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SUBORDINATION RESULTS FOR CERTAIN CLASSES OF ANALYTIC FUNCTIONS

ZHI-GANG WANG AND R. AGHALARY

Abstract. In the present paper, we derive some subordination results for certain classes of analytic functions by making use of a subordination theorem. Relevant connections of the results presented here with those obtained in earlier works are also pointed out.

1. Introduction and Preliminaries

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j, \tag{1.1}$$

which are *analytic* in the *open* unit disk

$$\mathbb{U} := \{z : \ z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

Also let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of all functions which are univalent in $\mathbb{U}$.

For $0 \leqq \alpha < 1$, we denote by $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ the usual subclasses of $\mathcal{S}$ consisting of functions which are starlike of order α and convex of order α in $\mathbb{U}$, respectively, that is,

$$\mathcal{S}^*(\alpha) := \left\{ f : \ f \in \mathcal{A} \quad \text{and} \quad \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \ z \in \mathbb{U} \right\},$$

and

$$\mathcal{K}(\alpha) := \left\{ f : \ f \in \mathcal{A} \quad \text{and} \quad \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \ z \in \mathbb{U} \right\}.$$

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Obviously, for any $0 \leqq \alpha < 1$, we have

$$f \in \mathcal{K}(\alpha) \Longleftrightarrow zf' \in \mathcal{S}^*(\alpha).$$

Let $\mathcal{T}(\lambda, \alpha)$ denote the class of functions in $\mathcal{A}$ satisfying the following inequality:

$$\Re\left(\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda z f'(z)}\right) > \alpha, \quad z \in \mathbb{U}$$

for some α $(0 \leqq \alpha < 1)$ and λ $(0 \leqq \lambda \leqq 1)$, and let $\mathcal{C}(\lambda, \alpha)$ denote the class of functions in $\mathcal{A}$ satisfying the following inequality:

$$\Re\left(z\frac{\lambda z^2 f'''(z) + (2\lambda+1)zf''(z) + f'(z)}{\lambda z^2 f''(z) + zf'(z)}\right) > \alpha, \quad z \in \mathbb{U}$$

for some α $(0 \leqq \alpha < 1)$ and λ $(0 \leqq \lambda \leqq 1)$. We note that

$$f \in \mathcal{C}(\lambda, \alpha) \Longleftrightarrow zf' \in \mathcal{T}(\lambda, \alpha).$$

The classes $\mathcal{T}(\lambda, \alpha)$ and $\mathcal{C}(\lambda, \alpha)$ were introduced and investigated by Altintas [1], and Kamali and Akbulut [2], respectively.

Let $\mathcal{M}(\beta)$ be the subclass of $\mathcal{A}$ consisting of functions f which satisfy the inequality:

$$\Re\left(\frac{zf'(z)}{f(z)}\right) < \beta, \quad z \in \mathbb{U}$$

for some β $(\beta > 1)$, and let $\mathcal{N}(\beta)$ be the subclass of $\mathcal{A}$ consisting of functions f which satisfy the inequality:

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) < \beta, \quad z \in \mathbb{U}$$

for some β $(\beta > 1)$. The classes $\mathcal{M}(\beta)$ and $\mathcal{N}(\beta)$ were introduced and investigated recently by Owa and Srivastava [5] (see also Nishiwaki and Owa [3], Owa and Nishiwaki [4], Srivastava and Attiya [7]).

Salagean [6] introduced the following operator:

$$D^0 f(z) = f(z), \qquad D^1 f(z) = Df(z) = zf'(z),$$

and

$$D^n f(z) = D(D^{n-1} f(z)) \qquad (n \in \mathbb{N} := \{1, 2, \ldots\}).$$

We note that

$$D^n f(z) = z + \sum_{j=2}^{\infty} j^n a_j z^j \qquad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}).$$

Motivated by the above mentioned function classes, we now introduce the following subclasses of $\mathcal{A}$ involving the Salagean operator.

Definition 1. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}_n(\lambda, \alpha)$ if it satisfies the following inequality:

$$\Re\left(\frac{(1-\lambda)D^{n+1}f(z)+\lambda D^{n+2}f(z)}{(1-\lambda)D^{n}f(z)+\lambda D^{n+1}f(z)}\right) > \alpha, \quad z \in \mathbb{U}, \tag{1.2}$$

where

$$n \in \mathbb{N}_0,\ 0 \leqq \alpha < 1 \quad \text{and} \quad 0 \leqq \lambda \leqq 1.$$

It is easy to see that the classes $\mathcal{T}(\lambda, \alpha)$ and $\mathcal{C}(\lambda, \alpha)$ are special cases of the class $\mathcal{S}_n(\lambda, \alpha)$.

Definition 2. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{M}_n(\lambda, \beta)$ if it satisfies the following inequality:

$$\Re\left(\frac{(1-\lambda)D^{n+1}f(z)+\lambda D^{n+2}f(z)}{(1-\lambda)D^{n}f(z)+\lambda D^{n+1}f(z)}\right) < \beta, \quad z \in \mathbb{U},$$

where

$$n \in \mathbb{N}_0,\ \beta > 1 \quad \text{and} \quad 0 \leqq \lambda \leqq 1.$$

It is also easy to see that the classes $\mathcal{M}(\beta)$ and $\mathcal{N}(\beta)$ are special cases of the class $\mathcal{M}_n(\lambda, \beta)$.

We now provide some coefficient sufficient conditions for functions belonging to the classes $\mathcal{S}_n(\lambda, \alpha)$ and $\mathcal{M}_n(\lambda, \beta)$, which will be used in the proofs of our main theorems.

Lemma 1. *Let $0 \leqq \alpha < 1$ and $0 \leqq \lambda \leqq 1$. If $f \in \mathcal{A}$ satisfies the following coefficient inequality:*

$$\sum_{j=2}^{\infty}(j^{n+1}-\alpha j^n)(1-\lambda+\lambda j)|a_j| \leqq 1-\alpha, \tag{1.3}$$

then $f \in \mathcal{S}_n(\lambda, \alpha)$.

Proof. To prove the claim, it suffices to show that

$$\left|\frac{(1-\lambda)D^{n+1}f(z)+\lambda D^{n+2}f(z)}{(1-\lambda)D^{n}f(z)+\lambda D^{n+1}f(z)} - 1\right| < 1-\alpha, \quad z \in \mathbb{U}.$$

By noting that for any $z \in \mathbb{U}$, we have

$$\begin{aligned} &\left|\frac{(1-\lambda)D^{n+1}f(z)+\lambda D^{n+2}f(z)}{(1-\lambda)D^{n}f(z)+\lambda D^{n+1}f(z)} - 1\right| \\ &\qquad = \left|\frac{\sum_{j=2}^{\infty}[(1-\lambda)(j^{n+1}-j^n)+\lambda(j^{n+2}-j^{n+1})]a_j z^{j-1}}{1+\sum_{j=2}^{\infty}[(1-\lambda)j^n+\lambda j^{n+1}]a_j z^{j-1}}\right| \\ &\qquad\qquad \leqq \frac{\sum_{j=2}^{\infty}[(1-\lambda)(j^{n+1}-j^n)+\lambda(j^{n+2}-j^{n+1})]\,|a_j|}{1-\sum_{j=2}^{\infty}[(1-\lambda)j^n+\lambda j^{n+1}]\,|a_j|}. \end{aligned}$$

It follows from (1.3) that the above last expression is bounded above by $1-\alpha$. This completes the proof of Lemma 1. $\square$

Lemma 2. *Let $\beta > 1$ and $0 \leqq \lambda \leqq 1$. If $f \in \mathcal{A}$ satisfies the following coefficient inequality:*

$$\sum_{j=2}^{\infty} \left[(1-\lambda)j^n + \lambda j^{n+1}\right](j + |j-2\beta|)\,|a_j| \leqq 2(\beta-1), \tag{1.4}$$

then $f \in \mathcal{M}_n(\lambda,\beta)$.

Proof. To prove $f \in \mathcal{M}_n(\lambda,\beta)$, it suffices to show that

$$\left|\frac{(1-\lambda)D^{n+1}f(z) + \lambda D^{n+2}f(z)}{(1-\lambda)D^{n}f(z) + \lambda D^{n+1}f(z)}\right| < \left|\frac{(1-\lambda)D^{n+1}f(z) + \lambda D^{n+2}f(z)}{(1-\lambda)D^{n}f(z) + \lambda D^{n+1}f(z)} - 2\beta\right|. \tag{1.5}$$

We consider $M \in \mathbb{R}$ defined by

$$\begin{aligned} M :=& \left|(1-\lambda)D^{n+1}f(z) + \lambda D^{n+2}f(z)\right| \\ &- \left|(1-\lambda)D^{n+1}f(z) + \lambda D^{n+2}f(z) - 2\beta\left[(1-\lambda)D^{n}f(z) + \lambda D^{n+1}f(z)\right]\right| \\ =& \left|z + \sum_{j=2}^{\infty}\left[(1-\lambda)j^{n+1} + \lambda j^{n+2}\right]a_j z^j\right| \\ &- \left|z + \sum_{j=2}^{\infty}\left[(1-\lambda)j^{n+1} + \lambda j^{n+2}\right]a_j z^j - 2\beta z - 2\beta\sum_{j=2}^{\infty}\left[(1-\lambda)j^{n} + \lambda j^{n+1}\right]a_j z^j\right|. \end{aligned}$$

Thus, for $|z| = r < 1$, we have

$$\begin{aligned} M \leqq& r + \sum_{j=2}^{\infty}\left[(1-\lambda)j^{n+1} + \lambda j^{n+2}\right]|a_j|\,r^j \\ &- \left[(2\beta-1)r - \sum_{j=2}^{\infty}\left|\left[(1-\lambda)j^{n+1} + \lambda j^{n+2}\right] - 2\beta\left[(1-\lambda)j^{n} + \lambda j^{n+1}\right]\right||a_j|\,r^j\right] \\ <& \Bigg(\sum_{j=2}^{\infty}\{\left[(1-\lambda)j^{n+1} + \lambda j^{n+2}\right] \\ &+ \left|\left[(1-\lambda)j^{n+1} + \lambda j^{n+2}\right] - 2\beta\left[(1-\lambda)j^{n} + \lambda j^{n+1}\right]\right|\}\,|a_j| - 2(\beta-1)\Bigg)r. \end{aligned}$$

It follows from (1.4) that $M < 0$, which implies that (1.5) holds true, hence $f \in \mathcal{M}_n(\lambda,\beta)$. □

In view of Lemmas 1 and 2, we now introduce the following subclasses:

$$\widetilde{\mathcal{S}_n}(\lambda,\alpha) \subset \mathcal{S}_n(\lambda,\alpha) \quad \text{and} \quad \widetilde{\mathcal{M}_n}(\lambda,\beta) \subset \mathcal{M}_n(\lambda,\beta),$$

which consist of functions $f \in \mathcal{A}$ whose coefficients of the series satisfy the inequalities (1.3) and (1.4), respectively.

The main purpose of the present paper is to derive some subordination results for the classes $\widetilde{\mathcal{S}_n}(\lambda,\alpha)$ and $\widetilde{\mathcal{M}_n}(\lambda,\beta)$. To prove our main results, we also need the following definitions and lemma.

Definition 3 (Hadamard Product or Convolution)**.** Given two functions $f,\ g \in \mathcal{A}$, where f is given by (1.1) and g is defined by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

the Hadamard product (or convolution) $f * g$ is defined by

$$(f*g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n =: (g*f)(z).$$

Definition 4 (Subordination Principle)**.** Given two functions f and g, analytic in $\mathbb{U}$, we say that the function f is subordinate to g in $\mathbb{U}$, and write

$$f(z) \prec g(z),$$

if there exists a Schwarz function ω, which is analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1$$

such that

$$f(z) = g(\omega(z)).$$

It is easy to see that

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Longrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence:

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Longleftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Definition 5 (Subordination Factor Sequence)**.** A sequence $\{b_j\}_{j=1}^{\infty}$ of complex numbers is said to be a subordination factor sequence if, whenever f of the form (1.1) is analytic, univalent and convex in $\mathbb{U}$, we have the subordination

$$\sum_{j=1}^{\infty} a_j b_j z^j \prec f(z), \quad a_1 = 1,\ z \in \mathbb{U}.$$

Lemma 3. (See Wilf [9]) *The sequence $\{b_j\}_{j=1}^{\infty}$ is a subordinating factor sequence if and only if*

$$\Re\left(1 + 2\sum_{j=1}^{\infty} b_j z^j\right) > 0, \quad z \in \mathbb{U}.$$

2. Subordination Result for the Class $\widetilde{\mathcal{S}_n}(\lambda,\alpha)$

We begin by presenting our first subordination result given by Theorem 6 below.

Theorem 6. *If* $f \in \widetilde{\mathcal{S}_n}(\lambda,\alpha)$ *and* $g \in \mathcal{K}(0)$, *then*

$$A_n(\lambda,\alpha)\cdot(f*g)(z) \prec g(z) \tag{2.1}$$

and

$$\Re(f) > -\frac{(1-\alpha)+2^n(1+\lambda)(2-\alpha)}{2^n(1+\lambda)(2-\alpha)} \tag{2.2}$$

for any $0 \leqq \lambda \leqq 1,\ 0 \leqq \alpha < 1$ *and* $n \in \mathbb{N}_0$, *where, for convenience,*

$$A_n(\lambda,\alpha) := \frac{2^{n-1}(1+\lambda)(2-\alpha)}{(1-\alpha)+2^n(1+\lambda)(2-\alpha)}. \tag{2.3}$$

The constant factor $A_n(\lambda,\alpha)$ *in the subordination result* (2.1) *is sharp, in the sense that* $A_n(\lambda,\alpha)$ *can not be replaced by a larger factor.*

Proof. Let $f \in \widetilde{\mathcal{S}_n}(\lambda,\alpha)$ and suppose that

$$g(z) = z + \sum_{j=2}^{\infty} c_j z^j \in \mathcal{K} := \mathcal{K}(0).$$

Then

$$A_n(\lambda,\alpha)\cdot(f*g)(z) = A_n(\lambda,\alpha)\cdot\left(z+\sum_{j=2}^{\infty} a_j c_j z^j\right), \tag{2.4}$$

where $A_n(\lambda,\alpha)$ is defined by (2.3). Thus, by Definition 4, the subordination result (2.1) holds true if

$$\{A_n(\lambda,\alpha)\cdot a_j\}_{j=1}^{\infty}$$

is a subordinating factor sequence, with $a_1 = 1$. By Lemma 3, this is equivalent to the following inequality:

$$\Re\left(1+\sum_{j=1}^{\infty}\frac{(1+\lambda)(2^{n+1}-\alpha 2^n)}{(1-\alpha)+(1+\lambda)(2^{n+1}-\alpha 2^n)}a_j z^j\right) > 0, \quad z \in \mathbb{U}. \tag{2.5}$$

Since

$$(1-\lambda+\lambda j)(j^{n+1}-\alpha j^n) \qquad (j \geqq 2;\ n \in \mathbb{N}_0)$$

is an increasing function of j, and using Lemma 1, we have

$$\begin{aligned}
&\Re\left(1+\sum_{j=1}^{\infty}\frac{(1+\lambda)(2^{n+1}-\alpha 2^n)}{(1-\alpha)+(1+\lambda)(2^{n+1}-\alpha 2^n)}a_j z^j\right)\\
&=\Re\Bigg(1+\frac{(1+\lambda)(2^{n+1}-\alpha 2^n)}{(1-\alpha)+(1+\lambda)(2^{n+1}-\alpha 2^n)}a_1 z\\
&\quad+\frac{1}{(1-\alpha)+(1+\lambda)(2^{n+1}-\alpha 2^n)}\cdot\sum_{j=2}^{\infty}(1+\lambda)(2^{n+1}-\alpha 2^n)a_j z^j\Bigg)\\
&\geqq 1-\frac{(1+\lambda)(2^{n+1}-\alpha 2^n)}{(1-\alpha)+(1+\lambda)(2^{n+1}-\alpha 2^n)}r\\
&\quad-\frac{1}{(1-\alpha)+(1+\lambda)(2^{n+1}-\alpha 2^n)}\cdot\sum_{j=2}^{\infty}(1+\lambda)(2^{n+1}-\alpha 2^n)\left|a_j\right| r^j\\
&>1-\frac{(1+\lambda)(2^{n+1}-\alpha 2^n)}{(1-\alpha)+(1+\lambda)(2^{n+1}-\alpha 2^n)}r-\frac{1-\alpha}{(1-\alpha)+(1+\lambda)(2^{n+1}-\alpha 2^n)}r\\
&=1-r>0 \qquad (|z|=r<1).
\end{aligned}$$

This evidently proves the inequality (2.5), and hence also the subordination result (2.1) asserted by Theorem 6. The inequality (2.2) asserted by Theorem 6 follows from (2.1) by setting

$$g(z)=\frac{z}{1-z}=\sum_{j=1}^{\infty}z^j\in\mathcal{K}.$$

Finally, we consider the function f_0 defined by

(2.6)
$$f_0(z):=z-\frac{1-\alpha}{(1+\lambda)(2^{n+1}-\alpha 2^n)}z^2 \qquad (n\in\mathbb{N}_0;\ 0\leqq\lambda\leqq 1;\ 0\leqq\alpha<1),$$

which belongs to the class $\widetilde{\mathcal{S}_n}(\lambda,\alpha)$. Thus, by (2.1), we know that

$$A_n(\lambda,\alpha)\cdot f_0(z)\prec\frac{z}{1-z},\quad z\in\mathbb{U}.$$

Furthermore, it can be easily verified for the function f_0 given by (2.6) that

$$\min_{z\in\mathbb{U}}\left\{\Re\left(A_n(\lambda,\alpha)\cdot f_0(z)\right)\right\}=-\frac{1}{2}.$$

This complete the proof of Theorem 6. □

Remark 1. Setting $\lambda=0$ in Theorem 6, we get the corresponding result obtained by Eker *et al.* [8].

3. Subordination Result for the Class $\widetilde{\mathcal{M}_n}(\lambda, \alpha)$

The proof of the following subordination result is similar to that of Theorem 6. We, therefore, choose to omit the analogous details involved.

Theorem 7. *If $f \in \widetilde{\mathcal{M}_n}(\lambda, \alpha)$ and $g \in \mathcal{K}(0)$, then*

$$B_n(\lambda, \beta) \cdot (f * g)(z) \prec g(z) \tag{3.1}$$

and

$$\Re(f) > -\frac{\beta - 1 + 2^n\beta(1+\lambda)}{2^n\beta(1+\lambda)}$$

for any $0 \leqq \lambda \leqq 1,\ 0 \leqq \alpha < 1$ and $n \in \mathbb{N}_0$, where, for convenience,

$$B_n(\lambda, \beta) := \frac{2^{n-1}\beta(1+\lambda)}{\beta - 1 + 2^n\beta(1+\lambda)}.$$

The constant factor $B_n(\lambda, \beta)$ in the subordination result (3.1) *is sharp, in the sense that $B_n(\lambda, \beta)$ can not be replaced by a larger factor.*

Remark 2. Putting $n = 0$ or 1 and $\lambda = 0$ in Theorem 7, we get the corresponding results obtained by Srivastava and Attiya [7].

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School of Mathematics and Computing Science,
Changsha University of Science and Technology,
Yuntang Campus,
Changsha 410114, Hunan, People's Republic of China
E-mail address: zhigwang@163.com

Department of Mathematics,
University of Urmia,
Urmia, Iran
E-mail address: raghalary@yahoo.com