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LEGENDRIAN WARPED PRODUCT SUBMANIFOLDS IN GENERALIZED SASAKIAN SPACE FORMS

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Abstract. Recently, K. Matsumoto and I. Mihai established a sharp inequality for warped products isometrically immersed in Sasakian space forms. As applications, they obtained obstructions to minimal isometric immersions of warped products into Sasakian space forms.

P. Alegre, D.E. Blair and A. Carriazo have introduced the notion of generalized Sasakian space form.

In the present paper, we obtain a sharp inequality for warped products isometrically immersed in generalized Sasakian space forms. Some applications are derived.

1. Introduction

Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds and f a positive differentiable function on M_1. The warped product of M_1 and M_2 is the Riemannian manifold

$$M_1 \times_f M_2 = (M_1 \times M_2, g),$$

where $g = g_1 + f^2 g_2$. The function f is called the warping function (see, for instance, [5]).

It is well-known that the notion of warped products plays some important role in Differential Geometry as well as in Physics. For a recent survey on warped products as Riemannian submanifolds, see [4].

Let $x\colon M_1 \times_f M_2 \to \widetilde{M}(c)$ be an isometric immersion of a warped product $M_1 \times_f M_2$ into a Riemannian manifold $\widetilde{M}(c)$ with constant sectional curvature c. We denote by h the second fundamental form of x and by $H_i = \frac{1}{n_i} trace h_i$ the partial mean curvatures, where $trace h_i$ is the trace of h restricted to M_i and $n_i = \dim M_i$ $(i = 1, 2)$.

The immersion x is said to be mixed totally geodesic if $h(X, Z) = 0$, for any vector fields X and Z tangent to M_1 and M_2 respectively.

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If $M_1 \times_\rho M_2$ is a warped product of two Riemannian manifolds and $\phi_i \colon N_i \to M_i$, $i = 1, 2$, are isometric immersions from Riemannian manifolds N_1, N_2 into Riemannian manifolds M_1, M_2 respectively. Define a positive function σ on N_1 by $\sigma = \rho \circ \phi_1$. Then the map

$$\phi \colon N_1 \times_\sigma N_2 \to M_1 \times_\rho M_2$$

given by $\phi(x_1, x_2) = (\phi_1(x_1), \phi_2(x_2))$ is an isometric immersion, which is called a warped product immersion [7].

In [6], K. Matsumoto and I. Mihai established the following sharp relationship between the warping function f of a warped product C-totally real isometrically immersed in a Sasakian space form $\widetilde{M}(c)$ and the squared mean curvature $||H||^2$.

Theorem 1. *Let x be a C-totally real isometric immersion of a n-dimensional warped product $M_1 \times_f M_2$ into a $(2m+1)$-dimensional Sasakian space form $\widetilde{M}(c)$. Then:*

$$\frac{\Delta f}{f} \le \frac{n^2}{4n_2}||H||^2 + n_1 \frac{c+3}{4}, \tag{1.1}$$

where $n_i = \dim M_i$, $i = 1, 2$, and Δ is the Laplacian operator of M_1. Moreover, the equality case of (1.1) *holds if and only if x is a mixed totally geodesic immersion and $n_1 H_1 = n_2 H_2$, where H_i, $i = 1, 2$, are the partial mean curvature vectors.*

The authors also gave an example of a submanifold satisfying the equality case of (1.1).

2. Preliminaries

In this section, we recall some definitions and basic formulas which we will use later.

A $(2m+1)$-dimensional Riemannian manifold $\left(\widetilde{M}, g\right)$ is said to be an almost contact metric manifold if there exist on $\widetilde{M}$ a $(1,1)$ tensor field ϕ, a vector field ξ (called the structure vector field) and a 1-form η such that $\eta(\xi) = 1$, $\phi^2(X) = -X + \eta(X)\xi$ and $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$, for any vector fields X, Y on $\widetilde{M}$. In particular, on an almost contact metric manifold we also have $\phi\xi = 0$ and $\eta \circ \phi = 0$.

We denote an almost contact metric manifold by $\left(\widetilde{M}, \phi, \xi, \eta, g\right)$.

A generalized Sasakian space form is an almost contact metric manifold $\left(\widetilde{M},\phi,\xi,\eta,g\right)$ whose curvature tensor is given by (see [1])

$$\begin{aligned}\widetilde{R}(X,Y)Z = {} & f_1\{g(Y,Z)X - g(X,Z)Y\} \\ & + f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ & + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi \\ & - g(Y,Z)\eta(X)\xi\},\end{aligned} \tag{2.1}$$

where f_1, f_2, f_3 are differentiable functions on $\widetilde{M}$. In such a case, we will write $\widetilde{M}(f_1,f_2,f_3)$.

This kind of manifold appears as a natural generalization of the well-known Sasakian space forms $\widetilde{M}(c)$, which can be obtained as particular cases of generalized Sasakian space forms, by taking $f_1=\frac{c+3}{4}$ and $f_2=f_3=\frac{c-1}{4}$.

For recent surveys on generalized Sasakian space forms see [1], [2].

Let M be a n-dimensional submanifold in a generalized Sasakian space form $\widetilde{M}(f_1,f_2,f_3)$.

We denote by $K(\pi)$ the sectional curvature of M associated with a plane section $\pi\subset T_pM$, $p\in M$, and ∇ the Riemannian connection of M, respectively. Also, let h be the second fundamental form and R the Riemann curvature tensor of M.

Then the equation of Gauss is given by

$$\begin{aligned}\widetilde{R}(X,Y,Z,W) = {} & R(X,Y,Z,W) \\ & + g(h(X,W),h(Y,Z)) - g(h(X,Z),h(Y,W)),\end{aligned} \tag{2.2}$$

for any vectors X, Y, Z, W tangent to M.

Let $p\in M$ and $\{e_1,\dots,e_n,\dots,e_{2m+1}\}$ an orthonormal basis of the tangent space $T_p\widetilde{M}(f_1,f_2,f_3)$, such that $e_1,\dots,e_n$ are tangent to M at p. We denote by H the mean curvature vector, that is

$$H(p)=\frac{1}{n}\sum_{i=1}^{n}h(e_i,e_i). \tag{2.3}$$

As is known, M is said to be minimal if H vanishes identically.

Also, we set

$$h_{ij}^r=g(h(e_i,e_j),e_r),\ i,j\in\{1,\dots,n\},\ r\in\{n+1,\dots,2m+1\} \tag{2.4}$$

the coefficients of the second fundamental form h with respect to

$$\{e_1,\dots,e_n,e_{n+1},\dots,e_{2m+1}\},$$

and

$$||h||^2=\sum_{i,j=1}^{n}g(h(e_i,e_j),h(e_i,e_j)). \tag{2.5}$$

Let M be a Riemannian n-manifold and $\{e_1, \ldots, e_n\}$ be an orthonormal basis of M. For a differentiable function f on M, the Laplacian Δf of f is defined by

$$\Delta f = \sum_{j=1}^{n} \{ \left(\nabla_{e_j} e_j \right) f - e_j e_j f \}. \tag{2.6}$$

We recall the following result of Chen for later use.

Lemma 2 ([3]). *Let $n \geq 2$ and $a_1, a_2, \ldots, a_n, b$ real numbers such that*

$$\left(\sum_{i=1}^{n} a_i \right)^2 = (n-1) \left(\sum_{i=1}^{n} a_i^2 + b \right).$$

Then $2a_1a_2 \geq b$, with equality holding if and only if

$$a_1 + a_2 = a_3 = \ldots = a_n.$$

3. Legendrian warped product submanifolds

K. Matsumoto and I. Mihai established a sharp relationship between the warping function f of a warped product C-totally real isometrically immersed in a Sasakian space form $\widetilde{M}(c)$ and the squared mean curvature $||H||^2$ (see [6]). We prove a similar inequality for Legendrian warped product submanifolds of a generalized Sasakian space form.

In this section, we investigate Legendrian warped product submanifolds in a generalized Sasakian space form $\widetilde{M}(f_1, f_2, f_3)$.

A submanifold M normal to ξ in a generalized Sasakian space form $\widetilde{M}(f_1, f_2, f_3)$ is said to be an anti-invariant submanifold if ϕ maps any tangent space of M into the normal space, that is $\phi(T_pM) \subset T_p^{\perp}M$, for every $p \in M$.

If the dimension of an anti-invariant submanifold M is maximum, then M is called a Legendrian submanifold.

If the dimension of the ambient space $\widetilde{M}(f_1, f_2, f_3)$ is $2n+1$, then any Legendrian submanifold is n-dimensional.

Theorem 3. *Let x be a Legendrian isometric immersion of a n-dimensional warped product $M_1 \times_f M_2$ into a $(2n+1)$-dimensional generalized Sasakian space form $\widetilde{M}(f_1, f_2, f_3)$. Then:*

$$\frac{\Delta f}{f} \leq \frac{n^2}{4n_2} ||H||^2 + n_1 f_1, \tag{3.1}$$

where $n_i = \dim M_i$, $i = 1, 2$, and Δ is the Laplacian operator of M_1. Moreover, the equality case of (3.1) *holds if and only if x is a mixed totally geodesic immersion and $n_1H_1 = n_2H_2$, where H_i, $i = 1, 2$, are the partial mean curvature vectors.*

Proof. Let $M_1 \times_f M_2$ be a Legendrian warped product submanifold into a generalized Sasakian space form $\widetilde{M}(f_1, f_2, f_3)$.

Since $M_1 \times_f M_2$ is a warped product, from the formula of the Riemannian connection ∇ it follows that

$$\nabla_X Z = \nabla_Z X = \frac{1}{f}(Xf)\, Z, \tag{3.2}$$

for any vector fields X, Z tangent to M_1, M_2, respectively.

Then, if X and Z are unit vector fields, the sectional curvature $K(X \wedge Z)$ of the plane section spanned by X and Z is given by

$$K(X \wedge Z) = g(\nabla_Z \nabla_X X - \nabla_X \nabla_Z X, Z) = \frac{1}{f}\{(\nabla_X X) f - X^2 f\}. \tag{3.2}$$

We choose a local orthonormal frame $\{e_1, \ldots, e_{n_1}, e_{n_1+1}, \ldots, e_n\}$ such that $e_1, \ldots, e_n$ are tangent to M_1, $e_{n+1}, \ldots, e_{2n+1}$ are tangent to M_2, e_{n+1} is parallel to the mean curvature vector H and $e_{2n+1} = \xi$.

Then, using (3.2), we get

$$\frac{\Delta f}{f} = \sum_{j=1}^{n_1} K(e_j \wedge e_s), \tag{3.3}$$

for each $s \in \{n_1 + 1, \ldots, 2n\}$.

From the equation of Gauss, we have

$$n^2 ||H||^2 = 2\tau + ||h||^2 - n(n-1) f_1, \tag{3.4}$$

where τ denotes the scalar curvature of $M_1 \times_f M_2$, that is,

$$\tau = \sum_{1 \le i < j \le n} K(e_i \wedge e_j).$$

We set

$$\delta = 2\tau - n(n-1) f_1 - \frac{n^2}{2} ||H||^2. \tag{3.5}$$

Then, (3.4) can be written as

$$n^2 ||H||^2 = 2\left(\delta + ||h||^2\right). \tag{3.6}$$

With respect to the above orthonormal frame, (3.6) takes the following form:

$$\left(\sum_{i=1}^{n} h_{ii}^{n+1}\right)^2 = 2\left[\delta + \sum_{i=1}^{n} \left(h_{ii}^{n+1}\right)^2 + \sum_{i \ne j} \left(h_{ij}^{n+1}\right)^2 + \sum_{r=n+2}^{2n} \sum_{i,j=1}^{n} \left(h_{ij}^{r}\right)^2\right].$$

If we put $a_1 = h_{11}^{n+1}$, $a_2 = \sum_{i=2}^{n_1} h_{ii}^{n+1}$ and $a_3 = \sum_{t=n_1+1}^{n} h_{tt}^{n+1}$, the above equation becomes

$$\left(\sum_{i=1}^{3} a_i\right)^2 = 2[\delta + \sum_{i=1}^{3} a_i^2 + \sum_{1\le i\ne j\le n} \left(h_{ij}^{n+1}\right)^2 + \sum_{r=n+2}^{2n}\sum_{i,j=1}^{n} \left(h_{ij}^{r}\right)^2 - \sum_{2\le j\ne k\le n_1} h_{jj}^{n+1}h_{kk}^{n+1} - \sum_{n_1+1\le s\ne t\le n} h_{ss}^{n+1}h_{tt}^{n+1}].$$

Thus a_1, a_2, a_3 satisfy the Lemma of Chen (for $n = 3$), i.e.,

$$\left(\sum_{i=1}^{3} a_i\right)^2 = 2\left(b + \sum_{i=1}^{3} a_i^2\right),$$

with

$$b = \delta + \sum_{1\le i\ne j\le n} \left(h_{ij}^{n+1}\right)^2 + \sum_{r=n+2}^{2n}\sum_{i,j=1}^{n} \left(h_{ij}^{r}\right)^2 - \sum_{2\le j\ne k\le n_1} h_{jj}^{n+1}h_{kk}^{n+1} - \sum_{n_1+1\le s\ne t\le n} h_{ss}^{n+1}h_{tt}^{n+1}.$$

Then $2a_1a_2 \ge b$, with equality holding if and only if $a_1 + a_2 = a_3$.

In the case under consideration, this means

$$\sum_{1\le j<k\le n_1} h_{jj}^{n+1}h_{kk}^{n+1} + \sum_{n_1+1\le s<t\le n} h_{ss}^{n+1}h_{tt}^{n+1} \ge \frac{\delta}{2} + \sum_{1\le\alpha<\beta\le n} \left(h_{\alpha\beta}^{n+1}\right)^2 + \frac{1}{2}\sum_{r=n+2}^{2n}\sum_{\alpha,\beta=1}^{n} \left(h_{\alpha\beta}^{r}\right)^2. \tag{3.7}$$

Equality holds if and only if

$$\sum_{i=1}^{n_1} h_{ii}^{n+1} = \sum_{t=n_1+1}^{n} h_{tt}^{n+1}. \tag{3.8}$$

Using again the Gauss equation, we have

$$n_2\frac{\Delta f}{f} = \tau - \sum_{1\le j<k\le n_1} K\left(e_j \wedge e_k\right) - \sum_{n_1+1\le s<t\le n} K\left(e_s \wedge e_t\right)$$

$$= \tau - \frac{n_1\left(n_1-1\right)f_1}{2} - \sum_{r=n+1}^{2n}\sum_{1\le j<k\le n_1} \left(h_{jj}^{r}h_{kk}^{r} - \left(h_{jk}^{r}\right)^2\right) \tag{3.9}$$

$$- \frac{n_2\left(n_2-1\right)f_1}{2} - \sum_{r=n+1}^{2n}\sum_{n_1+1\le s<t\le n} \left(h_{ss}^{r}h_{tt}^{r} - \left(h_{st}^{r}\right)^2\right).$$

Combining (3.7) and (3.9) and taking account of (3.3), we obtain

$$
\begin{aligned}
n_2\frac{\Delta f}{f} &\leq \tau - \frac{n(n-1)f_1}{2} + n_1 n_2 f_1 - \frac{\delta}{2} \\
&\quad - \sum_{\substack{1\leq j\leq n_1 \\ n_1+1\leq t\leq n}} \left(h_{jt}^{n+1}\right)^2 - \frac{1}{2}\sum_{r=n+2}^{2n}\sum_{\alpha,\beta=1}^{n}\left(h_{\alpha\beta}^r\right)^2 \\
&\quad + \sum_{r=n+2}^{2n}\sum_{1\leq j<k\leq n_1}\left(\left(h_{jk}^r\right)^2 - h_{jj}^r h_{kk}^r\right) \\
&\quad + \sum_{r=n+2}^{2n}\sum_{n_1+1\leq s<t\leq n}\left((h_{st}^r)^2 - h_{ss}^r h_{tt}^r\right) \qquad (3.10) \\
&= \tau - \frac{n(n-1)f_1}{2} + n_1 n_2 f_1 - \frac{\delta}{2} - \sum_{r=n+1}^{2n}\sum_{j=1}^{n_1}\sum_{t=n_1+1}^{n}\left(h_{jt}^r\right)^2 \\
&\quad - \frac{1}{2}\sum_{r=n+2}^{2n}\left(\sum_{j=1}^{n_1} h_{jj}^r\right)^2 - \frac{1}{2}\sum_{r=n+2}^{2n}\left(\sum_{t=n_1+1}^{n} h_{tt}^r\right)^2 \\
&\leq \tau - \frac{n(n-1)f_1}{2} + n_1 n_2 f_1 - \frac{\delta}{2} = \frac{n^2}{4}||H||^2 + n_1 n_2 f_1,
\end{aligned}
$$

which implies the inequality (3.1).

We see that the equality sign of (3.10) holds if and only if

$$h_{jt}^r = 0,\ 1\leq j\leq n_1,\ n_1+1\leq t\leq n,\ n+1\leq r\leq 2n, \tag{3.11}$$

and

$$\sum_{i=1}^{n_1} h_{ii}^r = \sum_{t=n_1+1}^{n} h_{tt}^r = 0,\ n+2\leq r\leq 2n. \tag{3.12}$$

Obviously (3.12) is equivalent to the mixed totally geodesicness of the warped product $M_1 \times_f M_2$ and (3.9) and (3.13) implies $n_1 H_1 = n_2 H_2$.

The converse statement is straightforward. □

Remark 4. The inequality (3.1) does not depend on the functions f_2 and f_3 of the generalized Sasakian space form $\widetilde{M}(f_1, f_2, f_3)$.

If we consider a submanifold tangent to ξ, then the corresponding inequality will depend on f_2 and f_3 too.

As applications, we derive certain obstructions to the existence of minimal Legendrian warped product submanifolds in generalized Sasakian space forms.

Let $x\colon M_1 \times_f M_2 \to \widetilde{M}(f_1, f_2, f_3)$ be a Legendrian minimal isometric immersion. Then the above theorem implies:

$$\frac{\Delta f}{f} \leq n_1 f_1.$$

Thus, if $f_1 < 0$, f cannot be a harmonic function or an eigenfunction of Laplacian with positive eigenvalue.

Corollary 5. *If f is a harmonic function, then $M_1 \times_f M_2$ admits no minimal Legendrian immersion into a generalized Sasakian space form $\widetilde{M}(f_1, f_2, f_3)$ with $f_1 < 0$.*

Proof. Let f be a harmonic function on M_1 and $x\colon M_1 \times_f M_2 \to \widetilde{M}(f_1, f_2, f_3)$ be a Legendrian minimal immersion. Then, the inequality (3.1) becomes $f_1 \geq 0$. $\square$

Corollary 6. *If f is an eigenfunction of Laplacian on M_1 with the corresponding eigenvalue $\lambda > 0$, then $M_1 \times_f M_2$ admits no minimal Legendrian immersion in a generalized Sasakian space form $\widetilde{M}(f_1, f_2, f_3)$ with $f_1 \leq 0$.*

References

[1] P. Alegre, D. E. Blair, and A. Carriazo. Generalized Sasakian-space-forms. *Israel J. Math.*, 141:157–183, 2004.
[2] P. Alegre and A. Carriazo. Generalized Sasakian-space-forms II. Preprint.
[3] B.-Y. Chen. Some pinching and classification theorems for minimal submanifolds. *Arch. Math. (Basel)*, 60(6):568–578, 1993.
[4] B.-Y. Chen. Geometry of warped products as Riemannian submanifolds and related problems. *Soochow J. Math.*, 28(2):125–156, 2002.
[5] B.-Y. Chen. On isometric minimal immersions from warped products into real space forms. *Proc. Edinb. Math. Soc. (2)*, 45(3):579–587, 2002.
[6] K. Matsumoto and I. Mihai. Warped product submanifolds in Sasakian space forms. *SUT J. Math.*, 38(2):135–144, 2002.
[7] B. O'Neill. *Semi-Riemannian geometry*, volume 103 of *Pure and Applied Mathematics*. Academic Press Inc., New York, 1983. With applications to relativity.

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