

Suppose that there exist $z_1,\ z_2 \in C(S, T)$ such that $Sz_1 = Tz_1 = p_1$ and $Sz_2 = Tz_2 = p_2$.

If $p_1 = p_2$, then $Sz_1 = Sz_2$ and since S is injective, it follows that $z_1 = z_2$.

If $p_1 \neq p_2$, then we have by the contractiveness condition (2.4) for S and T that

$$\begin{aligned} 0 &< ||p_1 - p_2|| = ||Tz_1 - Tz_2|| \\ &\leq \left(\frac{\varphi(||Sz_1 - Tz_1||) + a||Sz_1 - Sz_2||}{1 + L||Sz_1 - Tz_1||} \right) e^{k||Sz_1 - Tz_1||} \\ &\leq a||p_1 - p_2||, \end{aligned}$$

which leads to $(1 - a)||p_1 - p_2|| \leq 0$, from which it follows that $1 - a > 0$ since $a \in [0, 1)$, but $||p_1 - p_2|| \leq 0$, which is a contradiction since norm is nonnegative.

Therefore, we have that $||p_1 - p_2|| = 0$, that is, $p_1 = p_2 = p$. Since $p_1 = p_2$, then we have that $p_1 = Sz_1 = Tz_1 = Sz_2 = Tz_2 = p_2$, leading to $Sz_1 = Sz_2 \Rightarrow z_1 = z_2 = z$ (since S is injective).

Hence, $z \in C(S, T)$, that is, z is a unique coincidence point of S and T.

Now, we prove that $\{Sx_n\}_{n=0}^{\infty}$ converges strongly to p (where $Sz = Tz = p$) using again, condition (2.4). Therefore, we have

$$\begin{aligned} ||Sx_{n+1} - p|| &\leq (1 - \alpha_n)||Sx_n - p|| + \alpha_n||Tz - Tz_n|| \\ &\leq (1 - \alpha_n)||Sx_n - p|| + a\alpha_n||p - Sz_n||. \end{aligned} \tag{4.1}$$

Now, we have that

$$\begin{aligned} ||p - Sz_n|| &= ||(1 - \beta_n)(p - Sx_n) + \beta_n(p - Ty_n)|| \\ &\leq (1 - \beta_n)||Sx_n - p|| + a\beta_n||p - Sy_n||. \end{aligned} \tag{4.2}$$

Using (4.2) in (4.1) yields

$$||Sx_{n+1} - p|| \leq [1 - (1 - a)\alpha_n - a\alpha_n\beta_n]||Sx_n - p|| + a^2\alpha_n\beta_n||p - Sy_n||. \tag{4.3}$$

Furthermore, we have

$$\begin{aligned} |p - Sy_n|| &\leq (1 - \gamma_n)||Sx_n - p|| + \gamma_n||p - Tx_n|| \\ &\leq (1 - \gamma_n + a\gamma_n)||p - Sx_n||. \end{aligned} \tag{4.4}$$

Using (4.4) in (4.3) yields

$$
\begin{aligned}
||Sx_{n+1}-p|| &\leq [1-(1-a)\alpha_n-(1-a)a\alpha_n\beta_n-(1-a)a^2\alpha_n\beta_n\gamma_n]||Sx_n-p|| \\
&\leq [1-(1-a)\alpha_n]||Sx_n-p|| \\
&\leq \Pi_{j=0}^{n}[1-(1-a)\alpha_j]||Sx_0-p|| \\
&\leq e^{-(1-a)\sum_{j=0}^{n}\alpha_j}||Sx_0-p|| \to 0 \quad \text{as } n\to\infty. \qquad (4.5)
\end{aligned}
$$

Therefore, we obtain from (4.5) that $\lim\limits_{n\to\infty}||Sx_{n+1}-p|| = 0$, i.e. $\{Sx_n\}_{n=0}^{\infty}$ converges strongly to p. □

Remark 4.2. Theorem 4.1 is a generalization and extension of a multitude of results. In particular, Theorem 4.1 is a generalization and extension of both Theorem 1 and Theorem 2 of Berinde [6], Theorem 3 of Rafiq [30], Theorem 2 and Theorem 3 of Kannan [18], Theorem 3 of Kannan [19], Theorem 4 of Rhoades [31] as well as Theorem 8 of Rhoades [32]. Also, both Theorem 4 of Rhoades [31] and Theorem 8 of Rhoades [32] are Theorem 4.10 and Theorem 5.6 of Berinde [4] respectively. Our result also extends some similar ones in Berinde [7] and Olatinwo [23].

Remark 4.3. We have considered two new iteration processes to prove some unifying theorems for stability and convergence. These new iteration processes as well as the results obtained extend the frontiers of knowledge in the fixed point theory.

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