Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
25 (2009), 195–209
www.emis.de/journals
ISSN 1786-0091

ON DIFFERENTIAL SANDWICH THEOREMS FOR SOME SUBCLASS OF MULTIVALENT ANALYTIC FUNCTIONS

T. N. SHANMUGAM, J. PANDURANGAN, AND M. P. JEYARAMAN

ABSTRACT. In this present investigation we study certain application of differential subordination and superordination for the class of multivalent functions to be subordinated and superordinated by convex functions.

1. INTRODUCTION

Let $\mathcal{H}$ be the class of analytic functions in $\Delta := \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{H}[a, n]$ be a subclass of $\mathcal{H}$ consisting of functions of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots .$$

Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) := z^p + \sum_{n=p+1}^{\infty} a_n z^n \qquad (z \in \Delta), \tag{1.1}$$

and let $\mathcal{A} := \mathcal{A}_1$. Komatu [4] introduced the family of integral operators, defined by

$$I_a^\sigma f(z) := \frac{(1+a)^\sigma}{z^a \Gamma(\sigma)} \int_0^z \left(\log \frac{z}{t}\right)^{\sigma-1} t^{a-1} f(t) dt, \tag{1.2}$$

where $a > -1, \sigma > 0$ and $f \in \mathcal{A}$. It can be easily observed that

$$I_a^\sigma f(z) = z + \sum_{n=2}^{\infty} \left(\frac{1+a}{n+a}\right) a_n z^n. \tag{1.3}$$

From (1.2) and (1.3) it can be seen that

$$z(I_a^{\sigma+1} f(z))' = (1+a) I_a^\sigma f(z) - a I_a^{\sigma+1} f(z).$$

Let $p, h \in \mathcal{H}$ and let

$$\phi(r, s, t; z) \colon \mathbb{C}^3 \times \Delta \to \mathbb{C}.$$

2000 *Mathematics Subject Classification.* Primary 30C45, secondary 30C80.

Key words and phrases. Analytic functions, differential subordination, superordination, Komatu operator.

If $p(z)$ and $\phi(p(z), zp'(z), z^2p''(z); z)$ are univalent and if $p(z)$ satisfy the second order superordination

$$h(z) \prec \phi(p(z), zp'(z), z^2p''(z); z), \tag{1.4}$$

then p is the solution of the differential superordination (1.4). (If f is subordinate to F, then we say F is superordinate to f). An analytic function q is called a *subordinant* if $q \prec p$ for all p satisfying (1.4). A univalent subordinant $\widetilde{q}$ that satisfy $q \prec \widetilde{q}$ for all subordinants q of (1.4) is said to be best subordinant. Recently Miller and Mocanu [6] obtained conditions on h, q and ϕ for which the following implication holds:

$$h(z) \prec \phi(p(z), zp'(z), z^2p''(z); z) \Rightarrow q(z) \prec p(z).$$

Using the results of Miller and Mocanu [6], Bulboacă [3] have considered certain classes of first order differential superordinations as well as superordination preserving integral operators [2].

Over many years, several authors have studied the application of differential subordination and superordination for functionals like $\frac{zf'(z)}{f(z)}, 1 + \frac{zf''(z)}{f'(z)}, \frac{f(z)}{zf'(z)}$. Recently Obradović and Owa [7] obtained some subordination result in terms of $\left(\frac{f(z)}{z}\right)^{\mu}$.

In this present investigation we give some applications of first order differential subordination and superordination to obtain sufficient conditions for certain normalized analytic functions f to satisfy

$$q_1(z) \prec \frac{1}{p}\left(\frac{f(z)}{z^p}\right)^{\mu} \prec q_2(z)$$

where q_1 and q_2 are univalent in Δ. Interestingly various well known results are special cases of our results.

2. Preliminaries

For the present investigation we need the following definition and results.

Definition 2.1. [6, Definition 2, p. 817] Let $\mathcal{Q}$ be the set of all functions f that are analytic and injective on $\bar{\Delta} - E(f)$, where

$$E(f) = \left\{\zeta \in \partial\Delta : \lim_{z\to\zeta} f(z) = \infty\right\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\Delta - E(f)$.

Theorem 2.1 ([5, Theorem 3.4h, p. 132])**.** *Let q be univalent in the disk Δ and θ and ϕ be analytic in a domain D containing $q(\Delta)$ with $\phi(w) \neq 0$ when $w \in q(\Delta)$.*
Set $Q(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$. Suppose that

(1) *Q is starlike univalent in Δ and*
(2) *$\Re \frac{zh'(z)}{Q(z)} > 0$ for $z \in \Delta$.*

If ξ is analytic in Δ with $\xi(\Delta) \subseteq D$, and

$$\theta(\xi(z)) + z\xi'(z)\phi(\xi(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)), \tag{2.1}$$

then $\xi \prec q$ and q is the best dominant.

Theorem 2.2 ([3]). *Let q be univalent in the unit disk Δ and ϑ and ϕ be analytic in a domain D containing $q(\Delta)$, suppose that*

(1) $\Re \ \frac{\vartheta' q(z)}{\psi(q(z))} > 0$ *for all $z \in \Delta$ and*
(2) $\xi(z) = zq'(z)\psi(q(z))$ *is starlike univalent function in Δ.*

If $\xi \in \mathcal{H}[q(0), 1] \cap \mathcal{Q}$, with $\xi(\Delta) \subset D$, and $\vartheta(\xi(z)) + z\xi'(z)\psi(\xi(z))$ is univalent in Δ , and

$$\vartheta(q(z)) + zq'(z)\psi(q(z)) \prec \vartheta(\xi(z)) + z\xi'(z)\psi(\xi(z)), \tag{2.2}$$

then $q \prec \xi$ and q is the best subordinant.

Theorem 2.3 ([5, Lemma 1, p. 71]). *Let h be convex univalent in Δ with $h(0) = a$ and $0 \neq \gamma \in \mathbb{C}$ and $\Re\gamma > 0$. If $p \in \mathcal{H}[a, n]$ and*

$$p(z) + \frac{zp'(z)}{\gamma} \prec h(z)$$

then

$$p(z) \prec q(z) \prec h(z),$$

where

$$q(z) = \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z h(t)t^{\frac{\gamma}{n}-1}dt.$$

The function q is convex and is the best dominant.

3. Application of Differential Subordination

Theorem 3.1. *Let α, β and γ be complex numbers with $\gamma \neq 0$. Let q be convex univalent in Δ with $q(0) = 1$ and satisfy*

$$\Re\left\{\frac{\alpha + 2\beta q(z)}{\gamma}\right\} > 0. \tag{3.1}$$

Let $f \in \mathcal{A}_p$ and

$$\psi(z) := \frac{\alpha}{p}\left(\frac{f(z)}{z^p}\right)^{\mu} + \frac{\beta}{p^2}\left(\frac{f(z)}{z^p}\right)^{2\mu} + \gamma\mu\left(\frac{f(z)}{z^p}\right)^{\mu}\left[\frac{zf'(z)}{pf(z)} - 1\right]. \tag{3.2}$$

If

$$\psi(z) \prec \alpha q(z) + \beta q^2(z) + \gamma zq'(z)$$

then

$$\frac{1}{p}\left(\frac{f(z)}{z^p}\right)^{\mu} \prec q(z)$$

and q is the best dominant.

Proof. Define the function $\xi(z)$ by

$$\xi(z) := \frac{1}{p}\left(\frac{f(z)}{z^p}\right)^{\mu}. \tag{3.3}$$

A computation using (3.3) shows that

$$\frac{z\xi'(z)}{\xi(z)} = \frac{z\mu f'(z)}{f(z)} - \mu p.$$

Also we find that

$$\begin{aligned}\psi(z) &:= \alpha\xi(z) + \beta\xi^2(z) + \gamma z\xi'(z)\\ &= \frac{\alpha}{p}\left(\frac{f(z)}{z^p}\right)^{\mu} + \frac{\beta}{p^2}\left(\frac{f(z)}{z^p}\right)^{2\mu} + \gamma\mu\left(\frac{f(z)}{z^p}\right)^{\mu}\left[\frac{zf'(z)}{pf(z)} - 1\right].\end{aligned}$$

Since $\psi(z) \prec \alpha q(z) + \beta q^2(z) + \gamma zq'(z)$, this can be written as (2.1), when $\theta(w) := \alpha w + \beta w^2$ and $\phi(w) := \gamma$. Note that $\phi(w) \neq 0$ and $\theta(w), \phi(w)$ are analytic in $\mathbb{C}$. Set

$$\begin{aligned}Q(z) &:= \gamma zq'(z)\\ h(z) &:= \theta(q(z)) + Q(z)\\ &= \alpha q(z) + \beta q^2(z) + \gamma zq'(z).\end{aligned}$$

In light of the hypothesis of Theorem 2.1, we see that Q is starlike and

$$\Re\ \frac{zh'(z)}{Q(z)} = \Re\left\{\frac{\alpha + 2\beta q(z)}{\gamma} + (1 + \frac{zq''(z)}{q'(z)})\right\} > 0.$$

Hence the result follows as an application of Theorem 2.1. □

Theorem 3.2.

(1) *Let* $0 \neq \delta \in \mathbb{C}$ *and* q *be convex univalent in* Δ *with* $q(0) = 1$ *and satisfy*

$$\Re\left\{\frac{\mu}{\delta}\right\} > 0.$$

If $f \in \mathcal{A}$ *satisfy*

$$(1-\delta)\left(\frac{f(z)}{z}\right)^{\mu} + \delta\left(f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right) \prec q(z) + \frac{\delta}{\mu}zq'(z)$$

then

$$\left(\frac{f(z)}{z}\right)^{\mu} \prec q(z).$$

(2) *If* $f \in \mathcal{A}$ *satisfy*

$$f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} - \left(\frac{f(z)}{z}\right)^{\mu} \prec \frac{1}{\mu}zq'(z)$$

then

$$\left(\frac{f(z)}{z}\right)^{\mu} \prec q(z)$$

and q is the best dominant.

Proof. Proof of the first part follows from Theorem 3.1, by taking $\alpha = p = 1, \beta = 0$ and $\gamma = \frac{\delta}{\mu}$.

The proof of the second part follows from Theorem 3.1, by taking $\alpha = \beta = 0, p = 1$ and $\gamma = \frac{1}{\mu}$. □

By taking $\delta = \mu = n$ and $q(z) = \beta + (1-\beta)\left[-1 - \frac{2}{z}\log(1-z)\right]$ in first part of Theorem 3.2, we get the following result of Ponnusamy [8].

Corollary 3.3. *Let $f \in \mathcal{A}$. Then for a positive integer n, we have*

$$\Re\left\{(1-n)\left(\frac{f(z)}{z}\right)^n + nf'(z)\left(\frac{f(z)}{z}\right)^{n-1}\right\} > \beta$$

implies

$$\left(\frac{f(z)}{z}\right)^n \prec \beta + (1-\beta)(-1 - \frac{2}{z}\log(1-z))$$

and $\beta + (1-\beta)(-1 - \frac{2}{z}\log(1-z))$ is the best dominant.

By taking $\mu = 1$ and $q(z) = 1 + \left(\frac{A}{1+\delta}\right) z$ in Theorem 3.2 and $\mu = \delta = 1$ and $q(z) = \frac{A}{B} + \left(1 - \frac{A}{B}\right)\frac{\log(1+Bz)}{Bz}$ where δ, A and B are non zero complex numbers and $\Re\delta > 0$ and $-1 \leq B < A \leq 1$ in Theorem 3.2 we get the following result of Ponnusamy and Juneja [9].

Corollary 3.4. *Let $f \in \mathcal{A}$. Let δ be a complex number with $\Re\delta \geq 0$ and $-1 \leq B < A \leq 1$. Then*

$$(1-\delta)\frac{f(z)}{z} + \delta f'(z) \prec 1 + Az$$

implies

$$\frac{f(z)}{z} \prec 1 + \left(\frac{A}{1+\delta}\right) z$$

and the function $1 + \left(\frac{A}{1+\delta}\right) z$ is the best dominant. Also

$$f'(z) \prec \frac{1+Az}{1+Bz}$$

implies

$$\frac{f(z)}{z} \prec \frac{A}{B} + \left(1 - \frac{A}{B}\right)\frac{\log(1+Bz)}{Bz}$$

and the function $\frac{A}{B} + \left(1 - \frac{A}{B}\right)\frac{\log(1+Bz)}{Bz}$ is the best dominant.

By taking $\alpha = p = 1, \beta = 0$ and $\gamma = \frac{1}{\mu}$ in Theorem 3.1, we have the following result:

Corollary 3.5. *If $f \in \mathcal{A}$ satisfy*

$$f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} \prec q(z) + \frac{zq'(z)}{\mu}$$

implies

$$\left(\frac{f(z)}{z}\right)^{\mu} \prec q(z)$$

and q is the best dominant.

Theorem 3.6. *Let $\alpha, \beta, \gamma \in \mathbb{C}$ with $\gamma \neq 0$. Let q be convex univalent in Δ and $\frac{zq'(z)}{q(z)}$ be starlike univalent in Δ. Further assume that*

$$\Re\left\{\frac{\beta q(z)}{\gamma} - \frac{zq'(z)}{q(z)}\right\} > 0. \tag{3.4}$$

Let $f \in \mathcal{A}_p$ and if

$$\alpha + \frac{\beta}{p}\left(\frac{f(z)}{z^p}\right)^{\mu} + \gamma\mu\left[\frac{zf'(z)}{f(z)} - p\right] \prec \alpha + \beta q(z) + \frac{\gamma zq'(z)}{q(z)}$$

then

$$\frac{1}{p}\left(\frac{f(z)}{z^p}\right)^{\mu} \prec q(z)$$

where q is the best dominant.

Proof. Let $\theta(w) := \alpha + \beta w$ and $\phi(w) := \frac{\gamma}{w}$. Note that $\theta(w)$ and $\phi(w)$ are analytic in $\mathbb{C} \setminus \{0\}$. Hence the result follows as an application of Theorem 2.1 for $\xi(z) := \frac{1}{p}\left(\frac{f(z)}{z}\right)^{\mu}$. □

By taking $\alpha = p = 1, \beta = 0, \gamma = \frac{1}{\mu}$ and $q(z) = e^{\lambda A z}$, in Theorem 3.6 we get the following result obtained by Obradović and Owa [7].

Corollary 3.7. *Let $f \in \mathcal{A}$. If*

$$\frac{zf'(z)}{f(z)} \prec 1 + Az$$

then

$$\left(\frac{f(z)}{z}\right)^{\mu} \prec e^{\lambda A z},$$

where $e^{\lambda A z}$ is the best dominant.

We remark here that $q(z) = e^{\lambda A z}$ is univalent if and only if $|\lambda A| < \pi$.

For a special case when $q(z) = \frac{1}{(1-z)^{2b}}$ where $b \in \mathbb{C} \setminus \{0\}$, and $\alpha = \mu = p = 1, \beta = 0$ and $\gamma = \frac{1}{b}$ in Theorem 3.6, we have the following result obtained by the Srivatsava and Lashin [10].

Corollary 3.8. *Let $0 \neq b \in \mathbb{C}$. If $f \in \mathcal{A}$ and*

$$1+\frac{1}{b}\left[\frac{zf'(z)}{f(z)}-1\right] \prec \frac{1+z}{1-z}$$

then

$$\frac{f(z)}{z} \prec \frac{1}{(1-z)^{2b}},$$

where $\frac{1}{(1-z)^{2b}}$ is the best dominant.

By taking $q(z) = (1+Bz)^{\frac{\lambda(A-B)}{B}}$, $\alpha = p = 1, \beta = 0$ and $\gamma = \frac{1}{\mu}$ in Theorem 3.6, then we have the following result of Obradović and Owa [7].

Corollary 3.9. *Let $-1 \leq B < A \leq 1$. Let μ, A and B satisfy the relation either $\left|\frac{\lambda(A-B)}{B}-1\right| \leq 1$ or $\left|\frac{\lambda(A-B)}{B}+1\right| \leq 1$. If $f \in \mathcal{A}$ and*

$$\frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}$$

then

$$\left(\frac{f(z)}{z}\right)^{\mu} \prec (1+Bz)^{\frac{\lambda(A-B)}{B}}$$

and $(1+Bz)^{\frac{\lambda(A-B)}{B}}$ is the best dominant.

Theorem 3.10. *Let α, β and γ be complex numbers and $\gamma \neq 0$. Let $q(z)$ be univalent in Δ with $q(0) = 1$. Let $f \in \mathcal{A}_p$ satisfy (3.1). Let*

(3.5)
$$\psi(z) := \frac{\alpha}{p}\left(\frac{z^p}{f(z)}\right)^{\mu} + \frac{\beta}{p^2}\left(\frac{z^p}{f(z)}\right)^{2\mu} + \gamma\mu\left[\left(\frac{z^p}{f(z)}\right)^{\mu} - \frac{1}{p}\frac{zf'(z)}{f(z)}\left(\frac{z^p}{f(z)}\right)^{\mu}\right].$$

If

$$\psi(z) \prec \alpha q(z) + \beta q^2(z) + \gamma z q'(z)$$

then

$$\frac{1}{p}\left(\frac{z^p}{f(z)}\right)^{\mu} \prec q(z)$$

and q is the best dominant.

Proof. The proof is a straight forward application of Theorem 2.1. □

By putting $\alpha = p = 1, \beta = 0$ and $\gamma = \frac{\lambda}{\mu}$ in Theorem 3.10 we have the following result:

Corollary 3.11. *If $f(z) \in \mathcal{A}$ and*

$$(1+\lambda)\left(\frac{z}{f(z)}\right)^{\mu} - \lambda f'(z)\left(\frac{z}{f(z)}\right)^{\mu+1} \prec q(z) + \frac{\lambda}{\mu} z q'(z)$$

then

$$\left(\frac{z}{f(z)}\right)^{\mu} \prec q(z).$$

By taking $\lambda = -1$ in Corollary 3.11 we get the following result.

Corollary 3.12. *If $f \in \mathcal{A}$ and*

$$f'(z)\left(\frac{z}{f(z)}\right)^{\mu+1} \prec q(z) - \frac{zq'(z)}{\mu}$$

implies

$$\left(\frac{z}{f(z)}\right)^{\mu} \prec q(z)$$

and q is the best dominant.

4. Application of Superordination

Theorem 4.1. *Let α, β and γ be complex numbers and $\gamma \neq 0$. Let q be convex univalent in Δ with $q(0) = 1$ and satisfies*

$$\Re\left\{\left(\frac{\alpha + 2\beta q(z)}{\gamma}\right)\right\} > 0. \tag{4.1}$$

Let

$$\psi(z) := \frac{\alpha}{p}\left(\frac{f(z)}{z^p}\right)^{\mu} + \frac{\beta}{p^2}\left(\frac{f(z)}{z^p}\right)^{2\mu} + \gamma\mu\left(\frac{f(z)}{z^p}\right)^{\mu}\left[\frac{zf'(z)}{f(z)} - 1\right]$$

and is univalent in Δ. If $f \in \mathcal{A}_p$, $0 \neq \frac{1}{p}\left(\frac{f(z)}{z^p}\right)^{\mu} \in \mathcal{H}[1,1] \cap \mathcal{Q}$ then

$$\alpha q(z) + \beta q^2(z) + \gamma z q'(z) \prec \psi(z)$$

implies

$$q(z) \prec \frac{1}{p}\left(\frac{f(z)}{z^p}\right)^{\mu}$$

where q is the best subordinant.

Proof. Define the function $\xi(z)$ by

$$\xi(z) := \frac{1}{p}\left(\frac{f(z)}{z^p}\right)^{\mu}. \tag{4.2}$$

A computation using (4.2) shows that

$$\frac{z\xi'(z)}{\xi(z)} = \frac{z\mu f'(z)}{f(z)} - p\mu.$$

Note that

$$\begin{aligned}\psi(z) &:= \frac{\alpha}{p}\left(\frac{f(z)}{z^p}\right)^{\mu} + \frac{\beta}{p^2}\left(\frac{f(z)}{z^p}\right)^{2\mu} + \gamma\mu\left(\frac{f(z)}{z^p}\right)^{\mu}\left[\frac{zf'(z)}{f(z)} - 1\right] \\ &= \alpha\xi(z) + \beta\xi^2(z) + \gamma z\xi'(z).\end{aligned}$$

Since $\alpha q(z) + \beta q^2(z) + \gamma z q'(z) \prec \psi(z)$, this can be written as (2.2), when $\theta(w) := \alpha w + \beta w^2$ and $\phi(w) := \gamma$. Hence the result follows as an application of Theorem 2.2. $\square$

Corollary 4.2. *Let $f \in \mathcal{A}$ and δ be complex number with $\Re\delta > 0$ and $-1 \leq B < A \leq 1$.*
(i) If $(1-\delta)\frac{f(z)}{z} + \delta z f'(z)$ is univalent in Δ, then

$$1 + Az \prec (1-\delta)\frac{f(z)}{z} + \delta z f'(z) \Rightarrow 1 + \frac{A}{1+\delta}z \prec \frac{f(z)}{z}$$

and $\frac{A}{1+\delta}z$ is the best subordinant.
(ii) If $f'(z)$ is univalent in Δ then

$$\frac{1+Az}{1+Bz} \prec f'(z) \Rightarrow \frac{A}{B} + (1 - \frac{A}{B})\frac{\log(1+Bz)}{Bz} \prec \frac{f(z)}{z}$$

and $\frac{A}{B} + (1-\frac{A}{B})\frac{\log(1+Bz)}{Bz}$ is the best subordinant.

Proof. Proof of first part follows from Theorem 4.1 by taking $\alpha = p = 1$, $\beta = 0$, $\gamma = \delta$, $\mu = 1$ and $q(z) := 1 + \frac{A}{1+\delta}z$.

Proof of the second part follows from Theorem 4.1 by taking $\alpha = p = 1$, $\beta = 0$, $\gamma = \delta = 1$, $\mu = 1$ and $q(z) := \frac{A}{B} + (1-\frac{A}{B})\frac{\log(1+Bz)}{Bz}$. □

Theorem 4.3. *Let α, β and γ be complex numbers and $\gamma \neq 0$. Let q be convex univalent in Δ and $\frac{zq'(z)}{q(z)}$ be starlike univalent in Δ. Further assume that*

$$\Re\left\{\frac{\beta q(z)}{\gamma}\right\} > 0. \tag{4.3}$$

Let $\alpha + \frac{\beta}{p}\left(\frac{f(z)}{z^p}\right)^\mu + \gamma\mu\left[\frac{zf'(z)}{f(z)} - p\right]$ is univalent in Δ. If $f \in \mathcal{A}_p$, $\frac{1}{p}\left(\frac{f(z)}{z^p}\right)^\mu \in \mathcal{H}[1,1] \cap \mathcal{Q}$ then

$$\alpha + \beta q(z) + \frac{\gamma z q'(z)}{q(z)} \prec \alpha + \frac{\beta}{p}\left(\frac{f(z)}{z^p}\right)^\mu + \gamma\mu\left[\frac{zf'(z)}{f(z)} - p\right]$$

implies

$$q(z) \prec \frac{1}{p}\left(\frac{f(z)}{z^p}\right)^\mu$$

where q is the best subordinant.

Proof. Let $\theta(w) := \alpha + \beta w$ and $\phi(w) := \frac{\gamma}{w}$. Note that $\theta(w)$ and $\phi(w)$ are analytic in $\mathbb{C} \setminus \{0\}$. Hence the result follows as an application of Theorem 2.2, when $\xi(z) := \frac{1}{p}\left(\frac{f(z)}{z^p}\right)^\mu$. □

Note that by taking $\alpha = p = 1, \beta = 0, \gamma = \frac{1}{\mu}$ and $q(z) := e^{\lambda Az}$ we get the corresponding superordination result of Corollary 3.7. Also by taking $\alpha = \mu = p = 1, \beta = 0, \gamma = \frac{1}{b}$ and $q(z) := \frac{1}{(1-z)^{2b}}$ we obtain the superordination result of Corollary 3.8

Theorem 4.4. *Let $\alpha, \beta, \gamma \in \mathbb{C}$ and $\gamma \neq 0$. Let q be convex univalent in Δ with $q(0) = 1$ and satisfy*

$$\Re\left\{\left(\frac{\alpha + 2\beta q(z)}{\gamma}\right)\right\} > 0.$$

Let $\psi(z)$ as defined by (3.5) be univalent in Δ. If $f \in \mathcal{A}_p$ and $0 \neq \frac{1}{p}\left(\frac{z^p}{f(z)}\right)^\mu \in \mathcal{H}[1,1] \cap \mathcal{Q}$ then

$$\alpha q(z) + \beta q^2(z) + \gamma z q'(z) \prec \psi(z)$$

implies

$$q(z) \prec \frac{1}{p}\left(\frac{z^p}{f(z)}\right)^\mu$$

where q is the best subordinant.

By letting $\alpha = p = 1, \beta = 0$ and $\gamma = \frac{\lambda}{\mu}$ in Theorem 4.4, we get the following result:

Corollary 4.5. *Let $0 \neq \lambda \in \mathbb{C}$. Let q be convex univalent in Δ with $q(0) = 1$ and satisfy*

$$\Re\left\{\frac{\mu}{\lambda} q'(z)\right\} > 0.$$

Let

$$\psi_1(z) := (1+\lambda)\left(\frac{z}{f(z)}\right)^\mu - \lambda f'(z)\left(\frac{z}{f(z)}\right)^{\mu+1}$$

be univalent in Δ. If $f \in \mathcal{A}$ and $\left(\frac{z}{f(z)}\right)^\mu \in \mathcal{H}[1,1] \cap \mathcal{Q}$ then

$$q(z) + \frac{\lambda}{\mu} q'(z) \prec \psi_1(z)$$

implies that

$$q(z) \prec \left(\frac{z}{f(z)}\right)^\mu$$

and q is the best subordinant.

By taking $\lambda = -1$ in Corollary 4.5 we get the following result:

Corollary 4.6. *If $f \in \mathcal{A}$ and*

$$q(z) - \frac{q'(z)}{\mu} \prec f'(z)\left(\frac{z}{f(z)}\right)^\mu$$

implies

$$q(z) \prec \left(\frac{z}{f(z)}\right)^\mu$$

and q is the best subordinant.

5. Sandwich Results

By combining Theorem 3.1 and Theorem 4.1 we get the following sandwich type result.

Theorem 5.1. *Let q_1 and q_2 be convex univalent in Δ, satisfying (4.1) and (3.1) respectively. Let $\psi(z)$ as given by (3.2) be univalent in Δ. If $0 \neq \frac{1}{p}\left(\frac{f(z)}{z^p}\right)^{\mu} \in \mathcal{H}[1,1] \cap \mathcal{Q}$ then*

$$\alpha q_1(z) + \beta q_1^2(z) + \gamma z q_1'(z) \prec \psi(z) \prec \alpha q_2(z) + \beta q_2^2(z) + \gamma z q_2'(z)$$

implies

$$q_1(z) \prec \frac{1}{p}\left(\frac{f(z)}{z^p}\right)^{\mu} \prec q_2(z)$$

where q_1 and q_2 are respectively the best subordinant and best dominant.

Now by combining Theorem 3.6 and Theorem 4.3 with $p = 1$ we have the following result.

Theorem 5.2. *Let q_1 and q_2 be convex univalent in Δ, satisfying (4.3) and (3.4) respectively. Suppose $\frac{zq_i'(z)}{q_i(z)}$ be starlike univalent in Δ for $i = 1, 2$. Let*

$$\eta(z) := \alpha + \beta\left(\frac{f(z)}{z}\right)^{\mu} + \gamma\mu\left(\frac{zf'(z)}{f(z)} - 1\right)$$

be univalent in Δ. If $0 \neq \left(\frac{f(z)}{z}\right)^{\mu} \in \mathcal{H}[1,1] \cap \mathcal{Q}$ then

$$\alpha + \beta q_1(z) + \gamma\frac{zq_1'(z)}{q_1(z)} \prec \eta(z) \prec \alpha + \beta q_2(z) + \gamma\frac{zq_2'(z)}{q_2(z)}$$

implies

$$q_1(z) \prec \left(\frac{f(z)}{z}\right)^{\mu} \prec q_2(z)$$

where q_1 and q_2 are respectively best subordinant and best dominant.

Theorem 5.3. *Let q_1 and q_2 be convex univalent satisfying (4.1) and (3.1) respectively. Let $0 \neq \left(\frac{f(z)}{z}\right)^{\mu} \in \mathcal{H}[1,1] \cap \mathcal{Q}$.*

(i) Let $f \in \mathcal{A}$, and $(1-\delta)\left(\frac{f(z)}{z}\right)^{\mu} + \delta f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}$ is univalent in Δ then

$$\frac{1+(1-2\beta_1)z}{1-z} \prec (1-\delta)\left(\frac{f(z)}{z}\right)^{\mu} + \delta f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} \prec \frac{1+(1-2\beta_2)z}{1-z}$$

implies

$$1 + (-1-\beta_1)(1 - \frac{2}{z}\log(1-z) \prec \left(\frac{f(z)}{z}\right)^{\mu} \prec 1 + (-1-\beta_2)(1 - \frac{2}{z}\log(1-z)$$

where $1+(-1-\beta_1)(1-\frac{2}{z}\log(1-z)$ *and* $1+(-1-\beta_2)(1-\frac{2}{z}\log(1-z)$ *are respectively the best subordinant and best dominant.*
(ii) If $(1-\delta)\frac{f(z)}{z}+\delta f'(z)$ *is univalent in* Δ*, then*

$$1+A_1z \prec (1-\delta)\frac{f(z)}{z}+\delta f'(z) \prec 1+A_2z$$

implies

$$1+\frac{A_1}{1+\delta}z \prec \frac{f(z)}{z} \prec 1+\frac{A_2}{1+\delta}z$$

where $1+\left(\frac{A_1}{1+\delta}\right)z$ *and* $1+\left(\frac{A_2}{1+\delta}\right)z$ *are respectively the best subordinant and best dominant.*

Proof. The proof of the first part follows from Theorem 5.1 by taking $q_i(z) = 1+(1-\beta_i)(-1-2\log(1-z)$ for $i = 1, 2$ and by taking $\alpha = p = 1, \beta = 0$ and $\gamma = \frac{\delta}{\mu}$ and the proof of second part follows by taking $q_i(z) = 1+\left(\frac{A_i}{1+\delta}\right)z$ for $i = 1, 2$ and by taking $\alpha = \mu = p = 1, \beta = 0$ and $\gamma = \frac{\delta}{\mu}$. □

In a similar manner we may obtain the sandwich result by combining Theorem 4.4 and Theorem 3.10.

6. Application to Komatu operator

Theorem 6.1. *Let* $h \in \mathcal{H}, h(0) = 1, h'(0) \neq 0$ *and satisfy*

$$\Re\left\{1+\frac{zh''(z)}{h'(z)}\right\} > -\frac{1}{2} \qquad (z \in \Delta).$$

If $f \in \mathcal{A}_m$ *satisfy the differential subordination*

$$\frac{I_a^\sigma f(z)}{z} \prec h(z)$$

then

$$\frac{I_a^{\sigma+1} f(z)}{z} \prec g(z) \tag{6.1}$$

where

$$g(z) := \frac{1+a}{mz^{\frac{1+a}{m}}}\int_0^z h(t)t^{\frac{1+a}{m}-1}dt.$$

The function g *is convex and is the best dominant.*

Proof. Let the function $p(z)$ be defined by

$$p(z) := \frac{I_a^{\sigma+1} f(z)}{z}.$$

A simple computation shows that

$$\frac{zp'(z)}{p(z)} = \left[\frac{z(I_a^{\sigma+1} f(z))'}{I_a^{\sigma+1} f(z)} - 1\right].$$

By using the identity
$$z(I_a^{\sigma+1}f(z))' = (1+a)I_a^{\sigma}f(z) - aI_a^{\sigma+1}f(z),$$
we have
$$\frac{zp'(z)}{p(z)} = \left[\frac{(1+a)I_a^{\sigma}f(z)}{I_a^{\sigma+1}f(z)} - (a+1)\right],$$
and hence
$$p(z) + \frac{zp'(z)}{a+1} = \frac{I_a^{\sigma}f(z)}{z}.$$
The assertion (6.1) of Theorem 6.1 follows by an application of Theorem 2.3. □

Theorem 6.2. *Let the function $q(z)$ be convex univalent in Δ and $q(z) \neq 0$. Suppose that $\frac{zq'(z)}{q(z)}$ is starlike univalent in Δ and $q(z)$ satisfy*

$$\Re\left\{\frac{\beta}{\delta}q(z) + \frac{2\gamma}{\delta}q^2(z) - \frac{zq'(z)}{q(z)}\right\} > 0 \tag{6.2}$$

and let

$$\chi(z) := \alpha + \beta\left(\frac{I_a^{\sigma+1}f(z)}{z}\right)^{\mu} + \gamma\left(\frac{I_a^{\sigma+1}f(z)}{z}\right)^{2\mu} + \delta\mu(1+a)\left[\frac{I_a^{\sigma}f(z)}{I_a^{\sigma+1}f(z)} - 1\right]. \tag{6.3}$$

If

$$\chi(z) \prec \alpha + \beta q(z) + \gamma q^2(z) + \frac{\delta zq'(z)}{q(z)} \tag{6.4}$$

then
$$\left(\frac{I_a^{\sigma+1}f(z)}{z}\right)^{\mu} \prec q(z)$$
and q is the best dominant.

Proof. Define the function $p(z)$ by
$$p(z) := \left(\frac{I_a^{\sigma+1}f(z)}{z}\right)^{\mu}.$$

Note the function $p(z)$ is analytic in Δ. By a straight forward computation we have
$$\chi(z) := \alpha + \beta\left(\frac{I_a^{\sigma+1}f(z)}{z}\right)^{\mu} + \gamma\left(\frac{I_a^{\sigma+1}f(z)}{z}\right)^{2\mu} + \delta\mu(1+a)\left[\frac{I_a^{\sigma}f(z)}{I_a^{\sigma+1}f(z)} - 1\right]$$

$$= \alpha + \beta p(z) + \gamma p^2(z) + \frac{\delta zp'(z)}{p(z)}. \tag{6.5}$$

By using (6.5) in subordination (6.4), we have

$$\alpha+\beta p(z)+\gamma p^2(z)+\frac{\delta zp'(z)}{p(z)}\prec\alpha+\beta q(z)+\gamma q^2(z)+\frac{\delta zq'(z)}{q(z)}. \tag{6.6}$$

The subordination (6.6) is same as (2.1) with $\theta(w):=\alpha+\beta w+\gamma w^2$ and $\phi(w):=\frac{\delta}{w}$. Clearly $\theta(w)$ and $\phi(w)$ are analytic in $\mathbb{C}\setminus\{0\}$. Hence the result follows as an application of Theorem 2.1. $\square$

Theorem 6.3. *Let $q(z)$ be convex univalent in Δ and $\frac{zq'(z)}{q(z)}$ be starlike univalent in Δ. Further assume that*

$$\Re\left\{\frac{2\gamma}{\delta}q^2(z)+\frac{\beta}{\delta}q(z)\right\}>0.$$

Let $\chi(z)$ as defined by (6.3), is univalent in Δ. If $f(z)\in\mathcal{A}, 0\neq\left(\frac{I_a^{\sigma+1}f(z)}{z}\right)^\mu\in\mathcal{H}[1,1]\cap\mathcal{Q}$ then

$$\alpha+\beta q(z)+\gamma q^2(z)+\frac{\delta zq'(z)}{q(z)}\prec\chi(z)$$

implies

$$q(z)\prec\left(\frac{I_a^{\sigma+1}f(z)}{z}\right)^\mu$$

and q is the best subordinant.

By combining Theorem 6.2 and Theorem 6.3 we obtain the sandwich result, however the details are omitted.

References

[1] R. M. Ali, V. Ravichandran, M. H. Khan, and K. G. Subramanian. Differential sandwich theorems for certain analytic functions. *Far East J. Math. Sci. (FJMS)*, 15(1):87–94, 2004.

[2] T. Bulboacă. A class of superordination-preserving integral operators. *Indag. Math. (N.S.)*, 13(3):301–311, 2002.

[3] T. Bulboacă. Classes of first-order differential superordinations. *Demonstratio Math.*, 35(2):287–292, 2002.

[4] Y. Komatu. *Distortion theorems in relation to linear integral operators*, volume 385 of *Mathematics and its Applications*. Kluwer Academic Publishers Group, Dordrecht, 1996.

[5] S. S. Miller and P. T. Mocanu. *Differential subordinations*, volume 225 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker Inc., New York, 2000. Theory and applications.

[6] S. S. Miller and P. T. Mocanu. Subordinants of differential superordinations. *Complex Var. Theory Appl.*, 48(10):815–826, 2003.

[7] M. Obradović and S. Owa. On certain properties for some classes of starlike functions. *J. Math. Anal. Appl.*, 145(2):357–364, 1990.

[8] S. Ponnusamy. On Bazilevic functions. *Ann. Univ. Mariae Curie-Skłodowska Sect. A*, 42:115–127, 1988.

[9] S. Ponnusamy and O. P. Juneja. Some applications of first order differential subordinations. *Glas. Mat. Ser. III*, 25(45)(2):287–296, 1990.

[10] H. M. Srivastava and A. Y. Lashin. Some applications of the Briot-Bouquet differential subordination. *JIPAM. J. Inequal. Pure Appl. Math.*, 6(2):Article 41, 7 pp. (electronic), 2005.

Received on October 12, 2006; revised on June 8, 2009; accepted on June 13, 2009

Department of Mathematics
Anna University
Chennai - 600 025 India
E-mail address: `shan@annauni.edu`

Department of Mathematics
Anna University, M I T campus
Chennai - 600 044 India

Department of Mathematics
Easwari Engineering College
Ramapuram, Chennai - 600089 India
E-mail address: `jeyaraman_mp@yahoo.co.in`