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GROUPS WITH THE SAME PRIME GRAPH AS AN ALMOST SPORADIC SIMPLE GROUP

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The author dedicate this paper to his parents: Professor Amir Khosravi and Mrs. Soraya Khosravi for their unending love and support.

Abstract. Let G be a finite group. We denote by $\Gamma(G)$ the prime graph of G. Let S be a sporadic simple group. M. Hagie in (Hagie, M. (2003), The prime graph of a sporadic simple group, Comm. Algebra, 31: 4405-4424) determined finite groups G satisfying $\Gamma(G) = \Gamma(S)$. In this paper we determine finite groups G such that $\Gamma(G) = \Gamma(A)$ where A is an almost sporadic simple group, except $\mathrm{Aut}(McL)$ and $\mathrm{Aut}(J_2)$.

1. Introduction

If n is an integer, then we denote by $\pi(n)$ the set of all prime divisors of n. If G is a finite group, then the set $\pi(|G|)$ is denoted by $\pi(G)$. Also the set of order elements of G is denoted by $\pi_e(G)$. We construct the prime graph of G as follows:

The prime graph $\Gamma(G)$ of a group G is the graph whose vertex set is $\pi(G)$, and two distinct primes p and q are joined by an edge (we write $p \sim q$) if and only if G contains an element of order pq. Let $t(G)$ be the number of connected components of $\Gamma(G)$ and let $\pi_1(G)$, $\pi_2(G)$, $\ldots$, $\pi_{t(G)}(G)$ be the connected components of $\Gamma(G)$. We use the notation π_i instead of $\pi_i(G)$. If $2 \in \pi(G)$, then we always suppose $2 \in \pi_1$.

The concept of prime graph arose during the investigation of certain cohomological questions associated with integral representations of finite groups. It has been proved that for every finite group G we have $t(G) \leq 6$ [12, 22, 31] and the diameter of $\Gamma(G)$ is at most 5 [23]. In [20] and [19] finite groups with the same prime graph as a CIT simple group and $PSL(2, q)$ where $q = p^\alpha < 100$ are determined.

In [18] we introduced the following concept for finite groups:

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Definition 1.1. ([18]) A finite group G is called *recognizable by prime graph* (briefly, *recognizable by graph*) if $H \cong G$ for every finite group H with $\Gamma(H) = \Gamma(G)$. Also a finite simple nonabelian group P is called *quasirecognizable by prime graph*, if every finite group G with $\Gamma(G) = \Gamma(P)$ has a composition factor isomorphic to P.

It is proved that if $q = 3^{2n+1}$ $(n > 0)$, then the simple group ${}^2G_2(q)$ is uniquely determined by its prime graph [18, 32]. Also the authors in [21] proved that $PSL(2,p)$, where $p > 11$ is a prime number and $p \not\equiv 1 \pmod{12}$ is recognizable by prime graph. Hagie in [9] determined finite groups G satisfying $\Gamma(G) = \Gamma(S)$, where S is a sporadic simple group. In this paper, as the main result we determine finite groups G such that their prime graph is $\Gamma(A)$, where A is an almost sporadic simple group, except $\mathrm{Aut}(J_2)$ and $\mathrm{Aut}(McL)$.

In this paper, all groups are finite and by simple groups we mean non-abelian simple groups. All further unexplained notations are standard and refer to [5], for example. We use the results of J. S. Williams [31], N. Iiyori and H. Yamaki [12] and A. S. Kondrat'ev [22] about the prime graph of simple groups and the results of M. S. Lucido [24] about the prime graph of almost simple groups. We note that the structure of the almost sporadic simple groups are described in [5].

We denote by (a, b) the greatest common divisor of positive integers a and b. Let m be a positive integer and p be a prime number. Then $|m|_p$ denotes the $p-$part of m. In other words, $|m|_p = p^k$ if $p^k \| m$ (i.e. $p^k | m$ but $p^{k+1} \nmid m$).

2. Preliminary Results

First we give an easy remark:

Remark 2.1. Let N be a normal subgroup of G and $p \sim q$ in $\Gamma(G/N)$. Then $p \sim q$ in $\Gamma(G)$. In fact if $xN \in G/N$ has order pq, then there is a power of x which has order pq.

Definition 2.1. ([8]) A finite group G is called a 2-Frobenius group if it has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, where K and G/H are Frobenius groups with kernels H and K/H, respectively.

Lemma 2.1. *(*[31, Theorem A]*) If G is a finite group with its prime graph having more than one component, then G is one of the following groups:*

(a) *a Frobenius or a 2-Frobenius group;*
(b) *a simple group;*
(c) *an extension of a π_1-group by a simple group;*
(d) *an extension of a simple group by a π_1-group;*
(e) *an extension of a π_1-group by a simple group by a π_1-group.*

Lemma 2.2. *If G is a finite group with more than one prime graph component and has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups and K/H is simple, then H is a nilpotent group.*

Proof. The prime graph of G has more than one component. So let $q \in \pi_2$. Let $y \in G$ be an element of order q. Since, $H \lhd G$, y induces an automorphism $\sigma \in \mathrm{Aut}(H)$. If $\sigma(h) = h$, for some $1 \neq h \in H$, then $yh = hy$. From the assumption, H is a π_1−group and $o(y) = q$. So $(o(h), o(y)) = 1$, which implies that $o(hy) = o(h)o(y)$. Hence, $q \in \pi_1$, which is a contradiction. Therefore, σ is a fixed-point-free automorphism of order q. Thus, H is a nilpotent group, by Thompson's theorem ([7, Theorem 10.2.1]). □

The next lemma summarizes the basic structural properties of a Frobenius group [7, 25]:

Lemma 2.3. *Let G be a Frobenius group and let H, K be Frobenius complement and Frobenius kernel of G, respectively. Then $t(G) = 2$, and the prime graph components of G are $\pi(H)$, $\pi(K)$. Also the following conditions hold:*

(1) *$|H|$ divides $|K| - 1$.*
(2) *K is nilpotent and if $|H|$ is even, then K is abelian.*
(3) *Sylow p-subgroups of H are cyclic for odd p and are cyclic or generalized quaternion for $p = 2$.*
(4) *If H is a non-solvable Frobenius complement, then H has a normal subgroup H_0 with $|H : H_0| \leq 2$ such that $H_0 = SL(2, 5) \times Z$, where the Sylow subgroups of Z are cyclic and $(|Z|, 30) = 1$.*

Also the next lemma follows from [8] and the properties of Frobenius groups [10]:

Lemma 2.4. *Let G be a 2-Frobenius group, i.e. G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that K and G/H are Frobenius groups with kernels H and K/H, respectively. Then*

(a) *$t(G) = 2$, $\pi_1 = \pi(G/K) \cup \pi(H)$ and $\pi_2 = \pi(K/H)$;*
(b) *G/K and K/H are cyclic, $|G/K| \mid (|K/H| - 1)$ and $G/K \leq Aut(K/H)$;*
(c) *H is nilpotent and G is a solvable group.*

By using the above lemmas it follows that:

Lemma 2.5. *Let G be a finite group and let A be an almost sporadic simple group, i.e. there exists an sporadic simple group S such that $S \leq A \leq \mathrm{Aut}(S)$. If the prime graph of A is not connected and $\Gamma(G) = \Gamma(A)$, then one of the following holds:*

(a) *G is a Frobenius or a 2-Frobenius group;*
(b) *G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/K is a π_1-group, H is a nilpotent π_1-group, and K/H is a non-abelian simple group with $t(K/H) \geq 2$ and $G/K \leq Out(K/H)$. Also $\pi_2(A) = \pi_i(K/H)$ for some $i \geq 2$ and $\pi_2(A) \subseteq \pi(K/H) \subseteq \pi(S)$.*

The next lemma was introduced by Crescenzo and modified by Bugeaud:

Lemma 2.6. *([6, 17]) With the exceptions of the relations* $(239)^2-2(13)^4=-1$ *and* $(3)^5-2(11)^2=1$ *every solution of the equation*

$$p^m-2q^n=\pm 1;\quad p,q \quad prime\ ;\quad m,n>1,$$

has exponents $m=n=2$*; i.e. it comes from a unit* $p-q.2^{\frac{1}{2}}$ *of the quadratic field* $Q(2^{\frac{1}{2}})$ *for which the coefficients* p, q *are prime.*

Lemma 2.7. *([17]) The only solution of the equation* $p^m-q^n=1$*;* p, q *prime; and* $m,n>1$ *is* $3^2-2^3=1$.

Lemma 2.8 (Zsigmondy's Theorem [33])**.** *Let* p *be a prime and* n *be a positive integer. Then one of the following holds:*

(i) *there is a primitive prime* p' *for* p^n-1*, that is,* $p'|(p^n-1)$ *but* $p'\nmid(p^m-1)$*, for every* $1\leq m<n$,
(ii) $p=2$, $n=1$ *or 6,*
(iii) p *is a Mersenne prime and* $n=2$.

Definition 2.2. A group G is called a C_{pp} group if the centralizers in G of its elements of order p are $p-$groups.

Lemma 2.9. *([4]) (a) The* $C_{13,13}$*-simple groups are:* A_{13}, A_{14}, A_{15}*;* Suz, Fi_{22}*;* $L_2(q)$, $q=3^3$, 5^2, 13^n *or* $2\times 13^n-1$ *which is a prime,* $n\geq 1$*;* $L_3(3)$, $L_4(3)$, $O_7(3)$, $S_4(5)$, $S_6(3)$, $O_8^+(3)$, $G_2(q)$, $q=2^2$, *3;* $F_4(2)$,$U_3(q)$, $q=2^2$, 23*;* $Sz(2^3)$, ${}^3D_4(2)$, ${}^2E_6(2)$, ${}^2F_4(2)'$.

(b) The $C_{19,19}$*-simple groups are:* A_{19}, A_{20}, A_{21}*;* J_1, J_3, $O'N$, Th, HN*;* $L_2(q)$, $q=19^n$, $2\times 19^n-1$ *which is a prime,* $(n\geq 1)$*;* $L_3(7)$, $U_3(2^3)$, $R(3^3)$, ${}^2E_6(2)$.

Definition 2.3. By using the prime graph of G, the order of G can be expressed as a product of coprime positive integers m_i, $i=1,2,\ldots,t(G)$ where $\pi(m_i)=\pi_i(G)$. These integers are called *the order components* of G. The set of order components of G will be denoted by $OC(G)$. Also we call $m_2,\ldots,m_{t(G)}$ *the odd order components* of G.

The order components of non-abelian simple groups are listed in [13, Table 1].

Lemma 2.10. *([3, Lemma 8]) Let* G *be a finite group with* $t(G)\geq 2$ *and let* N *be a normal subgroup of* G*. If* N *is a* π_i-*group for some prime graph component* π_i *of* G *and* $m_1,m_2,\ldots,m_r$ *are some order components of* G *but not* π_i*-numbers, then* $m_1m_2\cdots m_r$ *is a divisor of* $|N|-1$.

3. Main Results

Let A be an almost sporadic simple group, that is $S\leq A\leq \mathrm{Aut}(S)$ where S is a sporadic simple group. Since $|Aut(S):S|\leq 2$ for sporadic simple groups S (see [5]), so $A=S$ or $A=\mathrm{Aut}(S)$. Hagie considered the case $A=S$. So in the sequel we only assume the case $A=\mathrm{Aut}(S)$.

We note that some of the sporadic simple groups have trivial outer automorphism groups. Also if S is one of the following groups: M_{12}, He, Fi_{22} or HN, then $\mathrm{Aut}(S) \neq S$. But, $\Gamma(S) = \Gamma(\mathrm{Aut}(S))$. Therefore, we consider the case $A = \mathrm{Aut}(S)$, where S is one of the following groups: M_{22}, J_3, HS, Suz, $O'N$ or Fi'_{24}.

Now, we consider the following Diophantine equations:

$$(i)\quad \frac{q^p-1}{q-1} = y^n, \qquad (ii)\quad \frac{q^p-1}{(q-1)(p,q-1)} = y^n,$$

$$(iii)\quad \frac{q^p+1}{q+1} = y^n, \qquad (iv)\quad \frac{q^p+1}{(q+1)(p,q+1)} = y^n.$$

These Diophantine equations have many applications in the theory of finite groups (for example see [16] or [17]). We note that the odd order components of some non-abelian simple groups of Lie type are of the form $(q^p \pm 1)/((q\pm1)(p,q\pm1))$ [13] and there exists some results about these Diophantine equations [15]. Now, we prove the following lemma about these Diophantine equations to determine some C_{pp}-simple groups.

Lemma 3.1. *Let $p \geq 3$ and p_0 be prime numbers and $q = p_0^{\alpha}$.*

(a) If $y = 11$ and $p_0 \in \{2,3,5,7\}$, then $(p,q,n) = (5,3,2)$ is the only solution of (i) and (ii). Also $(p,q,n) = (5,2,1)$ is the only solution of (iii) and (iv).

(b) If $y = 29$ and $p_0 \in \{2,3,5,7,11,13,17,19,23\}$, then the Diophantine equations (i)-(iv) have no solution.

(c) If $y = 31$ and $p_0 \in \{2,3,5,7,11,19\}$, then $(p,q,n) = (5,2,1)$ and $(3,5,1)$ are the only solutions of (i) and (ii). Also (iii) and (iv) have no solution.

Proof. Let $q = p_o^{\alpha}$ and $(q^p-1)/(q-1) = 11^n$ or $(q^p-1)/((q-1)(p,q-1)) = 11^n$. Then $11 \mid (p_0^{\alpha p} - 1)$, which implies that $p_0^{\alpha p} \equiv 1 \pmod{11}$ and hence $\beta := \mathrm{ord}_{11}(p_0)$ is a divisor of αp. Since, $p \geq 3$ and $(p_0^{\alpha p}-1)/(p_0^{\alpha}-1) = 11^n$ or $(p_0^{\alpha p}-1)/(p_0^{\alpha}-1)(p,p_0^{\alpha}-1)) = 11^n$, it follows that 11 is a primitive prime for $p_0^{\alpha p}-1$. Also 11 is a primitive prime for $p_o^{\beta}-1$, by the definition of $\mathrm{ord}_{11}(p_0)$. Therefore, $\beta = \alpha p$, by the definition of the primitive prime (see Lemma 2.8). Also by using the Fermat theorem we know that β is a divisor of 10. Hence, the only possibility for p is 5 and so $1 \leq \alpha \leq 2$. Now, by checking the possibilities for q it follows that $(p,q,n) = (5,3,2)$ is the only solution of the Diophantine equations (i) and (ii). Similarly consider the Diophantine equations

$$\frac{q^p+1}{q+1} = 11^n, \quad \text{and} \quad \frac{q^p+1}{(q+1)(p,q+1)} = 11^n,$$

Then 11 is a divisor of $p_o^{2\alpha p} - 1$ and in a similar manner it follows that $p = 5$ and $\alpha = 1$. Therefore, the only solution of these Diophantine equations is $(p,q,n) = (5,2,1)$.

The proof of (b) and (c) are similar and for convenience we omit the proof of them. $\square$

Now, by using Lemmas 2.6, 2.7 and 3.1, we can prove the following lemma:

Lemma 3.2. *Let M be a simple group of Lie type over $GF(q)$.*

(a) *If q is a power of 2, 3, 5 or 7 and M is a $C_{11,11}$-group, then M is one of the following simple groups: $L_2(11)$, $L_5(3)$, $L_6(3)$, $U_5(2)$, $U_6(2)$, $O_{11}(3)$, $S_{10}(3)$ or $O_{10}^+(3)$.*

(b) *If q is a power of 2, 3, 5, 7, 11, 13, 17, 19 or 23 and M is a $C_{29,29}$-group, then $M = L_2(29)$.*

(c) *If q is a power of 2, 3, 5, 7, 11 or 19 and M is a $C_{31,31}$-group, then M is $L_5(2)$, $L_3(5)$, $L_6(2)$, $L_4(5)$, $O_{10}^+(2)$, $O_{12}^+(2)$, $L_2(31)$, $L_2(32)$, $G_2(5)$ or $Sz(32)$.*

Proof. The odd order components of finite non-abelian simple groups are listed in Table 1 in [13]. Now, by using Lemmas 2.6, 2.7, 2.8 and 3.1 we get the result. For convenience we omit the proof. □

Theorem 3.1. *Let G be a finite group satisfying $\Gamma(G) = \Gamma(A)$.*

(a) *If $A = \mathrm{Aut}(J_3)$, then $G/O_\pi(G) \cong J_3$, where $2 \in \pi$, $\pi \subseteq \{2, 3, 5\}$ and $O_\pi(G) \neq 1$ or $G/O_\pi(G) \cong J_3.2$, where $\pi \subseteq \{2, 3, 5\}$.*

(b) *If $A = \mathrm{Aut}(M_{22})$, then $G/O_2(G) \cong M_{22}$ and $O_2(G) \neq 1$ or $G/O_\pi(G) \cong M_{22}.2$, where $\pi \subseteq \{2\}$.*

(c) *If $A = \mathrm{Aut}(HS)$, then $G/O_\pi(G) \cong U_6(2)$ or HS, where $2 \in \pi$, $\pi \subseteq \{2, 3, 5\}$ and $O_\pi(G) \neq 1$ or $G/O_\pi(G) \cong HS.2$, $U_6(2).2$ or McL, where $\pi \subseteq \{2, 3, 5\}$.*

(d) *If $A = \mathrm{Aut}(Fi'_{24})$, then $G/O_\pi(G) \cong Fi'_{24}$, where $2 \in \pi$, $\pi \subseteq \{2, 3\}$ and $O_\pi(G) \neq 1$ or $G/O_\pi(G) \cong Fi'_{24}.2$, where $\pi \subseteq \{2, 3\}$.*

(e) *If $A = \mathrm{Aut}(O'N)$, then $G/O_2(G) \cong O'N$, where $O_2(G) \neq 1$ or $G/O_\pi(G) \cong O'N.2$, where $\pi \subseteq \{2\}$.*

(f) *If $A = \mathrm{Aut}(Suz)$, then $G/O_\pi(G) \cong Suz$, where $2 \in \pi$, $\pi \subseteq \{2, 3, 5\}$ and $O_\pi(G) \neq 1$ or $G/O_\pi(G) \cong Suz.2$, where $\pi \subseteq \{2, 3, 5\}$.*

Proof. **(a)** Let $\Gamma(G) = \Gamma(\mathrm{Aut}(J_3))$. First, let G be a solvable group. Then G has a Hall $\{5, 17, 19\}$-subgroup H. Since, G is solvable, it follows that H is solvable. Hence, $t(H) \leq 2$, which is a contradiction, since there exists no edge between 5, 17 and 19 in $\Gamma(G)$. Thus, G is not solvable, and so G is not a 2-Frobenius group, by Lemma 2.4. If G is a non-solvable Frobenius group and H and K be the Frobenius complement and the Frobenius kernel of G, respectively, then by using Lemma 2.3 it follows that H has a normal subgroup H_0 with $|H : H_0| \leq 2$ such that $H_0 = SL(2, 5) \times Z$ where the Sylow subgroups of Z are cyclic and $(|Z|, 30) = 1$. We know that $3 \nsim 17$ and $3 \nsim 19$ in $\Gamma(G)$. Therefore, $Z = 1$. Hence, $\{17, 19\} \subseteq \pi(K)$. This is a contradiction, since K is nilpotent and $17 \nsim 19$ in $\Gamma(G)$. Hence, G is neither a Frobenius group nor a 2-Frobenius group. So by using Lemma 2.5, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a $C_{19,19}$ simple group. By using Lemma 2.9, K/H is A_{19}, A_{20}, A_{21}, J_1, J_3, $O'N$, Th, HN, $L_3(7)$, $U_3(8)$, $R(27)$, ${}^2E_6(2)$, $L_2(q)$, where $q = 19^n$ or $L_2(q)$, where $q = 2 \times 19^n - 1$ ($n \geq 1$) is a prime number. But, $\pi(K/H) \subseteq \pi(J_3)$ and $\pi(J_3) \cap \{7, 11, 13, 31\} = \emptyset$. Also $q = 2 \times 19^n - 1 > 19$.

Hence, the only possibilities for K/H are J_3 and $L_2(19^n)$, where $n \geq 1$. The orders of maximal tori of $A_m(q) = PSL(m+1, q)$ are

$$\frac{\prod_{i=1}^{k}(q^{r_i}-1)}{(q-1)(m+1,q-1)}; \quad (r_1,\dots,r_k) \in Par(m+1).$$

Therefore, every element of $\pi_e(PSL(2,q))$ is a divisor of q, $(q+1)/d$ or $(q-1)/d$, where $d = (2, q-1)$. If $q = 19^n$, then $3 \mid (19^n-1)/2$ and since $3 \sim 5$ and $3 \nsim 17$ in $\Gamma(G)$, it follows that if 5 divides $|G|$, then $5 \mid (19^n-1)$ and if 17 is a divisor of $|G|$, then $17 \mid (19^n+1)$. Note that $\pi(19-1) = \{2,3\}$, $\pi(19^2-1) = \{2,3,5\}$ and $17 \mid (19^4+1)$. Now by using the Zsigmondy's Theorem, Lemmas 2.6 and 2.7 it follows that the only possibility is $n = 1$.

Now, we consider these possibilities for K/H, separately.

Case 1. Let $K/H \cong J_3$.

We note that $Out(J_3) \cong \mathbb{Z}_2$ and hence G/H is isomorphic to J_3 or $J_3.2$. Also H is a nilpotent π_1-group. Hence, $\pi(H) \subseteq \{2,3,5,17\}$. If $17 \in \pi(H)$, then let T be a $\{3,17,19\}$ subgroup of G, since J_3 has a $19:9$ subgroup. Obviously, T is solvable and hence $t(T) \leq 2$, which is a contradiction. Therefore, $\pi = \pi(H) \subseteq \{2,3,5\}$ and $G/O_\pi(G) \cong J_3$ or $G/O_\pi(G) \cong J_3.2$. If $G/O_\pi(G) \cong J_3$, then $O_\pi(G) \neq 1$ and $2 \in \pi$, since $2 \nsim 17$ in $\Gamma(J_3)$.

Case 2. Let $K/H \cong L_2(19)$.

Since $Out(L_2(19)) \cong \mathbb{Z}_2$, it follows that $G/H \cong L_2(19)$ or $L_2(19).2$. But, in this case $\pi(K/H) = \{2,3,5,19\}$ and so $17 \mid |H|$. We know that $L_2(19)$ contains a $19:9$ subgroup and hence G has a $\{3,17,19\}$-subgroup T which is solvable and so $t(T) \leq 2$. But, this is a contradiction, since $t(T) = 3$. Therefore, $K/H \ncong L_2(19)$.

(b) Let $\Gamma(G) = \Gamma(\mathrm{Aut}(M_{22}))$.

If G is a solvable group, then let T be a Hall $\{3,5,7\}$-subgroup of G. Obviously T is solvable and hence $t(T) \leq 2$, which is a contradiction. If G is a non-solvable Frobenius group, then G has a Frobenius kernel K and a Frobenius complement H. By using Lemma 2.3, it follows that H has a normal subgroup $H_0 = SL(2,5) \times Z$, where $|H : H_0| \leq 2$ and $(|Z|, 30) = 1$. Since, $5 \nsim 7$ and $3 \nsim 11$ in $\Gamma(G)$, it follows that $Z = 1$ and so $\pi(K) = \{7,11\}$, which is a contradiction since K is nilpotent and $7 \nsim 11$ in $\Gamma(G)$. Therefore, G is not a Frobenius group or a 2-Frobenius group. By using Lemma 2.5, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a $C_{11,11}$-simple group. If K/H is an alternating group or a sporadic simple group which is a $C_{11,11}$-group, then K/H is: A_{11}, A_{12}, M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, McL, HS, Sz, $O'N$, Co_2 or J_1. Also $\Gamma(K/H)$ is a subgraph of $\Gamma(G)$, by Remark 2.1. Therefore, $3 \nsim 5$ in $\Gamma(K/H)$ and $\pi(K/H) \subseteq \{2,3,5,7,11\}$, which implies that the only possibilities for K/H are $L_2(11)$, M_{11}, M_{12} and M_{22}. If $K/H \cong M_{11}$, M_{12} or $L_2(11)$, then K/H has a $11:5$ subgroup by [5]. Also in these cases $7 \notin \pi(K/H)$ and hence $7 \in \pi(H)$. Now, consider the $\{5,7,11\}$ subgroup T of G which is solvable and hence $t(T) \leq 2$, a contradiction. Therefore, $K/H \cong M_{22}$ and since $Out(M_{22}) \cong \mathbb{Z}_2$ it follows that $G/H \cong M_{22}$ or $M_{22}.2$. Also H is a nilpotent

π_1-group and so $\pi(H) \subseteq \{2,3,5,7\}$. By using [5] we know that M_{22} has a $11:5$ subgroup. If $3 \in \pi(H)$, then let T be a $\{3,5,11\}$ subgroup of G which is solvable and hence $t(T) \le 2$, which is a contradiction, since there exists any edge between 3, 5 and 11 in $\Gamma(G)$. Therefore, $3 \notin \pi(H)$. Similarly, it follows that $7 \notin \pi(H)$. Let $5 \in \pi(H)$ and $Q \in Syl_5(H)$. Also let $P \in Syl_3(K)$. We know that H is nilpotent and hence Q *char* H. Since $H \lhd K$ it follows that $Q \lhd K$. Therefore P acts by conjugation on Q and since $3 \nsim 5$ in $\Gamma(G)$ it follows that P acts fixed point freely on Q. Hence, QP is a Frobenius group with Frobenius kernel Q and Frobenius complement P. Now by using Lemma 2.3 it follows that P is a cyclic group which implies that a Sylow 3-subgroup of M_{22} is cyclic. But, this is a contradiction since a 3-Sylow subgroup of M_{22} are elementary abelian by [5]. Therefore, H is a 2-group. Then $G/O_2(G) \cong M_{22}$, where $O_2(G) \neq 1$ or $G/O_\pi(G) \cong M_{22}.2$, where $\pi \subseteq \{2\}$.

(C) Let $\Gamma(G) = \Gamma(\mathrm{Aut}(HS))$.

If G is solvable, then G has a Hall $\{3,7,11\}$-subgroup T. Hence, T is solvable and so $t(T) \le 2$, which is a contradiction. Hence, G is not a 2-Frobenius group. If G is a non-solvable Frobenius group, then by using Lemma 2.3, H, the Frobenius complement of G, has a normal subgroup $H_0 = SL(2,5) \times Z$, where $(|Z|, 30) = 1$ and $|H : H_0| \le 2$. Since, $5 \nsim 7$ and $5 \nsim 11$ in $\Gamma(G)$, it follows that $Z = 1$ and hence 77 is a divisor of $|K|$, where K is the Frobenius kernel of G. But, this is a contradiction. Since, $7 \nsim 11$ in $\Gamma(G)$ and K is nilpotent.

Now, similar to (b), G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is one of the following groups: M_{11}, M_{12}, M_{22}, McL, HS, $U_5(2)$, $U_6(2)$ and $L_2(11)$.

Case 1. Let $K/H \cong M_{11}$, M_{12}, $U_5(2)$ or $L_2(11)$.

By using [5] we know that $|Out(K/H)|$ is a divisor of 2. Therefore, $7 \notin \pi(G/H)$, and hence $7 \in \pi(H)$. Since in each case, K/H has a $11:5$ subgroup it follows that G has a $\{5,7,11\}$ subgroup T, which is solvable and hence $t(T) \le 2$. But, this is a contradiction and so this case is impossible.

Case 2. Let $K/H \cong M_{22}$.

We note that $out(M_{22}) \cong \mathbb{Z}_2$. Hence, $G/H \cong M_{22}$ or $M_{22}.2$. First let $G/H \cong M_{22}$, where H is a π_1-group and $\pi_1 = \{2,3,5,7\}$. We know that M_{22} has a $11:5$ subgroup (see [5]). If $2 \in \pi(H)$, then G has a $\{2,5,11\}$ subgroup T which is solvable and hence $t(T) \le 2$, a contradiction. Therefore, $2 \notin \pi(H)$. If $3 \in \pi(H)$ or $7 \in \pi(H)$, then let T be a $\{3,5,11\}$ or $\{5,7,11\}$ subgroup of G, respectively. Then $t(T) \le 2$, which is a contradiction. If $5 \in \pi(H)$, then let P be a Sylow 5-subgroup of H. If $Q \in Syl_3(G)$, then Q acts fixed point freely on P, since $3 \nsim 5$ in $\Gamma(G)$. Therefore, PQ is a Frobenius group which implies that Q be a cyclic group and it is a contradiction. Hence $H = 1$ and so $G = M_{22}$. But, $\Gamma(M_{22}) \neq \Gamma(\mathrm{Aut}(HS))$, since $2 \nsim 5$ in $\Gamma(M_{22})$. Therefore, this case is impossible.

Now, let $G/H \cong M_{22}.2$. By using [5], M_{22} has a $11:5$ subgroup. Similar to the above discussion we conclude that $\{3,5,7\} \cap \pi(H) = \emptyset$, and hence H is a 2-group. But, in this case 3 and 5 are not joined which is a contradiction. Therefore, Case 2 is impossible, too.

Case 3. Let $K/H \cong U_6(2)$.

By using [5], it follows that $Out(K/H) \cong S_3$. We know that $U_6(2).3$ has an element of order 21. Therefore, $G/H \cong U_6(2)$ or $U_6(2).2$. Also $7 \notin \pi(H)$, since $U_6(2)$ has a $11:5$ subgroup. Therefore, if $G/H \cong U_6(2)$, then $2 \in \pi$, $\pi \subseteq \{2,3,5\}$ and $G/O_\pi(G) \cong U_6(2)$, where $O_\pi(G) \neq 1$. Similarly, if $G/H \cong U_6(2).2$, then $G/O_\pi(G) \cong U_6(2).2$, where $\pi \subseteq \{2,3,5\}$.

Case 4. Let $K/H \cong McL$.

Note that $Out(McL) = 2$. But, $G/H \ncong McL.2$, since $McL.2$ has an element of order 22. Similar to the above proof it follows that $G/O_\pi(G) \cong McL$ and $\pi \subseteq \{2,3,5\}$, since McL has a $11:5$ subgroup.

Case 5. Let $K/H \cong HS$.

There exists a $11:5$ subgroup in HS. Similar to Case 3, it follows that $G/O_\pi(G) \cong HS$, where $2 \in \pi$, $\pi \subseteq \{2,3,5\}$ and $O_\pi(G) \neq 1$, or $G/O_\pi(G) \cong HS.2$, where $\pi \subseteq \{2,3,5\}$.

(d) Let $\Gamma(G) = \Gamma(\mathrm{Aut}(Fi'_{24}))$.

We claim that G is not solvable, otherwise let T be a Hall $\{7,17,23\}$-subgroup of G, which is solvable but $t(T) = 3$, a contradiction. If G is a non-solvable Frobenius group, then $\{11,13,17,23,29\} \subseteq \pi(K)$, where K is the Frobenius kernel of G. But, this is a contradiction since $11 \nsim 13$ and K is nilpotent. Hence, by using Lemma 2.5, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, where K/H is a $C_{29,29}$-simple group and $\pi(K/H) \subseteq \pi(G)$. Therefore, K/H is $L_2(29)$, Ru or Fi'_{24}. If $K/H \cong L_2(29)$ or Ru, then $\{17,23\} \subseteq \pi(H)$, which is a contradiction. Since, H is nilpotent and $17 \nsim 23$ in $\Gamma(G)$. Therefore, $K/H \cong Fi'_{24}$ and so $G/H \cong Fi'_{24}$ or $Fi'_{24}.2$. By using [5], we know that Fi'_{24} has a $23:11$ subgroup. Therefore, $\pi(H) \cap \{5,7,13,17\} = \emptyset$. Also Fi'_{24} has a $29:7$ subgroup, and hence $\pi(H) \cap \{11,13\} = \emptyset$. Therefore, $\pi(H) \subseteq \{2,3\}$ and so $G/O_\pi(G) \cong Fi'_{24}$, where $2 \in \pi$, $\pi \subseteq \{2,3\}$ and $O_\pi(G) \neq 1$; or $G/O_\pi(G) \cong Fi'_{24}.2$, where $\pi \subseteq \{2,3\}$.

(e) Let $\Gamma(G) = \Gamma(\mathrm{Aut}(O'N))$.

If G is solvable, then G has a Hall $\{3,11,31\}$-subgroup T, which has three components and this is a contradiction. If G is a non-solvable Frobenius group, then the Frobenius kernel of G has elements of order 7 and 11. But, $77 \notin \pi_e(G)$, which is a contradiction. Therefore, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, where K/H is a $C_{31,31}$-simple group and $\pi(K/H) \subseteq \pi(G)$. Hence, K/H is $L_3(5)$, $L_5(2)$, $L_6(2)$, $L_2(31)$, $L_2(32)$, $G_2(5)$ or $O'N$. If $K/H \cong L_2(5)$, $L_6(2)$, $L_2(31)$ or $G_2(5)$, then $11, 19 \in \pi(H)$, which is a contradiction. Since, $209 \notin \pi_e(G)$ and H is nilpotent. If $K/H \cong L_3(5)$ or $L_2(32)$, then $\{7,19\} \subseteq \pi(H)$, which is a contradiction. Since, $7 \nsim 19$ in $\Gamma(G)$. Therefore, $K/H \cong O'N$ and $Out(O'N) = 2$, which implies that $G/H \cong O'N$ or $O'N.2$. We know that $O'N$ has a $11 : 5$ subgroup by [5] and if we consider $\{5, 11, p\}$-subgroup of G, where $p \in \{7, 19, 31\}$, it follows that $\pi(H) \cap \{7, 19, 31\} = \emptyset$. Therefore, $\pi(H) \subseteq \{2, 3, 5, 11\}$. Also $O'N$ has a $19 : 3$ subgroup, which implies that $\pi(H) \cap \{11\} = \emptyset$. Let $p \in \{3, 5\}$. If $p \in \pi(H)$, then let P be the p-Sylow subgroup of H. If $Q \in Syl_7(G)$, then Q acts fixed point freely on P, since $7 \nsim 3$ and $7 \nsim 5$ in $\Gamma(G)$. Therefore, PQ is a Frobenius group and hence Q is a cyclic group. But, this is a contradiction. Since, Sylow 7-subgroups of $O'N$ are elementary abelian by [5]. Therefore, $\pi(H) \cap \{3, 5\} = \emptyset$. Hence, $\pi(H)$ is a 2-group. Then $G/O_2(G) \cong O'N$, where $O_2(G) \neq 1$; or $G/O_\pi(G) \cong O'N.2$ where $\pi \subseteq \{2\}$.

(f) Let $\Gamma(G) = \Gamma(\mathrm{Aut}(Suz))$.

Since, $7 \nsim 11$, $11 \nsim 13$ and $7 \nsim 13$, it follows that G is not a solvable group. If G is a 2-Frobenius group, then $\{11, 13\} \subseteq \pi(K)$, where K is the Frobenius kernel of G. Then $11 \sim 13$, since K is nilpotent. But, this is a contradiction. Therefore, G is neither a Frobenius group nor a 2-Frobenius group. Hence, there exists a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that K/H is a $C_{13,13}$ simple group and $\pi(K/H) \subseteq \pi(G)$. Therefore, K/H is $Sz(8)$, $U_3(4)$, ${}^3D_4(2)$, Suz, Fi_{22}, ${}^2F_4(2)'$, $L_2(27)$, $L_2(25)$, $L_2(13)$, $L_3(3)$, $L_4(3)$, $O_7(3)$, $O_8^+(3)$, $S_6(3)$, $G_2(4)$, $S_4(5)$ or $G_2(3)$.

If $K/H \cong {}^2F_4(2)'$, $U_3(4)$, $L_2(25)$, $L_4(3)$, $S_4(5)$ or $G_2(3)$, then $\{7, 11\} \subseteq \pi(H)$, which implies that $7 \sim 11$, since H is nilpotent. But, this is a contradiction. If $K/H \cong {}^3D_4(2)$, $L_2(27)$, $L_2(13)$ or $L_3(3)$, then $\{5, 11\} \subseteq \pi(H)$ and we get a contradiction similarly. Since, $5 \nsim 11$.

If $K/H \cong G_2(4)$, $S_6(3)$, $O_7(3)$ or $O_8^+(3)$, then $11 \in \pi(H)$ and K/H has a $13 : 3$ subgroup by [5]. Let T be a $\{3, 11, 13\}$-subgroup of G. It follows that $t(T) = 3$, which is a contradiction. Since, T is solvable.

If $K/H \cong Fi_{22}$, then $G/H \cong Fi_{22}$ or $Fi_{22}.2$, where $\pi(H) \subseteq \{2, 3, 5, 7, 11\}$. Since, Fi_{22} has $11 : 5$ and $13 : 3$ subgroups it follows that $\{7, 11\} \cap \pi(H) = \emptyset$. Therefore, $G/O_\pi(G) \cong Fi_{22}$ or $Fi_{22}.2$, where $\pi \subseteq \{2, 3, 5\}$.

Let $K/H \cong Sz(8)$. It is known that $Out(Sz(8)) \cong \mathbb{Z}_3$ and so $G/H \cong Sz(8)$ or $Sz(8).3$. If $G/H \cong Sz(8)$, then $\{3, 11\} \subseteq \pi(H)$ which is a contradiction. Since, $3 \nsim 11$. If $G/H \cong Sz(8).3$, then let T be $\{3, 7, 11\}$-subgroup of G. Since, $Sz(8)$ has a $7 : 6$ subgroup. Then $t(T) = 3$, which is a contradiction.

If $K/H \cong Suz$, then $G/H \cong Suz$ or $Suz.2$. If $G/K \cong Suz$, then $\pi(H) \subseteq \{2, 3, 5, 7, 11\}$. Since, Suz has a $11 : 5$ and $13 : 3$ subgroups it follows that $7, 11 \notin \pi(H)$. Therefore, $G/O_\pi(G) \cong Suz$, where $2 \in \pi$ and $\pi \subseteq \{2, 3, 5\}$ and $O_\pi(G) \neq 1$. If $G/H \cong Suz.2$, then it follows that $G/O_\pi(G) \cong Suz.2$, where $\pi \subseteq \{2, 3, 5\}$. $\square$

Remark 3.1. W. Shi and J. Bi in [29] put forward the following conjecture:

Let G be a group and M be a finite simple group. Then $G \cong M$ if and only if (i) $|G| = |M|$, and, (ii) $\pi_e(G) = \pi_e(M)$.

This conjecture is valid for sporadic simple groups [27], alternating groups and some simple groups of Lie type [28, 26, 29]. As a consequence of the main theorem, we prove the validity of this conjecture for the groups under discussion.

Theorem 3.2. *Let G be a finite group and A be an almost sporadic simple group, except* $\mathrm{Aut}(J_2)$ *and* $\mathrm{Aut}(McL)$. *If $|G| = |A|$ and $\pi_e(G) = \pi_e(A)$, then $G \cong A$.*

We note that Theorem 3.2 was proved in [14] by using the characterization of almost sporadic simple groups with their order components. Now, we give a new proof for this theorem. In fact we prove the following result which is a generalization of the Shi-Bi Conjecture and so Theorem 3.2 is an immediate consequence of Theorem 3.3. Also note that Theorem 3.3 is a generalization of a result in [1].

Theorem 3.3. *Let A be an almost sporadic simple group, except* $\mathrm{Aut}(J_2)$ *and* $\mathrm{Aut}(McL)$. *If G is a finite group satisfying $|G| = |A|$ and $\Gamma(G) = \Gamma(A)$, then $G \cong A$.*

Proof. First, let $A = \mathrm{Aut}(M_{22})$. By using Theorem 3.1, it follows that $G/O_2(G) \cong M_{22}$ or $G/O_\pi(G) \cong M_{22}.2$, where $\pi \subseteq \{2\}$. If $G/O_2(G) \cong M_{22}$, then $|O_2(G)| = 2$. Hence, $O_2(G) \subseteq Z(G)$ which is a contradiction. Since, G has more than one component and hence $Z(G) = 1$. Therefore, $G/O_\pi(G) \cong M_{22}.2$, where $2 \in \pi$, which implies that $O_\pi(G) = 1$ and hence $G \cong M_{22}.2$

Let $A = \mathrm{Aut}(HS)$. By using Theorem 3.1, it follows that $G/O_\pi(G) \cong U_6(2)$ or HS, where $2 \in \pi$, $\pi \subseteq \{2, 3, 5\}$ and $O_\pi(G) \neq 1$; or $G/O_\pi(G) \cong U_6(2).2$, McL or $HS.2$, where $\pi \subseteq \{2, 3, 5\}$.

By using [5], it follows that 3^6 divides the orders of $U_6(2)$, $U_6(2).2$ and McL, but $3^6 \nmid |G|$.

Therefore, $G/O_\pi(G) \cong HS$ or $HS.2$. Now, we get the result similarly to the last case.

For convenience we omit the details of the proof of other cases. □

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