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NOTE ON AN INEQUALITY OF F. QI AND L. DEBNATH

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Abstract. In this paper, a similar result of F. Qi and L. Debnath's inequality is given, and a generalization of Alzer's inequality is established.

1. Introduction

It is well-known that the following inequality

$$\frac{n}{n+1} < \left(\frac{(1/n)\sum_{i=1}^{n} i^r}{(1/(n+1))\sum_{i=1}^{n+1} i^r}\right)^{1/r} \leq \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}} \tag{1.1}$$

holds for $r > 0$ and $n \in \mathbb{N}$. We call the left-hand side of this inequality Alzer's inequality [2], and the right-hand side Martins's inequality [4].

Alzer's inequality has invoked the interest of several mathematicians, we refer the reader to [3, 4, 5, 6, 7] and the references therein.

In [6] F. Qi and L. Debnath gave a further generalization of (1.1), they proved the following result:

Theorem 1.1. *Let n and m be natural numbers. Suppose $\{a_1, a_2, \cdots\}$ is a positive and increasing sequence satisfying*

$$\frac{(k+2)a_{k+2}^r - (k+1)a_{k+1}^r}{(k+1)a_{k+1}^r - ka_k^r} \geq \left(\frac{a_{k+2}}{a_{k+1}}\right)^r \tag{1.2}$$

for any given positive real number r and $k \in \mathbb{N}$. Then we have the inequality

$$\frac{a_n}{a_{n+m}} \leq \left(\frac{(1/n)\sum_{i=1}^{n} a_i^r}{(1/(n+m))\sum_{i=1}^{n+m} a_i^r}\right)^{1/r}. \tag{1.3}$$

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Chen and F. Qi in [3] show that Alzer's inequality (1.1) is valid for $r < 0$.

Motivated by approach of [3], a natural question is does (1.3) still hold for $r < 0$. In this paper, we show that (1.3) is no longer valid for $r < 0$. But, we found another similar result.

2. Main results

Theorem 2.1. *Let n and m be natural numbers. Suppose $\{a_1, a_2, \cdots\}$ is a positive and decreasing sequence satisfying*

$$\frac{(k+2)a_{k+1}^s - (k+1)a_{k+2}^s}{(k+1)a_k^s - ka_{k+1}^s} \ge \left(\frac{a_{k+1}}{a_k}\right)^s, \tag{2.1}$$

for any given positive real number s and $k \in \mathbb{N}$. Then we have the inequality

$$\frac{a_{m+n}}{a_n} \le \left(\frac{(1/n)\sum_{i=1}^{n}\frac{1}{a_i^s}}{(1/(n+m))\sum_{i=1}^{n+m}\frac{1}{a_i^s}}\right)^{1/s}. \tag{2.2}$$

The lower bound of (2.2) is best possible.

Proof. The inequality (2.2) is equivalent to

$$\frac{a_{n+m}^s}{a_n^s} \le \frac{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{a_i^s}}{\frac{1}{n+m}\sum_{i=1}^{n+m}\frac{1}{a_i^s}}. \tag{2.3}$$

This is also equivalent to

$$\frac{a_{n+1}^s}{a_n^s} \le \frac{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{a_i^s}}{\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{a_i^s}}. \tag{2.4}$$

That is,

$$\frac{a_{n+1}^s}{(n+1)}\sum_{i=1}^{n+1}\frac{1}{a_i^s} \le \frac{a_n^s}{n}\sum_{i=1}^{n}\frac{1}{a_i^s}. \tag{2.5}$$

Since,

$$\sum_{i=1}^{n+1}\frac{1}{a_i^s} = \sum_{i=1}^{n}\frac{1}{a_i^s} + \frac{1}{a_{n+1}^s}. \tag{2.6}$$

Inequality (2.5) reduces to

$$\sum_{i=1}^{n}\frac{1}{a_i^s} \ge \frac{n}{(n+1)a_n^s - na_{n+1}^s}. \tag{2.7}$$

Since, $\{a_1, a_2, \cdots\}$ is a positive and decreasing sequence, it is easy to see that inequality (2.7) holds for $n = 1$.

Assume that (2.7) holds for $n > 1$. Using the principle of induction, it is easy to show that (2.7) holds for $n+1$. Using equality (2.6), the induction can be written as (2.1) for $k = n$. Thus, inequality (2.7) holds.

It can easily be shown that

$$\lim_{s\to+\infty}\left(\frac{(1/n)\sum_{i=1}^{n}\frac{1}{a_i^s}}{(1/(n+m))\sum_{i=1}^{n+m}\frac{1}{a_i^s}}\right)^{1/s}=\frac{a_{m+n}}{a_n} \tag{2.8}$$

Hence, the lower bound of (2.2) is best possible. The proof is complete. □

The following example shows that the sequence satisfying (2.1) is exists.

Example 2.2. Let $a_k=\frac{1}{k}, (k=1,2,\cdots)$, then

$$\frac{(k+2)a_{k+1}^s-(k+1)a_{k+2}^s}{(k+1)a_k^s-ka_{k+1}^s}=\frac{k^s}{(k+2)^s}\cdot\frac{(k+2)^{s+1}-(k+1)^{s+1}}{(k+1)^{s+1}-k^{s+1}}$$

Define function $f,g\colon [k,k{+}1]\to R$, where $f(x)=(x{+}1)^{s+1}, g(x)=x^{s+1}, s>0$. Applying the Cauchy's mean-value theorem, it turns out that there exists one point $\xi\in(k,k+1)$ such that

$$\begin{aligned}\frac{k^s}{(k+2)^s}\cdot\frac{(k+2)^{s+1}-(k+1)^{s+1}}{(k+1)^{s+1}-k^{s+1}}&=\frac{k^s}{(k+2)^s}\cdot\frac{f(k+1)-f(k)}{g(k+1)-g(k)}\\&=\frac{k^s}{(k+2)^s}\cdot\frac{f'(\xi)}{g'(\xi)}\\&=\frac{k^s}{(k+2)^s}\cdot\left(\frac{1+\xi}{\xi}\right)^s\\&\ge\frac{k^s}{(k+2)^s}\cdot\left(1+\frac{1}{k+1}\right)^s\\&=\left(\frac{k}{k+1}\right)^s=\left(\frac{a_{k+1}}{a_k}\right)^s\end{aligned}$$

Hence,

$$\frac{(k+2)a_{k+1}^s-(k+1)a_{k+2}^s}{(k+1)a_k^s-ka_{k+1}^s}\ge\left(\frac{a_{k+1}}{a_k}\right)^s$$

Let $a_k=\frac{1}{k}$ in Theorem 2.1, then we have

Corollary 2.1.

$$\frac{n}{n+m}<\left(\frac{(1/n)\sum_{i=1}^{n}i^s}{(1/(n+m))\sum_{i=1}^{n+m}i^r}\right)^{1/s}, \tag{2.9}$$

where $s>0$ *and* $m,n\in\mathbb{N}$.

Let $m=1$ in (2.9), then we get (1.1).

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