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COMMUTATIVITY OF PRIME Γ-NEAR RINGS WITH $\Gamma-(\sigma,\tau)$-DERIVATION

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Abstract. Let N be a prime Γ-near ring with multiplicative center Z. Let σ and τ be automorphisms of N and δ be a $\Gamma-(\sigma,\tau)$-derivation of N such that N is 2-torsion free. In this paper the following results are proved:

(1) If $\sigma\gamma\delta=\delta\gamma\sigma$ and $\tau\gamma\delta=\delta\gamma\tau$ and $\delta(N)\subseteq Z$, or $[\delta(x),\delta(y)]_\gamma=0$, for all $x,y\in N$ and $\gamma\in\Gamma$, then N is a commutative ring.
(2) If δ_1 is a Γ-derivation, δ_2 is a $\Gamma-(\sigma,\tau)$ derivation of N such that $\tau\gamma\delta_1=\delta_1\gamma\tau$ and $\tau\gamma\delta_2=\delta_2\gamma\tau$, then $\delta_1(\delta_2(N))=0$ implies $\delta_1=0$ or $\delta_2=0$.
(3) The condition for a $\Gamma-(\sigma,\tau)$-derivation to be zero in prime Γ-near ring is also investigated.

1. Introduction

Throughout this paper N denotes a zero symmetric left Γ-near ring with multiplicative center Z. A Γ-near ring is a triple $(N,+,\Gamma)$ which satisfies the following conditions.

(1) $(N,+)$ is a group.
(2) Γ is a non-empty set of binary operators on N such that for each $\gamma\in\Gamma$, $(N,+,\gamma)$ is a near ring.
(3) $x\beta(y\gamma z)=(x\beta y)\gamma z$ for all $x,y,z\in N$ and $\beta,\gamma\in\Gamma$.

N is called a prime Γ-near ring if $x\Gamma N\Gamma y=\{0\}$ implies $x=0$ or $y=0$; $x,y\in N$. Recall that N is called a prime near ring if $xNy=0$ implies $x=0$ or $y=0; x,y\in N$.

For a Γ-near ring N, the set $N_0=\{x\in N : 0\gamma x=0,\ \text{for all}\ \gamma\in\Gamma\}$ is called zero symmetric part of N. If $N=N_0$, then N is called zero symmetric. Recall that as in [8, 3, 9]; a Γ-derivation on N is an additive endomorphism δ on N satisfying the product rule $\delta(x\gamma y)=\delta(x)\gamma y+x\gamma\delta(y)$ for all $x,y\in N$

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and $\gamma \in \Gamma$. An additive mapping $\delta : N \to N$ is called a $\Gamma-(\sigma,\tau)$-derivation if there exists automorphisms $\sigma, \tau : N \to N$ such that

$$\delta(x\gamma y) = \delta(x)\gamma\sigma(y) + \tau(x)\gamma\delta(y) \text{ for all } x, y \in N \text{ and } \gamma \in \Gamma.$$

For all $x, y \in N$ and $\gamma \in \Gamma$, the symbol $[x, y]^{\gamma}_{\sigma,\tau}$ denotes $\tau(x)\gamma y - y\gamma\sigma(x)$. The other commutators are; $[x, y]_{\gamma} = x\gamma y - y\gamma x$ and $(x, y) = x + y - x - y$, the additive group commutator. An element $c \in N$ for which $\delta(c) = 0$ is called a constant.

The purpose of this paper is to study and generalize some results of [9] and [1] on commutativity of prime Γ-near rings. Some recent results on rings deal with commutativity of prime and semi-prime rings admitting suitably-constrained derivations. For further details on prime near rings we refer the reader to [5, 6, 3, 2, 10, 12].

As a generalization of near rings, Γ-near rings were introduced by Satyanarayana [11]. Booth together with Groenewald [7] studied several aspects of Γ-near rings. In this paper we investigate the condition for a $\Gamma-(\sigma,\tau)$ derivation to be zero in prime Γ-near rings.

2. Main Result

We begin with the following Lemma.

Lemma 2.1. *An additive endomorphism δ on a Γ-near ring N is a $\Gamma-(\sigma,\tau)$-derivation if and only if $\delta(x\gamma y) = \tau(x)\gamma\delta(y) + \delta(x)\gamma\sigma(y)$, for all $x, y \in N$ and $\gamma \in \Gamma$.*

Proof. Let δ be a $\Gamma-(\sigma,\tau)$-derivation on a Γ near ring.

Since, $x\gamma(y + y) = x\gamma y + x\gamma y$, we have

$$\begin{aligned} \delta(x\gamma(y+y)) &= \delta(x)\gamma\sigma(y+y) + \tau(x)\gamma\delta(y+y) \\ &= \delta(x)\gamma\sigma(y) + \delta(x)\gamma\sigma(y) + \tau(x)\gamma\delta(y) + \tau(x)\gamma\delta(y), \end{aligned} \tag{2.1}$$

for all $x, y \in N$ and $\gamma \in \Gamma$.

Also,

$$\begin{aligned} \delta(x\gamma y + x\gamma y) &= \delta(x\gamma y) + \delta(x\gamma y) \\ &= \delta(x)\gamma\sigma(y) + \tau(x)\gamma\delta(y) + \delta(x)\gamma\sigma(y) + \tau(x)\gamma\delta(y), \end{aligned} \tag{2.2}$$

for all $x, y \in N$ and $\gamma \in \Gamma$. Comparing (2.1) and (2.2), we have

$$\delta(x)\gamma\sigma(y) + \tau(x)\gamma\delta(y) = \tau(x)\gamma\delta(y) + \delta(x)\gamma\sigma(y),$$

for all $x, y \in N$ and $\gamma \in \Gamma$.

Hence, we have,

$$\delta(x\gamma y) = \tau(x)\gamma\delta(y) + \delta(x)\gamma\sigma(y), \text{ for all } x, y \in N \text{ and } \gamma \in \Gamma.$$

Conversely, suppose for all $x, y \in N$ and $\gamma \in \Gamma$

$$\delta(x\gamma y) = \tau(x)\gamma\delta(y) + \delta(x)\gamma\sigma(y)$$

Then,

$$\begin{aligned}\delta(x\gamma(y+y)) &= \tau(x)\gamma\delta(y+y)+\delta(x)\gamma\sigma(y+y)\\ &= \tau(x)\gamma\delta(y)+\tau(x)\gamma\delta(y)+\delta(x)\gamma\sigma(y)+\delta(x)\gamma\sigma(y), \qquad (2.3)\end{aligned}$$

for all $x, y \in N$ and $\gamma \in \Gamma$.

Also,

$$\begin{aligned}\delta(x\gamma y+x\gamma y) &= \delta(x\gamma y)+\delta(x\gamma y)\\ &= \tau(x)\gamma\delta(y)+\delta(x)\gamma\sigma(y)+\tau(x)\gamma\delta(y)+\delta(x)\gamma\sigma(y), \qquad (2.4)\end{aligned}$$

for all $x, y \in N$ and $\gamma \in \Gamma$. Comparing (2.3) and (2.4), we have

$$\tau(x)\gamma\delta(y)+\delta(x)\gamma\sigma(y)=\delta(x)\gamma\sigma(y)+\tau(x)\gamma\delta(y),$$

for all $x, y \in N$ and $\gamma \in \Gamma$. Thus, for all $x, y \in N$ and $\gamma \in \Gamma$, we have

$$\delta(x\gamma y)=\delta(x)\gamma\sigma(y)+\tau(x)\gamma\delta(y).$$

□

Lemma 2.2. *Let δ be a $\Gamma-(\sigma,\tau)$-derivation on a near ring N. Then for all $x, y, z \in N$ and $\beta, \gamma \in \Gamma$;*

$$(\delta(x)\gamma\sigma(y)+\tau(x)\gamma\delta(y))\beta\sigma(z)=\delta(x)\gamma\sigma(y)\beta\sigma(z)+\tau(x)\gamma\delta(y)\beta\sigma(z).$$

Proof. For all $x, y, z \in N$ and $\beta, \gamma \in \Gamma$

$$\begin{aligned}\delta((x\gamma y)\beta z) &= \delta(x\gamma y)\beta\sigma(z)+\tau(x\gamma y)\beta\delta(z)\\ &= (\delta(x)\gamma\sigma(y)+\tau(x)\gamma\delta(y))\beta\sigma(z)+\tau(x)\gamma\tau(y)\beta\delta(z). \qquad (2.5)\end{aligned}$$

Also, for all $x, y, z \in N$ and $\beta, \gamma \in \Gamma$

$$\begin{aligned}\delta(x\gamma(y\beta z)) &= \delta(x)\gamma\sigma(y\beta z)+\tau(x)\gamma\delta(y\beta z)\\ &= \delta(x)\gamma\sigma(y)\beta\sigma(z)+\tau(x)\gamma(\delta(y))\beta\sigma(z)+\tau(y)\beta\delta(z))\\ &= \delta(x)\gamma\sigma(y)\beta\sigma(z)+\tau(x)\gamma\delta(y)\beta\sigma(z)+\tau(x)\gamma\tau(y)\beta\delta(z). \qquad (2.6)\end{aligned}$$

Comparing (2.5) and (2.6), we get

$$\delta(x)\sigma(y)+\tau(x)\delta(y))\sigma(z)=\delta(x)\sigma(y)\sigma(z)+\tau(x)\delta(y)\sigma(z)$$

for all $x, y, z \in N$ and $\beta, \gamma \in \Gamma$. □

Lemma 2.3. *Let N be a Γ-prime near ring with multiplicative center Z.*

(1) *If there exists a nonzero element $z \in Z$ such that $z+z \in Z$, then $(N,+)$ is abelian.*
(2) *Let δ be a nonzero $\Gamma-(\sigma,\tau)$-derivation of N and $a \in N$. If $\delta(N)\gamma\sigma(a)=0$ or $a\gamma\delta(N)=0$, then $a=0$.*

Proof. (1) Let $a \in N$ such that $0 \neq z=\delta(a) \in Z$. Then $z+z \in Z-\{0\}$. Now, Z is the multiplicative center of N. Therefore, for all $x, y \in Z$ and $\gamma \in \Gamma$, we have $(x+y)\gamma(z+z)=(z+z)\gamma(x+y)$. It implies that,

$$x\gamma z+x\gamma z+y\gamma z+y\gamma z=z\gamma x+z\gamma y+z\gamma x+z\gamma y,$$

and $z \in Z$, implies $z\gamma(x-y)=0$. Now, N is a Γ-prime near ring and $z \neq 0$. Therefore, $(x-y)=0$. Hence, N is abelian.

(2) By hypothesis, $\delta(N)\gamma\sigma(a)=0$, where $a \in N$ and $\gamma \in \Gamma$. Therefore, for all $x, y \in N$ and $\beta, \gamma \in \Gamma$

$$\delta(x\beta y)\gamma\sigma(a)=0.$$

Now, by Lemma (2.2), we have

$$\delta(x)\beta\sigma(y)\gamma\sigma(a)+\tau(x)\beta\delta(y)\gamma\sigma(a)=0,$$

which implies that

$$\delta(x)\beta\sigma(y)\gamma\sigma(a)=0, \text{ or } \delta(x)\Gamma N\Gamma\sigma(a)=0.$$

But, N is a prime Γ-near ring, δ a nonzero $\Gamma-(\sigma,\tau)$-derivation of N and σ is an automorphism. Therefore, $a=0$.

Now, let $a\gamma\delta(N)=0$. Then for all $x, y \in N$ and $\beta, \gamma \in \Gamma$,

$$a\gamma\delta(x\beta y)=0,$$

which implies that

$$a\gamma(\delta(x)\beta\sigma(y)+\tau(x)\beta\delta(y))=0,$$

i.e.

$$a\gamma\delta(x)\beta\sigma(y)+a\gamma\tau(x)\beta\delta(y)=0.$$

Therefore, for all $x, y \in N$ and $\beta, \gamma \in \Gamma$, we have $a\gamma\tau(x)\beta\delta(y)=0$.

Now, τ is an automorphism of N so, $a\Gamma N\Gamma\delta(N)=0$. Also N is prime and $\delta(N) \neq 0$ imply that $a=0$. □

Lemma 2.4. *Let N be a 2-torsion free prime Γ-near ring, and δ be a $\Gamma-(\sigma,\tau)$-derivation of N. If $\delta^2=0$, and σ, τ commute with δ, then $\delta=0$.*

Proof. For all $x, y \in N$ and $\gamma \in \Gamma$, $\delta^2(x\gamma y)=0$. So, we have

$$\begin{aligned}
0 &= \delta(\delta(x\gamma y)) = \delta(\delta(x)\gamma\sigma(y)+\tau(x)\gamma\delta(y)) \\
&= \delta(\delta(x)\gamma\sigma(y))+\delta(\tau(x)\gamma\delta(y)) \\
&= \delta(\delta(x))\gamma\sigma(\sigma(y))+\tau(\delta(x))\gamma\delta(\sigma(y))+\delta(\tau(x))\gamma\sigma(\delta(y)) \\
&\quad +\tau(\tau(x))\gamma\delta(\delta(y)) \\
&= \delta^2(x)\gamma\sigma^2(y)+\tau(\delta(x))\gamma\delta(\sigma(y))+\delta(\tau(x))\gamma\sigma(\delta(y))+\tau^2(x)\gamma\delta^2(y) \\
&= 2\delta(\tau(x))\gamma\delta(\sigma(y)) \quad \text{(By hypothesis).}
\end{aligned}$$

Therefore, for all $x, y \in N$ and $\gamma \in \Gamma$; $\delta(\tau(x))\gamma\delta(\sigma(y))=0$.

Now, as N is 2-torsion free near ring and σ is an automorphism of N, we get $\delta(\tau(x))\delta(N)=0$. Hence, by Lemma (2.3), $\delta=0$. □

Now, we are in a position to generalize some results of Oznur Golbasi and Neset Aydin [9] and Mohammad, Ashraf., Ali, Asma and Ali, Sakir [1] in Prime Γ-near rings.

Theorem 2.5. *Let δ be a $\Gamma-(\sigma,\tau)$-derivation of a Γ-near-ring N. If $a \in N$ is not a left zero divisor and $[a,\delta(a)]^{\gamma}_{(\sigma,\tau)} = 0$, then (x,a) is constant for all $x \in N$ and $\gamma \in \Gamma$.*

Proof. Let $x \in N$ and $\gamma \in \Gamma$. We have, $\delta(a\gamma(x+a)) = \delta(a\gamma x + a\gamma a)$ Expanding the equation, we have

$$\begin{aligned}&\delta(a)\gamma\sigma(x) + \delta(a)\gamma\sigma(a) + \tau(a)\gamma\delta(x) + \tau(a)\gamma\delta(a)\\ &\quad = \delta(a)\gamma\sigma(x) + \tau(a)\gamma\delta(x) + \delta(a)\gamma\sigma(a) + \tau(a)\gamma\delta(a).\end{aligned}$$

Therefore,

$$\delta(a)\gamma\sigma(a) + \tau(a)\gamma\delta(x) = \tau(a)\gamma\delta(x) + \delta(a)\gamma\sigma(a).$$

This implies,

$$0 = \tau(a)\gamma\delta(x) + \delta(a)\gamma\sigma(a) - \tau(a)\gamma\delta(x) - \delta(a)\gamma\sigma(a).$$

But, $[a,\delta(a)]^{\gamma}_{\sigma,\tau} = 0$, which implies that

$$\tau(a)\gamma\delta(a) - \delta(a)\gamma\sigma(a) = 0.$$

Thus,

$$0 = \tau(a)\gamma\delta(x) + \tau(a)\gamma\delta(a) - \tau(a)\gamma\delta(x) - \tau(a)\gamma\delta(a),$$

which implies that $\tau(a)\gamma\delta(x,a) = 0$.

But, τ is an automorphism of N, and $\tau(a)$ is not a left zero divisor. Therefore, $\delta(x,a) = 0$. Hence, (x,a) is constant for all $x \in N$. ☐

Theorem 2.6. *Let N have no non-zero divisors of zero. If N admits a non-trivial (σ,τ)-commuting $\Gamma-(\sigma,\tau)$-derivation δ, then $(N,+)$ is abelian.*

Proof. Let c be any additive commutator. Then Theorem (2.5) implies, c is a constant. Also, for any $x \in N$ and $\gamma \in \Gamma$, $x\gamma c$ is also an additive commutator and hence a constant. Thus, for all $x \in N$ and $\gamma \in \Gamma$

$$0 = \delta(x\gamma c) = \delta(x)\gamma\sigma(c) + \tau(x)\gamma\delta(c).$$

This implies $\delta(x)\gamma\sigma(c) = 0$ for all $x \in N$ and $\gamma \in \Gamma$.

Since, $\delta(x) \neq 0$ for some $x \in N$ and $\gamma \in \Gamma$. Therefore, $\sigma(c) = 0$. Thus, $c = 0$ for all additive commutators c. Hence, $(N,+)$ is abelian. ☐

Theorem 2.7. *Let N be a prime Γ-near ring with a nonzero $\Gamma-(\sigma,\tau)$-derivation δ such that $\sigma\gamma\delta = \delta\gamma\sigma$ and $\tau\gamma\delta = \delta\gamma\tau$ for all $\gamma \in \Gamma$. If $\delta(N) \subseteq Z$, then $(N,+)$ is abelian. Moreover, if N is 2-torsion free, then N is a commutative ring.*

Proof. By hypothesis $\delta(N) \subseteq Z$ and δ is non-trivial. Therefore, there exists $0 \neq a \in N$ such that $z = \delta(a) \in Z - \{0\}$ and $z + z = \delta(a+a) \in Z - \{0\}$.

Therefore, by Lemma (2.3), $(N,+)$ is abelian.

Again by hypothesis, for all $a,b,c \in N$ and $\beta,\gamma \in \Gamma$, we have

$$\sigma(c)\gamma\delta(a\beta b) = \delta(a\beta b)\gamma\sigma(c).$$

Now, using Lemma (2.2) and the fact that N is a left near-ring, we have

$$\sigma(c)\gamma\delta(a)\beta\sigma(b)+\sigma(c)\gamma\tau(a)\beta\delta(b)=\delta(a)\beta\sigma(b)\gamma\sigma(c)+\tau(a)\beta\delta(b)\gamma\sigma(c),$$

for all $a,b,c\in N$ and $\beta,\gamma\in\Gamma$.

Now, $\delta(N)\subseteq Z, \sigma\gamma\delta=\delta\gamma\sigma$ and $\tau\gamma\delta=\delta\gamma\tau$ for all $\gamma\in\Gamma$, we get

$$\delta(a)\ \gamma\sigma(c)\beta\sigma(b)+\delta(b)\gamma\sigma(c)\beta\tau(a)=\delta(a)\gamma\sigma(b)\beta\sigma(c)+\delta(b)\gamma\tau(a)\beta\sigma(c)$$

for all $a,b,c\in N$ and $\beta,\gamma\in\Gamma$.

Comparing the two sides and using the fact that $(N,+)$ is abelian, we get

$$\delta(a)\ \gamma\sigma(c)\beta\sigma(b)-\delta(a)\gamma\sigma(b)\beta\sigma(c)=\delta(b)\gamma\tau(a)\beta\sigma(c)-\delta(b)\gamma\sigma(c)\beta\tau(a)$$

or

$$\delta(a)\gamma\sigma([c,b]_\beta)=\delta(b)\gamma([\tau(a),\sigma(c)]_\beta),$$

for all $a,b,c\in N$ and $\beta,\gamma\in\Gamma$.

Now, suppose that N is not commutative, and choose $b,c\in N$ such that $[c,b]\neq 0$, and $a=\delta(x)\in Z$.

Then for all $x\in N$ and $\gamma\in\Gamma$, we get $\delta^2(x)\gamma\sigma([c,b])=0$.

Now, by Lemma (2.3), we see that central element $\delta^2(x)$ can not be a divisor of zero, which implies that $\delta^2(x)=0$ for all $x\in N$. By Lemma (2.4), this can not happen for non trivial δ. Thus, $\sigma([c,b])=0$, for all $b,c\in N$. Hence, N is a commutative ring, as σ is an automorphism of N. □

Theorem 2.8. *Let N be a prime Γ-near ring with a nonzero $\Gamma-(\sigma,\tau)$-derivation δ such that $\sigma\gamma\delta=\delta\gamma\sigma$ and $\tau\gamma\delta=\delta\gamma\tau$. If $[\delta(x),\delta(y)]_\gamma=0$, for all $x,y\in N$ and $\gamma\in\Gamma$, then $(N,+)$ is abelian. Moreover, if N is 2-torsion free, then N is a commutative ring.*

Proof. By hypothesis we have, $\delta(x+x)\gamma\delta(x+y)=\delta(x+y)\gamma\delta(x+x)$ for all $x,y\in N$ and $\gamma\in\Gamma$. This implies that

$$\delta(x)\gamma\delta(x)+\delta(x)\gamma\delta(y)=\delta(x)\gamma\delta(x)+\delta(y)\gamma\delta(x)$$

for all $x,y\in N$ and $\gamma\in\Gamma$. Hence, $\delta(x)\gamma\delta(x,y)=0$ for all $x,y\in N$ and $\gamma\in\Gamma$, which implies $\delta(x)\gamma\delta(c)=0$ for all $x\in N$, $\gamma\in\Gamma$ and the additive commutator c. Now, by Lemma (2.3), we have $\delta(c)=0$, for all additive commutators c. Now, N is a left near ring and c an additive commutator. Therefore, $x\gamma c$ is also an additive commutator for all $x\in N$. Therefore, $\delta(x\gamma c)=0$ for all $x\in N$, $\gamma\in\Gamma$ and for all additive commutators c. Therefore, by Lemma (2.3), $c=0$. Hence, $(N,+)$ is abelian.

Now, assume that N is 2-torsion free, $\sigma\gamma\delta=\delta\gamma\sigma$ and $\tau\gamma\delta=\delta\gamma\tau$.

Then by Lemma (2.1) and Lemma (2.2) we have

$$\begin{aligned}\delta(\delta(x)\gamma y)\gamma\delta(z)&=\delta^2(x)\gamma\sigma(y)\gamma\delta(z)+\tau(\delta(x))\gamma\delta(y))\gamma\delta(z)\\ \delta^2(x)\gamma\sigma(y)\gamma\delta(z)&=\delta(\delta(x)\gamma y)\gamma\delta(z)-\tau(\delta(x))\gamma\delta(y))\gamma\delta(z).\end{aligned}\tag{2.7}$$

Now; $\delta(x)\gamma\delta(y) = \delta(y)\gamma\delta(x)$, for all $x, y, z \in N$, and $\gamma \in \Gamma$. Therefore,

$$\begin{aligned} \delta(\delta(x)\gamma y)\gamma\delta(z) &= \delta(z)\gamma\delta(\delta(x)\gamma y) \\ &= \delta(z)\gamma\delta^2(x)\gamma\sigma(y) + \delta(z)\gamma\tau(\delta(x))\gamma\delta(y) \\ &= \delta^2(x)\gamma\delta(z)\gamma\sigma(y) + \tau(\delta(x))\gamma\delta(y)\gamma\delta(z) \end{aligned} \tag{2.8}$$

for all $x, y, z \in N$ and $\gamma \in \Gamma$.

Combining (2.7) and (2.8), we have for all $x, y, z \in N$ and $\gamma \in \Gamma$

$$\delta^2(x)\gamma\sigma(y)\gamma\delta(z) - \delta^2(x)\gamma\delta(z)\gamma\sigma(y) = 0$$

or

$$\delta^2(x)\gamma(\sigma(y)\gamma\delta(z) - \delta(z)\gamma\sigma(y)) = 0$$

Now, replacing y by $y\gamma a$, we have for all $a, x, y, z \in N$ and $\gamma \in \Gamma$

$$\delta^2(x)\gamma(\sigma(y\gamma a)\gamma\delta(z) - \delta(z)\gamma\sigma(y\gamma a)) = 0$$

or

$$\delta^2(x)\gamma\sigma(y)\gamma(\sigma(a)\gamma\delta(z) - \delta(z)\gamma\sigma(a)) = 0$$

Thus, $\delta^2(x)\gamma N(\sigma(a)\gamma\delta(z) - \delta(z)\gamma\sigma(a)) = 0$ for all $a, x, y, z \in N$ and $\gamma \in \Gamma$.

Since, N is prime and σ is an automorphism. Therefore for all $a, x, z \in N$ and $\gamma \in \Gamma$

$$\delta^2(x) = 0, \text{ or } \sigma(a)\gamma\delta(z) - \delta(z)\gamma\sigma(a)) = 0$$

But, by Lemma 2.4 $\delta^2(x) = 0$ is not possible. Hence,

$$\sigma(a)\gamma\delta(z) - \delta(z)\gamma\sigma(a)) = 0,$$

for all $a, z \in N$ and $\gamma \in \Gamma$.

Therefore, $\delta(N) \subseteq Z$. Hence, by Theorem (2.7), N is commutative. □

Theorem 2.9. *Let N be a 2-torsion free prime Γ-near ring N, δ_1 be a $\Gamma-(\sigma,\tau)$-derivation of N and δ_2 be a Γ derivation of N. If $\delta_1(\delta_2(N)) = 0$, then $\delta_1 = 0$, or $\delta_2 = 0$.*

Proof. By hypothesis for all $a, b \in N$ and$\gamma \in \Gamma$ $\delta_1(\delta_2(a\gamma b)) = 0$. Therefore, we have

$$\begin{aligned} 0 &= \delta_1(\delta_2(a)\gamma b) + a\gamma\delta_2(b)) = \delta_1(\delta_2(a)\gamma b) + \delta_1(a\gamma\delta_2(b)) \\ &= \delta_1(\delta_2(a))\gamma\sigma(b) + \tau(\delta_2(a))\gamma\delta_1(b)\delta_1(a)\sigma(\delta_2(b)) + \tau(a)\gamma\delta_1(\delta_2(b)). \end{aligned}$$

Now, for all $a, b \in N$ and $\gamma \in \Gamma$, we have

$$\tau(\delta_2(a))\gamma\delta_1(b) + \delta_1(a)\gamma\sigma(\delta_2(b)) = 0.$$

Replacing a by $\delta_2(a)$, then for all $a, b \in N$ and $\gamma \in \Gamma$, we have

$$\tau(\delta_2^2(a))\gamma\delta_1(b) = 0.$$

Now, Lemma (2.3), implies that $\delta_1 = 0$ or $\delta_2^2 = 0$. If $\delta_2^2 = 0$, then by Lemma (2.4), $\delta_2 = 0$. Hence, this theorem is proved. □

Theorem 2.10. *Let N be a 2-torsion free prime Γ-near ring N, δ_1 be a Γ-derivation of N and δ_2 be a $\Gamma-(\sigma,\tau)$-derivation of N such that $\tau\gamma\delta_1=\delta_1\gamma\tau$ and $\tau\gamma\delta_2=\delta_2\gamma\tau$. If $\delta_1(\delta_2(N))=0$, then $\delta_1=0$ or $\delta_2=0$.*

Proof. By hypothesis $\delta_1(\delta_2(a\gamma b))=0$, for all $a,b\in N$ and $\gamma\in\Gamma$.
Therefore, we have

$$\begin{aligned}0&=\delta_1(\delta_2(a)\gamma\sigma(b)+\tau(a)\gamma\delta_2(b))=\delta_1(\delta_2(a)\gamma\sigma(b))+\delta_1(\tau(a)\gamma\delta_2(b))\\&=\delta_1(\delta_2(a))\gamma\sigma(b)+\delta_2(a)\gamma\delta_1(\sigma(b))+\delta_1(\tau(a))\gamma\delta_2(b)+\tau(a)\gamma\delta_1(\delta_2(b)).\end{aligned}$$

This implies that

$$\delta_2(a)\gamma\delta_1(\sigma(b))+\delta_1(\tau(a))\gamma\delta_2(b)=0,$$

for all $a,b\in N$ and $\gamma\in\Gamma$. Replacing a by $\delta_2(a)$, and using the fact that $\tau\gamma\delta_1=\delta_1\gamma\tau$ and $\tau\gamma\delta_2=\delta_2\gamma\tau$, we have

$$\delta_2^2(a)\gamma\delta_1(\sigma(b))=0,\ \text{for all } a,b\in N \text{ and } \gamma\in\Gamma.$$

Applying Lemma (2.3), we have $\delta_1=0$, or $\delta_2^2=0$. If $\delta_2^2=0$, then by Lemma (2.4), $\delta_2=0$. The proof is complete. □

Lastly, we generalize a result of Yong Uk Cho and Young Bae Jun [8, Proposition 3.9] in Prime Γ-near rings.

Theorem 2.11. *Let δ be a $\Gamma-(\sigma,\tau)$-derivation on a zero symmetric prime Γ-near ring N. If there exists a nonzero element $x\in N$ such that $x\gamma\delta(y)=0$ for all $y\in N$ and $\gamma\in\Gamma$, then $\delta=o$.*

Proof. Let x be a nonzero element of N such that

$$x\gamma\delta(y)=0 \text{ for all } y\in N \text{ and } \gamma\in\Gamma.$$

Replacing y by $y\beta z$ we get,

$$\begin{aligned}0&=x\gamma\delta(y\beta z)=x\gamma(\delta(y)\beta\sigma(z)+\tau(y)\beta\delta(z))\\&=x\gamma\delta(y)\beta\sigma(z)+x\gamma\tau(y)\beta\delta(z)=x\gamma\tau(y)\beta\delta(z),\end{aligned}$$

for all $y,z\in N$ and $\beta,\gamma\in\Gamma$.
Therefore, $x\Gamma N\Gamma\delta(z)=0$. Since, N is prime, implies $\delta(z)=0$ for all $z\in N$. Hence, $\delta=0$. □

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